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Dynamical behavior for fractional-order shunting inhibitory cellular neural networks

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Abstract

This paper deals with a class of fractional-order shunting inhibitory cellular neural networks. Applying the contraction mapping principle, Krasnoselskii fixed point theorem and the inequality technique, some very verifiable criteria on the existence and uniqueness of nontrivial solution are obtained. Moreover, we also investigate the uniform stability of the fractional-order shunting inhibitory cellular neural networks. Finally, an example is given to illustrate our main theoretical findings. Our results are new and complement previously known results.

Keywords: Shunting inhibitory cellular neural networks, fractional order, uniform stability.
2010 MSC: 34B15, 34G20, 34D20.

1. Introduction

Since the work of Biouzerdoum and Pinter [1–3], shunting inhibitory cellular neural networks have been extensively applied in various fields such as psychophysics, speech, robotics, perception, adaptive pattern recognition, vision, image processing and so on. It is well known that the unique globally stable equilibrium plays an important role in solving some optimization problems. Thus considerable effort has been devoted to investigate the existence, uniqueness and stability of the equilibrium for neural networks and many results on this topic are reported (see [7, 12, 20]).

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Many scholars argue that fractional-order calculus is a valuable tool in modelling of many physics and engineering phenomena since it can describe memory and hereditary properties of the systems, while the integer order can not deal with this problem [4, 11]. During the past decades, the dynamical behavior of fractional-order neural networks has attracted tremendous attention of numerous authors. For example, Wang et al. [18] investigated the global stability analysis of fractional-order Hopfield neural networks with time delay, Zhang et al. [22] considered the Mittag-Leffler stability of fractional-order Hopfield neural networks, Wang et al. [16] discussed the asymptotic stability of delayed fractional-order neural networks with impulsive effects, Wang et al. [17] focused on the stability analysis of fractional-order Hopfield neural networks with time delays. For more detailed work, we refer the readers to [5, 6, 8, 9, 14, 19].

Here we would like to point that the existence of nontrivial solution should not be ignored when we study the stability of the equilibrium or synchronization behavior of fractional-order neural networks. Inspired by the viewpoint, in this paper, we investigate the following fractional-order shunting inhibitory cellular neural networks

$$\begin{cases} {}^cD^{\alpha}x_{ij} = -a_{ij}x_{ij}(t) - \sum\limits_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(t))x_{ij}(t) + L_{ij}, t \geq 0, \\ x_{ij}(0) = x_{ij0}, \end{cases} \tag{1.1}$$

where $i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$, $0 < \alpha < 1, {}^c D^{\alpha}$ is the Caputo fractional derivative, C_{ij} denotes the cell at the (i, j) position of the lattice, the r-neighborhood $N_r(i, j)$ of C_{ij} is given by

$$N_r(i, j) = \{C_{kl} : \max(|k - i|, |l - j|) \leq r, 1 \leq k \leq m, 1 \leq l \leq n\}.$$

x_{ij} denotes the activity of the cell C_{ij}, L_{ij} denotes the external input to C_{ij}, the constant $a_{ij} > 0$ represents the passive decay rate of the cell activity, $C_{ij}^{kl} \geq 0$ stands for the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij}, and the activity function $f(.)$ is a continuous function representing the output or firing rate of the cell C_{kl}.

In this paper, we make an attempt to discuss the dynamics of system (1.1). The rest of this paper is arranged as follows. In Section 2, we present some definitions and lemmas. In Section 3, we establish some sufficient criteria on the existence and uniqueness of the nontrivial solution and uniform stability of the fractional-order neural networks in a finite time interval. In Section 4, an example is given to illustrate the efficiency of the theoretical findings. In Section 5, a brief conclusion is presented.

2. Preliminaries

In this section, we will present some preliminaries on fractional calculus. In details, one can see [13]. We know that there are several definitions for fractional derivatives such as Grünwald-Letnikov, Riemann-Liouville and Caputo. Throughout this paper, the Caputo derivative is used since its initial conditions take the same the integer order differential equation.

Definition 2.1 ([13])**.** The Riemann-Liouville fractional integral operator of $\alpha > 0$ of the function h is defined by

$$I^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} h(s)ds,$$

where Γ is the gamma function.

Definition 2.2 ([13])**.** The Caputo fractional-order derivative of order $\alpha > 0$ of a function $h(t)$ is defined by

$${}^cD^{\alpha}h(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} h^{(n)}(s)ds, n - 1 < \alpha < n, n \in N^+.$$

Lemma 2.3 ([21])**.** *Let $\alpha > 0$, then the differential equation ${}^cD^{\alpha}y(t) = h(t)$ has solutions*

$$y(t) = I^{\alpha}h(t) + c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in R, n = [\alpha] + 1$.

Lemma 2.4 ([10]). *Let D be a closed convex and nonempty subset of a Banach space X. Let ϕ_1, ϕ_2 be the operators such that*

(i) $\phi_1 x + \phi_2 y \in D$, *whenever* $x, y \in D$;

(ii) ϕ_1 *is compact and continuous;*

(iii) ϕ_2 *is a contraction mapping.*

Then there exists $x \in D$ such that $\phi_1 x + \phi_2 x = x$.

As a consequence of Lemma 2.3, we define the solution of (1.1).

Lemma 2.5. *The continuous function $x_{ij}(t)$ is said to be a solution of the system* (1.1) *if the following condition*

$$x_{ij} = x_{ij0} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[-a_{ij}x_{ij}(s) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(s))x_{ij}(s) + L_{ij} \right] ds,$$

is satisfied.

3. Existence, uniqueness and uniform stability of solution

In this section, we will investigate the existence, uniqueness and uniform stability of solution. In order to obtain our main results, we firstly make the following assumption.

(H1): There exist positive constants L and M such that for any $u, v \in R$,

$$|f(u) - f(v)| \le L|u - v|, \quad |f(u)| \le M.$$

Let $X = \{x | x = (x_{11}, x_{12}, \cdots, x_{1n}, x_{21}, x_{22}, \cdots, x_{mn})^T, x_{ij} \in C[0,T]\}$. Obviously, X is a Banach space with the norm

$$||x|| = \sup_{0\le t\le T} \left(\sum_{i=1,2,\cdots,m,j=1,2,\cdots,n}^{mn} |x_{ij}(t)|^p \right)^{\frac{1}{p}}.$$

Now we are ready to present our the first result.

Theorem 3.1. *In addition to* (H1), *if there exists a real number $p > 1$ such that*

(H2): $a^2 - 4\Pi_0 c > 0$, *where*

$$a = \left\{ \frac{a_{ij}^+ T^\alpha (mn)^{\frac{1}{p}}}{\Gamma(\alpha+1)} + \frac{L^+ T^\alpha (mn)^{\frac{1}{p}}}{\Gamma(\alpha+1)} + \frac{|f(0)|T^\alpha}{\Gamma(\alpha+1)} \left[\sum_{ij=11}^{mn} C_{ij0}^p \right]^{\frac{1}{p}} - 1 \right\},$$

$$c = \frac{LT^\alpha}{\Gamma(\alpha+1)} \left[\sum_{C_{kl}\in N_r(i,j)} |C_{ij}^{kl}|^{\frac{p}{p-1}} \right]^{\frac{p-1}{p}}, \Pi_0 = \left[\sum_{ij=11}^{mn} |x_{ij0}| \right]^{\frac{1}{p}},$$

$$C_{ij0} = \sum_{C_{kl}\in N_r(i,j)} |C_{ij}^{kl}|, a^+ = \max_{ij\in\Lambda} a_{ij}, \Lambda = \{11, 12, \cdots, 1n, 21, 22, \cdots, mn\},$$

(H3):

$$a^+ T^\alpha (mn)^{\frac{1}{p}} + (\sum_{C_{kl}\in N_r(i,j)} |C_{ij}^{kl}| + \varrho L) T^\alpha (mn)^{\frac{1}{p}} < \Gamma(\alpha+1),$$

where $\varrho \ge a + \sqrt{a^2 - 4\Pi_0 c}$, then system (1.1) *has a unique solution on* $[0, T]$.

Proof. Define $F : X \to X$ as follows

$$(Fx)(t) = ((Fx_{11})(t), (Fx_{12})(t), \cdots, (Fx_{mn})(t))^T, \tag{3.1}$$

where

$$(Fx_{ij})(t) = x_{ij0} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[-a_{ij}x_{ij}(s) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(s))x_{ij}(s) + L_{ij} \right] ds. \tag{3.2}$$

Firstly, we prove that $FB_\varrho \subset B_\varrho$, where $B_\varrho = \{x \in X : ||x|| \le \varrho\}$ and $\varrho \ge a + \sqrt{a^2 - 4\Pi_0 c}$. It follows from (3.1) that

$$||Fx|| = \sup_{0\le t\le T} \left\{ \sum_{ij=11}^{mn} \left| x_{ij0} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[-a_{ij}x_{ij}(s) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(s))x_{ij}(s) + L_{ij} \right] ds \right|^p \right\}^{\frac{1}{p}}. \tag{3.3}$$

By the Minkowski inequality

$$\left[\sum_{i=1}^{n} (a_i + b_i + \cdots + s_i)^p \right]^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p \right)^{\frac{1}{p}} + \cdots + \left(\sum_{i=1}^{n} s_i^p \right)^{\frac{1}{p}},$$

where $a_i, b_i, \cdots, s_i \ge 0, p > 1, i = 1, 2, \cdots, n$, we have

$$\begin{aligned}
||Fx|| \le & \left[\sum_{ij=11}^{mn} |x_{ij0}|^p \right]^{\frac{1}{p}} + \sup_{0\le t\le T} \left[\sum_{ij=11}^{mn} \left(\int_0^t \frac{(t-s)^{\alpha-1} a_{ij} |x_{ij}(s)|}{\Gamma(\alpha)} ds \right)^p \right]^{\frac{1}{p}} \\
& + \sup_{0\le t\le T} \left[\sum_{ij=11}^{mn} \left(\int_0^t \frac{(t-s)^{\alpha-1} |L_{ij}|}{\Gamma(\alpha)} ds \right)^p \right]^{\frac{1}{p}} \\
& + \sup_{0\le t\le T} \left[\sum_{ij=11}^{mn} \left(\int_0^t \sum_{C_{kl}\in N_r(i,j)} \frac{|C_{ij}^{kl}|(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(0)||x_{ij}(s)ds \right)^p \right]^{\frac{1}{p}} \\
& + \sup_{0\le t\le T} \left[\sum_{ij=11}^{mn} \left(\int_0^t \sum_{C_{kl}\in N_r(i,j)} \frac{|C_{ij}^{kl}|(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(x_{kl}(s)) - f(0)||x_{ij}(s)|ds \right)^p \right]^{\frac{1}{p}} \\
= & \Pi_0 + \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4,
\end{aligned} \tag{3.4}$$

where

$$\Pi_0 = \left[\sum_{ij=11}^{mn} |x_{ij0}|^p \right]^{\frac{1}{p}},$$

$$\Pi_1 = \sup_{0\le t\le T} \left[\sum_{ij=11}^{mn} \left(\int_0^t \frac{(t-s)^{\alpha-1} a_{ij} |x_{ij}(s)|}{\Gamma(\alpha)} ds \right)^p \right]^{\frac{1}{p}},$$

$$\Pi_2 = \sup_{0\le t\le T} \left[\sum_{ij=11}^{mn} \left(\int_0^t \frac{(t-s)^{\alpha-1} |L_{ij}|}{\Gamma(\alpha)} ds \right)^p \right]^{\frac{1}{p}},$$

$$\Pi_3 = \sup_{0\le t\le T} \left[\sum_{ij=11}^{mn} \left(\int_0^t \sum_{C_{kl}\in N_r(i,j)} \frac{|C_{ij}^{kl}|(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(0)||x_{ij}(s)|ds \right)^p \right]^{\frac{1}{p}},$$

$$\Pi_4 = \sup_{0\le t\le T}\left[\sum_{ij=11}^{mn}\left(\int_0^t \sum_{C_{kl}\in N_r(i,j)} \frac{|C_{ij}^{kl}|(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(x_{kl}(s))-f(0)||x_{ij}(s)|ds\right)^p\right]^{\frac{1}{p}}.$$

Then

$$\begin{aligned}
\Pi_1 &= \sup_{0\le t\le T}\left[\sum_{ij=11}^{mn}\left(\int_0^t \frac{(t-s)^{\alpha-1}a_{ij}|x_{ij}(s)|}{\Gamma(\alpha)}ds\right)^p\right]^{\frac{1}{p}}\\
&\le \frac{a_{ij}^+T^\alpha(mn)^{\frac{1}{p}}}{\Gamma(\alpha+1)}\varrho,\\
\Pi_2 &= \sup_{0\le t\le T}\left[\sum_{ij=11}^{mn}\left(\int_0^t \frac{(t-s)^{\alpha-1}|L_{ij}|}{\Gamma(\alpha)}ds\right)^p\right]^{\frac{1}{p}}\\
&\le \frac{L^+T^\alpha(mn)^{\frac{1}{p}}}{\Gamma(\alpha+1)},\\
\Pi_3 &= \sup_{0\le t\le T}\left[\sum_{ij=11}^{mn}\left(\int_0^t \sum_{C_{kl}\in N_r(i,j)} \frac{|C_{ij}^{kl}|(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(0)||x_{ij}(s)|ds\right)^p\right]^{\frac{1}{p}}\\
&\le \frac{|f(0)|T^\alpha}{\Gamma(\alpha+1)}\left[\sum_{ij=11}^{mn}C_{ij0}^p\right]^{\frac{1}{p}}\varrho,\\
\Pi_4 &= \sup_{0\le t\le T}\left[\sum_{ij=11}^{mn}\left(\int_0^t \sum_{C_{kl}\in N_r(i,j)} \frac{|C_{ij}^{kl}|(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(x_{kl}(s))-f(0)||x_{ij}(s)|ds\right)^p\right]^{\frac{1}{p}}\\
&\le \sup_{0\le t\le T}\left[\sum_{ij=11}^{mn}\left(\int_0^t \sum_{C_{kl}\in N_r(i,j)} \frac{|C_{ij}^{kl}|(t-s)^{\alpha-1}}{\Gamma(\alpha)}L|x_{kl}(s))||x_{ij}(s)|ds\right)^p\right]^{\frac{1}{p}}\\
&\le \sup_{0\le t\le T}\left\{\sum_{ij=11}^{mn}\left[\int_0^t\left(\sum_{C_{kl}\in N_r(i,j)}\left(\frac{|C_{ij}^{kl}|L(t-s)^{\alpha-1}|x_{ij}(s)|}{\Gamma(\alpha)}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right.\right.\\
&\qquad\left.\left.\times\left(\sum_{C_{kl}\in N_r(i,j)}|x_{kl}(s)|^p\right)^{\frac{1}{p}}ds\right]^p\right\}^{\frac{1}{p}}\\
&\le \frac{LT^\alpha}{\Gamma(\alpha+1)}\left[\sum_{C_{kl}\in N_r(i,j)}|C_{ij}^{kl}|^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}}\varrho^2.
\end{aligned}$$

Here we shall point out that the upper bound Π_4 is derived by applying Holder inequality

$$\sum_{i=1}^n a_ib_i \le \left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}},$$

where

$$a_i, b_i \ge 0,\quad p,q>0,\quad \frac{1}{p}+\frac{1}{q}=1.$$

It follows from (3.4) that

$$||Fx|| \le \left[\sum_{ij=11}^{mn}|x_{ij0}|\right]^{\frac{1}{p}} + \left\{\frac{a_{ij}^+T^\alpha(mn)^{\frac{1}{p}}}{\Gamma(\alpha+1)}\right.$$

$$
+ \frac{L^{+}T^{\alpha}(mn)^{\frac{1}{p}}}{\Gamma(\alpha+1)} + \frac{|f(0)|T^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{ij=11}^{mn} C_{ij0}^{p}\right]^{\frac{1}{p}}\Bigg\}\varrho
+ \frac{LT^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{C_{kl}\in N_r(i,j)} |C_{ij}^{kl}|^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}} \varrho^2. \tag{3.5}
$$

In view of (H2), we get $||Fx|| \leq \varrho$.

Secondly, we prove that $F : X \to X$ ia a contraction mapping. Let $x, y \in X$, we get

$$
\begin{aligned}
||Fx - Fy|| &= \sup_{0\leq t\leq T}\left\{\sum_{ij=11}^{mn}\left|\int_0^t \frac{-a_{ij}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Bigg[(x_{ij}(s)-y_{ij}(s))\right.\right.\\
&\quad \left.\left. - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(f(x_{kl}(s))x_{ij}(s) - f(y_{kl}(s))y_{ij}(s))\Bigg]ds\right|^p\right\}^{\frac{1}{p}}\\
&\leq \sup_{0\leq t\leq T}\left\{\sum_{ij=11}^{mn}\left[\int_0^t \frac{a_{ij}|x_{ij}(s)-y_{ij}(s)|(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right]^p\right\}^{\frac{1}{p}}\\
&\quad + \sup_{0\leq t\leq T}\left\{\sum_{ij=11}^{mn}\left[\int_0^t \frac{\sum_{C_{kl}\in N_r(i,j)}|C_{ij}^{kl}||x_{ij}(s)-y_{ij}(s)|(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right]^p\right\}^{\frac{1}{p}}\\
&\quad + \sup_{0\leq t\leq T}\left\{\sum_{ij=11}^{mn}\left[\int_0^t \frac{\sum_{C_{kl}\in N_r(i,j)}|C_{ij}^{kl}|\varrho L|x_{kl}(s)-y_{kl}(s)|(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right]^p\right\}^{\frac{1}{p}}\\
&\leq \left[\frac{a_{ij}^{+}T^{\alpha}(mn)^{\frac{1}{p}}}{\Gamma(\alpha+1)} + \frac{\sum_{C_{kl}\in N_r(i,j)}|C_{ij}^{kl}|MT^{\alpha}(mn)^{\frac{1}{p}}}{\Gamma(\alpha+1)}\right.\\
&\quad \left. + \frac{\sum_{C_{kl}\in N_r(i,j)}|C_{ij}^{kl}|\varrho LT^{\alpha}(mn)^{\frac{1}{p}}}{\Gamma(\alpha+1)}\right]||x-y||.
\end{aligned} \tag{3.6}
$$

In view of (H3), we can conclude that F is a contraction mapping. The proof of Theorem 3.1 is completed. □

Theorem 3.2. *In addition to* (H1)*, if there exists a real number* $p > 1$ *such that*

(H4):

$$
a_{ij}^{+}T^{\alpha}(mn)^{\frac{1}{p}} < \Gamma(\alpha+1),
$$

then system (1.1) *has at least one solution on* $[0, T]$.

Proof. Now we can define two operators L and N on $B_{\varrho}(B_{\varrho} = \{x \in X : ||x|| \leq \varrho\})$ by

$$
(Lx)(t) = ((Lx_{11})(t), (Lx_{12})(t), \cdots, (Lx_{mn})(t))^T,
$$
$$
(Nx)(t) = ((Nx_{11})(t), (Nx_{12})(t), \cdots, (Nx_{mn})(t))^T,
$$

where

$$
(Lx_{ij})(t) = x_{ij0} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[-a_{ij}x_{ij}(s) + L_{ij}\right]ds,
$$

$$(Nx_{ij})(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[-\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}f(x_{kl}(s))x_{ij}(s)\right]ds.$$

We firstly prove that for any $x,y\in B_\varrho$, $Lx+Ny\in B_\varrho$. In fact, by Minkowski inequality, we have

$$\begin{aligned}||Lx+Ny||&=\sup_{0\le t\le T}\left\{\sum_{ij=11}^{mn}\left|x_{ij0}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[-a_{ij}x_{ij}(s)\right.\right.\right.\\&\qquad\left.\left.\left.-\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}f(x_{kl}(s))x_{ij}(s)+L_{ij}\right]ds\right|^p\right\}^{\frac{1}{p}}.\\&\le\left[\sum_{ij=11}^{mn}|x_{ij0}|\right]^{\frac{1}{p}}+\sup_{0\le t\le T}\left[\sum_{ij=11}^{mn}\left(\int_0^t\frac{(t-s)^{\alpha-1}a_{ij}|x_{ij}(s)|}{\Gamma(\alpha)}ds\right)^p\right]^{\frac{1}{p}}\\&\quad+\sup_{0\le t\le T}\left[\sum_{ij=11}^{mn}\left(\int_0^t\frac{(t-s)^{\alpha-1}|L_{ij}|}{\Gamma(\alpha)}ds\right)^p\right]^{\frac{1}{p}}\\&\quad+\sup_{0\le t\le T}\left[\sum_{ij=11}^{mn}\left(\int_0^t\sum_{C_{kl}\in N_r(i,j)}\frac{|C_{ij}^{kl}|(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(0)||x_{ij}(s)ds\right)^p\right]^{\frac{1}{p}}\\&\quad+\sup_{0\le t\le T}\left[\sum_{ij=11}^{mn}\left(\int_0^t\sum_{C_{kl}\in N_r(i,j)}\frac{|C_{ij}^{kl}|(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(x_{kl}(s))-f(0)||x_{ij}(s)|ds\right)^p\right]^{\frac{1}{p}}\\&\le\varrho.\end{aligned}$$

Then we can conclude that $Lx+Ny\in B_\varrho$.
Secondly, for any $x,y\in B_\varrho$, we have

$$\begin{aligned}||Lx-Ly||&=\sup_{0\le t\le T}\left[\sum_{ij=11}^{mn}\int_0^t\left(\frac{a_{ij}(t-s)^{\alpha-1}|x_{ij}(s)-y_{ij}(s)|}{\Gamma(\alpha)}ds\right)^p\right]^{\frac{1}{p}}\\&\le\frac{a_{ij}^{+}T^\alpha(mn)^{\frac{1}{p}}}{\Gamma(\alpha+1)}.\end{aligned}$$

By (H4), we know that L is a contraction mapping.

Now we prove that N is continuous and compact. Since f is continuous, then N is continuous. Let $x\in B_\varrho$, we get

$$\begin{aligned}||(Nx)(t)||&=\sup_{0\le t\le T}\left\{\sum_{ij=11}^{mn}\left[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sum_{C_{kl}\in N_r(i,j)}|C_{ij}^{kl}||f(x_{kl}(s))x_{ij}(s)|ds\right]^p\right\}^{\frac{1}{p}}\\&=\frac{MT^\alpha}{\Gamma(\alpha+1)}\left[\sum_{C_{kl}\in N_r(i,j)}|C_{ij}^{kl}|^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}}.\end{aligned}$$

This implies that N is uniformly bounded on B_ϱ.

In the sequel, we prove that $(Nx)(t)$ is equicontinuous. In fact, for $x\in B_\varrho, 0<t_2<t_1$, we get

$$||(Nx)(t_1)-(Nx)(t_2)||=\left[\left(\sum_{ij=11}^{mn}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\sum_{C_{kl}\in N_r(i,j)}|C_{ij}^{kl}||f(x_{kl}(s))x_{ij}(s)|ds\right.\right.$$

$$
\begin{aligned}
&- \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{C_{kl}\in N_r(i,j)} |C_{ij}^{kl}||f(x_{kl}(s))x_{ij}(s)|ds\Bigg)^p\Bigg]^{\frac{1}{p}} \\
&= \Bigg[\sum_{ij=11}^{mn}\Bigg(\sum_{C_{kl}\in N_r(i,j)} |C_{ij}^{kl}|M\Bigg(\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds \qquad (3.7)\\
&+ \int_{t_2}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Bigg)\Bigg)^p\Bigg]^{\frac{1}{p}} \\
&\le \frac{(M+\varrho)T^\alpha}{\Gamma(\alpha+1)}\Bigg[\sum_{C_{kl}\in N_r(i,j)} |C_{ij}^{kl}|^{\frac{p}{p-1}}\Bigg]^{\frac{p-1}{p}} |(2(t_1-t_2)^\alpha + t_2^\alpha - t_1^\alpha|.
\end{aligned}
$$

Let $t_2 \to t_1$, then the right-hand side of (3.7) tends to zero. Thus $N(B_\varrho)$ is relatively compact. In view of Arzalá-Ascoli theorem, we can conclude that N is compact. Thus it follows from Lemma 2.4 that system (1.1) has at least one solution. The proof of Theorem 3.2 is completed. □

Theorem 3.3. *In addition to the conditions* (H1)–(H3)*, if*

(H5):

$$
\frac{a_{ij}^+ T^\alpha (mn)^{\frac{1}{p}}}{\Gamma(\alpha+1)} + \frac{(M+\varrho)T^\alpha}{\Gamma(\alpha+1)}\Bigg[\sum_{C_{kl}\in N_r(i,j)} |C_{ij}^{kl}|^{\frac{p}{p-1}}\Bigg]^{\frac{p-1}{p}} < 1,
$$

is satisfied, then the solution of system (1.1) *is uniformly stable on* $[0, T]$.

Proof. Assume that $x_{ij}(t)$ and $y_{ij}(t)$ are any two solutions of system (1.1) with initial condition $x_{ij}(0) = x_{ij0}, y_{ij}(0) = y_{ij0}$ and $\left(\sum_{ij=11}^{mn} |x_{ij0} - y_{ij0}|^p\right)^{\frac{1}{p}} \le \varrho$.

Then we have

$$
\begin{aligned}
x_{ij}(t) &= x_{ij0} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Bigg[-a_{ij}x_{ij}(s) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(s))x_{ij}(s) + L_{ij}\Bigg]ds,\\
y_{ij}(t) &= y_{ij0} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Bigg[-a_{ij}y_{ij}(s) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(y_{kl}(s))y_{ij}(s) + L_{ij}\Bigg]ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
||x-y|| &= \sup_{0\le t\le T}\left(\sum_{ij=11}^{mn} |x_{ij}(t) - y_{ij}(t)|^p\right)^{\frac{1}{p}} \\
&\le \left(\sum_{ij=11}^{mn} |x_{ij0} - y_{ij0}|^p\right)^{\frac{1}{p}} \\
&+ \sup_{0\le t\le T}\Bigg[\sum_{ij=11}^{mn}\int_0^t\left(\frac{a_{ij}(t-s)^{\alpha-1}|x_{ij}(s)-y_{ij}(s)|}{\Gamma(\alpha)}ds\right)^p\Bigg]^{\frac{1}{p}} \\
&+ \sup_{0\le t\le T}\Bigg[\sum_{ij=11}^{mn}\int_0^t\left(\frac{\sum_{C_{kl}\in N_r(i,j)}|C_{ij}^{kl}|(t-s)^{\alpha-1}|f(x_{kl}(s))x_{ij}(s) - f(y_{kl}(s))y_{ij}(s)|}{\Gamma(\alpha)}ds\right)^p\Bigg]^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \varrho + \sup_{0\leq t\leq T}\left[\sum_{ij=11}^{mn}\int_0^t\left(\frac{a_{ij}(t-s)^{\alpha-1}|x_{ij}(s)-y_{ij}(s)|}{\Gamma(\alpha)}ds\right)^p\right]^{\frac{1}{p}}\\
&\quad+\sup_{0\leq t\leq T}\left[\sum_{ij=11}^{mn}\int_0^t\left(\frac{\sum_{C_{kl}\in N_r(i,j)}|C_{ij}^{kl}|(t-s)^{\alpha-1}M|x_{ij}(s)-y_{ij}(s)|}{\Gamma(\alpha)}ds\right)^p\right]^{\frac{1}{p}}\\
&\quad+\sup_{0\leq t\leq T}\left[\sum_{ij=11}^{mn}\int_0^t\left(\frac{\sum_{C_{kl}\in N_r(i,j)}|C_{ij}^{kl}|(t-s)^{\alpha-1}\varrho|x_{kl}(s)-y_{kl}(s)|}{\Gamma(\alpha)}ds\right)^p\right]^{\frac{1}{p}}\\
&\leq \varrho + \left\{\frac{a_{ij}^{+}T^{\alpha}(mn)^{\frac{1}{p}}}{\Gamma(\alpha+1)}+\frac{(M+\varrho)T^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{C_{kl}\in N_r(i,j)}|C_{ij}^{kl}|^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}}\right\}||x-y||.
\end{aligned}
$$

Then

$$
||x-y||\leq \frac{1}{1-\left\{\frac{a_{ij}^{+}T^{\alpha}(mn)^{\frac{1}{p}}}{\Gamma(\alpha+1)}+\frac{(M+\varrho)T^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{C_{kl}\in N_r(i,j)}|C_{ij}^{kl}|^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}}\right\}}\varrho\ .
$$

Thus, for any $\varepsilon > 0$, there exists

$$
\varrho = 1-\left\{\frac{a_{ij}^{+}T^{\alpha}(mn)^{\frac{1}{p}}}{\Gamma(\alpha+1)}+\frac{(M+\varrho)T^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{C_{kl}\in N_r(i,j)}|C_{ij}^{kl}|^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}}\right\},
$$

such that $||x-y||\leq \varepsilon$, which implies that the solution of system (1.1) is uniformly stable on $[0,T]$. The proof of Theorem 3.3 is completed. □

4. An example

In this section, we give an example to illustrate our main results. Consider the following fractional-order shunting inhibitory cellular neural networks

$$
\begin{cases}
{}^{c}D^{\alpha}x_{11} = -a_{11}x_{11}(t) - \sum\limits_{C_{kl}\in N_r(1,1)} C_{11}^{kl}f(x_{kl}(t))x_{11}(t)+L_{11}, t\geq 0,\\
{}^{c}D^{\alpha}x_{12} = -a_{12}x_{12}(t) - \sum\limits_{C_{kl}\in N_r(1,2)} C_{12}^{kl}f(x_{kl}(t))x_{12}(t)+L_{12}, t\geq 0,\\
{}^{c}D^{\alpha}x_{21} = -a_{21}x_{21}(t) - \sum\limits_{C_{kl}\in N_r(2,1)} C_{21}^{kl}f(x_{kl}(t))x_{21}(t)+L_{21}, t\geq 0,\\
{}^{c}D^{\alpha}x_{22} = -a_{22}x_{22}(t) - \sum\limits_{C_{kl}\in N_r(2,2)} C_{22}^{kl}f(x_{kl}(t))x_{22}(t)+L_{22}, t\geq 0,\\
x_{11}(0)=x_{110}=0.2,\\
x_{12}(0)=x_{120}=0.1,\\
x_{21}(0)=x_{210}=0.1,\\
x_{22}(0)=x_{220}=0.1,
\end{cases}
\tag{4.1}
$$

where

$$
\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} = \begin{bmatrix} 0.5 & 0.4\\ 0.45 & 0.52\end{bmatrix},
$$

$$
\begin{bmatrix} L_{11} & L_{12}\\ L_{21} & L_{22}\end{bmatrix} = \begin{bmatrix} 0.25 & 0.32\\ 0.41 & 0.28\end{bmatrix},
$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0 \end{bmatrix}.$$

Let $T = 1, p = 2, \alpha = 0.7, f(x) = \tanh(x), r = 1$. Then $I_0 = 0.41$. It is easy to check that all the conditions in Theorem 3.3 are fulfilled. Hence we can conclude that system (4.1) has a unique solution which uniformly stable on $[0, T]$. The result is shown in Figure 1.

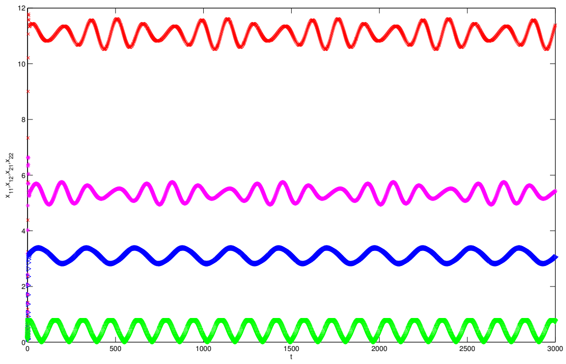

Figure 1: Time response of state variables $x_{ij}(i, j = 1, 2)$ where the red line stands for $x_{11}(t)$ and the magenta line stands for $x_{12}(t)$, the blue line stands for $x_{21}(t)$ and the green line stands for $x_{22}(t)$.

5. Conclusions

In this paper, we investigate a class of fractional-order shunting inhibitory cellular neural networks. Some very verifiable criteria on the existence and uniqueness of nontrivial solution are established by applying the contraction mapping principle, Krasnoselskii fixed point theorem and the inequality technique. Further, the uniform stability of the fractional-order shunting inhibitory cellular neural networks in fixed time-intervals is analyzed. At last, an example is given to illustrate our main theoretical predictions. In [15], Shao only investigated the anti-periodic solution of shunting inhibitory cellular neural networks which is integer order. In this paper, we consider the existence and uniqueness of nontrivial solution of fractional-order shunting inhibitory cellular neural networks. From this viewpoint, our results are new and complement previously known results in [15].

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