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On a degenerate λ-q-Daehee polynomials

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Abstract

Daehee numbers and polynomials are introduced by Kim [T. Kim, Integral Transforms Spec. Funct., **13** (2002), 65–69] and [D. S. Kim, T. Kim, Appl. Math. Sci. (Ruse), **7** (2013), 5969–5976], and those polynomials and numbers are generalized by many researchers. In this paper, we make an attempt to degenerate λ-q-Daehee polynomials, and derive some new and interesting identities and properties of those polynomials and numbers.

Keywords: λ-Daehee polynomials, q-Daehee polynomials, degenerate λ-q-Daehee polynomials.
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1. Introduction

Let p be a given prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denotes the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normally defined by $|p|_p = \frac{1}{p}$, and let q be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$ so that $q^x = e^{x \log q}$ for each $x \in \mathbb{Z}_p$. The q-extension of number x is defined by

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$.

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Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *p-adic bosonic integral on* $\mathbb{Z}_p$ is defined by Kim (see [5, 6]) to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x. \tag{1.1}$$

If we put $f_1(x) = f(x+1)$, then, by (1.1), we can derive the following very useful integral identity;

$$qI_q(f_1) - I_q(f) = (q-1)f(0) + \frac{q-1}{\log q} f'(0), \tag{1.2}$$

where $f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0}$.

As is well-known, the *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l) x^l, \tag{1.3}$$

and the *Stirling numbers of the second kind* is defined by the generating function to be

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!}, \tag{1.4}$$

where n is an nonnegative integer (see [3, 16]). Note that

$$(\log(x+1))^n = n! \sum_{l=n}^{\infty} S_1(l,n) \frac{x^l}{l!}, \ (n \geq 0), \ \text{(see [3, 16])}. \tag{1.5}$$

The *q-Bernoulli polynomials of order k* are defined as follows:

$$\left(\frac{q-1+\frac{(q-1)}{\log q} t}{qe^t - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x) \frac{t^n}{n!}, \ \text{(see [2, 13])}. \tag{1.6}$$

When $x = 0$, $B_{n,q} = B_{n,q}(0)$ are called the *n-th q-Bernoulli numbers.*

The *Daehee polynomials of the first kind* are defined by the generating function to be

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!},$$

and the *Daehee polynomials of the second kind* are given by the generating function to be

$$\frac{\log(1+t)}{1-(1+t)^{-1}}(1+t)^x = \sum_{n=0}^{\infty} \widehat{D}_n(x) \frac{t^n}{n!},$$

(see [8, 11, 15, 17]). In [2], Cho et. al. defined the *q-Daehee polynomials* as follows:

$$\frac{1-q+\frac{1-q}{\log q}\log(1+t)}{1-q(1+t)}(1+t)^x = \sum_{n=0}^{\infty} D_{n,q}(x) \frac{t^n}{n!},$$

and, in [14], Park generalized the q-Daehee polynomials which are called the *λ-q-Daehee polynomials* as follows:

$$\frac{q-1+\frac{q-1}{\log q}\lambda\log(1+t)}{q(1+t)^\lambda - 1}(1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda,q}(x) \frac{t^n}{n!}. \tag{1.7}$$

In [1], Carlitz consider the degenerate Bernoulli polynomials which are defined by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{x}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_n(x|\lambda)\frac{t^n}{n!}. \tag{1.8}$$

When $x=0$, $\beta_n(\lambda)=\beta_n(0|\lambda)$ are called the *degenerate Bernoulli numbers*. Note that $\lim_{\lambda\to 0}\beta_n(\lambda)=B_n$.

It is well-known fact that

$$e^t=\lim_{u\to 0}(1+ut)^{\frac{1}{u}},\ \text{(see [1, 3])}.$$

Thus, the function $(1+ut)^{\frac{1}{u}}$ is called the *degenerate function of* e^t, and so for $t=\log e^t$, we have $\log(1+ut)^{\frac{1}{u}}$ as the degenerate function.

Recently, many authors have studied special polynomials related to Daehee polynomials and Changhee polynomials (see [1]–[18]). These polynomials are useful to study number theory, special function theory, umbral calculus, combinatorics and other applied mathematics and mathematical physics. In particular, in [11], authors considered the λ-Daehee polynomials and investigated their properties, and in [14], author attempted generalization of those polynomials.

In this paper, we attempt the q-analogue of degenerate λ-Daehee polynomials which are called λ-q-Daehee polynomials, and find some new and interesting identities and properties of those polynomials and numbers.

2. Degenerate λ-q-Daehee polynomials of the first kind

From now on, we assume that $t\in\mathbb{C}$ with $|t|_p<p^{-\frac{1}{p-1}}$ and $\lambda\in\mathbb{Z}_p$, and consider the *degenerate λ-q-Daehee polynomials* which are a generalization of Daehee polynomials as follows:

$$\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda}-1}\left(1+\frac{1}{u}\log(1+ut)\right)^{x}=\sum_{n=0}^{\infty}D_{n,\lambda,q}(x|u)\frac{t^n}{n!}. \tag{2.1}$$

When $x=0$, $D_{n,\lambda,q}(u)=D_{n,\lambda,q}(0|u)$ are called the *degenerate λ-q-Daehee numbers*. Note that $\lim_{u\to 0}D_{n,\lambda,q}(x|u)=D_{n,\lambda,q}(x)$ and $D_{n,1,q}(0)=D_{n,q}$.

Let us take $f(x)=\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda x}$. From (1.2), we have

$$\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda x}d\mu_q(x)=\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda}-1}. \tag{2.2}$$

By (2.1) and (2.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}D_{n,\lambda,q}(x|u)\frac{t^n}{n!}&=\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda}-1}\left(1+\frac{1}{u}\log(1+ut)\right)^{x}\\&=\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda y+x}d\mu_q(y),\end{aligned} \tag{2.3}$$

and, by (1.5),

$$\begin{aligned}\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda y+x}d\mu_q(y)&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\lambda y+x}{n}u^{-n}\left(\log(1+ut)\right)^n d\mu_q(y)\\&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\lambda y+x}{n}u^{-n}n!\sum_{k=n}^{\infty}S_1(k,n)\frac{u^kt^k}{k!}d\mu_q(y)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}u^{n-k}S_1(n,k)\int_{\mathbb{Z}_p}(\lambda y+x)_k d\mu_q(y)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

Thus, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$D_{n,\lambda,q}(x|u) = \sum_{k=0}^{n} u^{n-k} S_1(n,k) \int_{\mathbb{Z}_p} (\lambda y + x)_k d\mu_q(y).$$

By replacing t by $\frac{1}{u}\left(e^{ut}-1\right)$ in (2.1), we obtain the equation

$$\begin{aligned} \frac{q-1+\frac{q-1}{\log q}\lambda\log(1+t)}{q(1+t)^{\lambda}-1}(1+t)^x &= \sum_{n=0}^{\infty} D_{n,\lambda,q}(x|u)\frac{1}{n!}\left(\frac{1}{u}\left(e^{ut}-1\right)\right)^n \\ &= \sum_{n=0}^{\infty} D_{n,\lambda,q}(x|u)\frac{1}{n!}n!\sum_{l=n}^{\infty} S_2(l,n)u^{-n}\frac{(ut)^l}{l!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{D_{m,\lambda,q}(x|u)S_2(n,m)u^{n-m}}{m!}\right)\frac{t^n}{n!}. \end{aligned} \tag{2.5}$$

On the other hand, by replacing t by $\frac{1}{u}\left(e^{u(e^t-1)}-1\right)$ in (2.1), we have

$$\begin{aligned} \frac{q-1+\frac{q-1}{\log q}\lambda t}{qe^{\lambda t}-1}e^{tx} &= \sum_{n=0}^{\infty} D_{n,\lambda,q}(x|u)\frac{u^{-n}}{n!}\left(e^{u(e^t-1)}-1\right)^n \\ &= \sum_{n=0}^{\infty} D_{n,\lambda,q}(x|u)\frac{u^{-n}}{n!}n!\sum_{l=n}^{\infty} S_2(l,n)\frac{u^l}{l!}\left(e^t-1\right)^l \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} D_{m,\lambda,q}(x|u)S_2(n,m)\frac{u^{n-m}}{n!}\left(e^t-1\right)^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m} D_{k,\lambda,q}(x|u)u^{m-k}S_2(m,k)S_2(n,m)\right)\frac{t^n}{n!}, \end{aligned} \tag{2.6}$$

and

$$\frac{q-1+\frac{q-1}{\log q}\lambda t}{qe^{\lambda t}-1}e^{tx} = \frac{q-1+\frac{q-1}{\log q}\lambda t}{qe^{\lambda t}-1}e^{\lambda t\left(\frac{\lambda}{x}\right)} = \sum_{n=0}^{\infty}\lambda^n B_{n,q}\left(\frac{x}{\lambda}\right)\frac{t^n}{n!}. \tag{2.7}$$

By (1.3) and Theorem 2.1,

$$\begin{aligned} D_{n,\lambda,q}(x|u) &= \sum_{k=0}^{n} u^{n-k}S_1(n,k)\int_{\mathbb{Z}_p}(\lambda y+x)_k d\mu_q(y) \\ &= \sum_{k=0}^{n} u^{n-k}S_1(n,k)\sum_{l=0}^{k} S_1(k,l)\lambda^l\int_{\mathbb{Z}_p}\left(y+\frac{x}{\lambda}\right)d\mu_q(y) \\ &= \sum_{k=0}^{n}\sum_{l=0}^{k} u^{n-k}\lambda^l S_1(n,k)S_1(k,l)B_{l,q}\left(\frac{x}{\lambda}\right). \end{aligned} \tag{2.8}$$

Therefore, by (1.7), (2.5), (2.6) and (2.8), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$D_{n,\lambda,q}(x) = \sum_{m=0}^{n} D_{m,\lambda,q}(x|u)S_2(n,m)u^{-n},$$

and

$$D_{n,\lambda,q}(x|u) = \sum_{k=0}^{n}\sum_{l=0}^{k} u^{n-k}\lambda^l S_1(n,k)S_1(k,l)B_{l,q}\left(\frac{x}{\lambda}\right).$$

In addition,

$$\lambda^n B_{n,q}\left(\frac{x}{\lambda}\right) = \sum_{m=0}^{n}\sum_{k=0}^{m} D_{k,\lambda,q}(x|u)u^{m-k}S_2(m,k)S_2(n,m).$$

From now on, we consider the *higher order degenerate λ-q-Daehee polynomials of the first kind* as follows:

$$D^{(k)}_{n,\lambda,q}(x|u) = \sum_{l=0}^{n} u^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda y_1+\cdots+\lambda y_k+x)_l d\mu_q(x_1)\cdots d\mu_q(x_k). \tag{2.9}$$

From (2.9), we can derive the generating function of $D^{(k)}_{n,\lambda,q}(x)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} D^{(k)}_{n,\lambda,q}(x|u)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n} u^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda y_1+\cdots+\lambda y_k+x)_l d\mu_q(y_1)\cdots d\mu_q(y_k)\\
&= \sum_{l=0}^{\infty} u^{-l}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda y_1+\cdots+\lambda y_k+x)_l d\mu_q(y_1)\cdots d\mu_q(y_k)\frac{1}{l!}l!\sum_{m=l}^{\infty}S_1(m,l)\frac{(ut)^m}{m!}\\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{l=0}^{\infty}\binom{\lambda y_1+\cdots+\lambda y_k+x}{l}u_{-l}\left(\log(1+ut)\right)^l d\mu_q(y_1)\cdots d\mu_q(y_k)\\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda y_1+\cdots+\lambda y_k+x} d\mu_q(y_1)\cdots d\mu_q(y_k).
\end{aligned} \tag{2.10}$$

Note that by (1.3),

$$\begin{aligned}
D^{(k)}_{n,\lambda,q}(x|u) &= \sum_{l=0}^{n} u^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda y_1+\cdots+\lambda y_k+x)_l d\mu_q(y_1)\cdots d\mu_q(y_k)\\
&= \sum_{l=0}^{n} u^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{s=0}^{l}S_1(l,s)(\lambda y_1+\cdots+\lambda y_k+x)^s d\mu_q(y_1)\cdots d\mu_q(y_k)\\
&= \sum_{l=0}^{n}\sum_{s=0}^{l} u^{n-l}S_1(n,l)S_1(l,s)\lambda^s\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(y_1+\cdots+y_k+\frac{x}{\lambda}\right)^s d\mu_q(y_1)\cdots d\mu_q(y_k)\\
&= \sum_{l=0}^{n}\sum_{s=0}^{l} u^{n-l}S_1(n,l)S_1(l,s)\lambda^s B^{(k)}_{s,q}\left(\frac{x}{\lambda}\right).
\end{aligned} \tag{2.11}$$

From (2.6) and (2.10), we get

$$\begin{aligned}
\left(\frac{q-1+\frac{q-1}{\log q}\lambda t}{qe^{\lambda t}-1}\right)^k e^{tx} &= \sum_{n=0}^{\infty} D^{(k)}_{n,\lambda,q}(x|u)\frac{u^{-n}}{n!}\left(e^{u(e^t-1)}-1\right)^n\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{s=0}^{m} D^{(k)}_{s,\lambda,q}(x|u)u^{m-s}S_2(m,s)S_2(n,m)\right)\frac{t^n}{n!}
\end{aligned} \tag{2.12}$$

and

$$\left(\frac{q-1+\frac{q-1}{\log q}\lambda t}{qe^{\lambda t}-1}\right)^k e^{\lambda t\left(\frac{x}{\lambda}\right)} = \sum_{n=0}^{\infty} \lambda^n B_{n,q}^{(k)}\left(\frac{x}{\lambda}\right)\frac{t^n}{n!}. \tag{2.13}$$

Thus, by (2.11), (2.12) and (2.13), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, $k \in \mathbb{N}$, we have*

$$\lambda^n B_{n,q}^{(k)}\left(\frac{x}{\lambda}\right) = \sum_{m=0}^{n}\sum_{s=0}^{m} D_{s,\lambda,q}^{(k)}(x|u)u^{m-s}S_2(m,s)S_2(n,m),$$

and

$$\begin{aligned} D_{n,\lambda,q}^{(k)}(x|u) &= \sum_{l=0}^{n}\sum_{s=0}^{l} u^{n-l}S_1(n,l)S_1(l,s)\lambda^s B_{s,q}^{(k)}\left(\frac{x}{\lambda}\right) \\ &= \sum_{l=0}^{n}\sum_{s=0}^{l}\sum_{m=0}^{s}\sum_{r=0}^{m} u^{n-l+m-r}\lambda^{-s} D_{r,\lambda,q}^{(k)}(x|u)S_1(n,l)S_1(l,s)S_2(s,m)S_2(m,r). \end{aligned}$$

3. Degenerate λ-q-Daehee polynomials of the second kind

Let us define the *degenerate λ-q-Daehee polynomials of the second kind* as follows:

$$\frac{q-1-\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda}-1}\left(1+\frac{1}{u}\log(1+ut)\right)^x = \sum_{n=0}^{\infty}\widehat{D}_{n,\lambda,q}(x|u)\frac{t^n}{n!}. \tag{3.1}$$

In the special case, $\lambda = 1$, $\widehat{D}_{n,q}(x|u) = \widehat{D}_{n,1,q}(x|u)$ are called the *degenerate q-Daehee polynomials of the second kind*, and if $x = 0$, then $\widehat{D}_{n,q}(u) = \widehat{D}_{n,1,q}(0|u)$ are called the *degenerate q-Daehee numbers of the second kind.*

Let us take $f(x) = (1+t)^{-\lambda x}$. Then, by (1.2), we get

$$\int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda x} d\mu_q(x) = \frac{q-1-\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda}-1}, \tag{3.2}$$

and so

$$\begin{aligned} \int_{\mathbb{Z}_p}\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda y+x} d\mu_q(y) &= \frac{q-1-\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q\left(1+\frac{1}{u}\log(1+ut)\right)^{-\lambda}-1}\left(1+\frac{1}{u}\log(1+ut)\right)^x \\ &= \sum_{n=0}^{\infty}\widehat{D}_{n,\lambda,q}(x|u)\frac{t^n}{n!}. \end{aligned} \tag{3.3}$$

By (3.3), we have

$$\sum_{n=0}^{\infty}\widehat{D}_{n,\lambda,q}(x|u)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} u^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}(-\lambda y+x)_m\, d\mu_q(y)\frac{t^n}{n!}. \tag{3.4}$$

By (3.2) and (3.4), we get

$$\begin{aligned} \widehat{D}_{n,\lambda,q}(x|u) &= \sum_{m=0}^{n} u^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}(-\lambda y+x)_m\, d\mu_q(y) \\ &= \sum_{m=0}^{n}\sum_{l=0}^{m} u^{n-m}(-\lambda)^l S_1(n,m)S_1(m,l)\int_{\mathbb{Z}_p}\left(y-\frac{x}{\lambda}\right)^l d\mu_q(y) \\ &= \sum_{m=0}^{n}\sum_{l=0}^{m} u^{n-m}(-\lambda)^l S_1(n,m)S_1(m,l)B_{l,q}\left(-\frac{x}{\lambda}\right), \end{aligned} \tag{3.5}$$

and, by (3.3), we have

$$\begin{aligned}\frac{q-1-\frac{q-1}{\log q}\lambda t}{q-e^{\lambda t}}e^{(\lambda+x)t} &= \sum_{n=0}^{\infty}\widehat{D}_{n,\lambda,q}(x|u)\frac{1}{n!}\left(\frac{1}{u}e^{u(e^t-1)}-1\right)^n \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\widehat{D}_{n,\lambda,q}(x|u)u^{m-n}S_2(m,n)\right)\frac{t^m}{m!},\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\frac{q-1-\frac{q-1}{\log q}\lambda t}{q-e^{\lambda t}}e^{(\lambda+x)t} &= \frac{q-1-\frac{q-1}{\log q}\lambda t}{q-e^{\lambda t}}e^{\left(1+\frac{x}{\lambda}\right)\lambda t} \\ &= \sum_{m=0}^{\infty}(-\lambda)^m B_{m,q}\left(-\frac{x}{\lambda}\right)\frac{t^m}{m!}.\end{aligned} \tag{3.7}$$

Therefore, by (3.5), (3.6) and (3.7), we obtain the following theorem.

Theorem 3.1. *For $m \geq 0$, we have*

$$\widehat{D}_{m,\lambda,q}(x|u) = \sum_{n=0}^{m}\sum_{l=0}^{n}u^{m-n}(-\lambda)^l S_1(m,n)S_1(n,l)B_{l,q}\left(-\frac{x}{\lambda}\right),$$

and

$$\begin{aligned}(-\lambda)^m B_{m,q}\left(-\frac{x}{\lambda}\right) &= \sum_{n=0}^{m}\widehat{D}_{n,\lambda,q}(x|u)u^{m-n}S_2(m,n) \\ &= \sum_{n=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{k}u^{m-k}(-\lambda)^l S_1(n,k)S_1(k,l)S_2(m,n)B_{l,q}\left(-\frac{x}{\lambda}\right).\end{aligned}$$

By the Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *For $m \geq 0$, we have*

$$\widehat{D}_{m,\lambda,q}(u) = \sum_{n=0}^{m}\sum_{l=0}^{n}u^{m-n}(-\lambda)^l S_1(m,n)S_1(n,l)B_{l,q},$$

and

$$\begin{aligned}B_{m,q} &= (-\lambda)^{-m}\sum_{n=0}^{m}\widehat{D}_{n,\lambda,q}(u)u^{m-n}S_2(m,n) \\ &= \sum_{n=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{k}u^{m-k}(-\lambda)^{l-m}S_1(n,k)S_1(k,l)S_2(m,n)B_{l,q}.\end{aligned}$$

As the special case of the Corollary 3.2, $\lambda = 1$ and $u = 1$, we have

$$\widehat{D}_{m,q} = \sum_{n=0}^{m}\sum_{l=0}^{n}(-1)^l S_1(m,n)S_1(n,l)B_{l,q}$$

and

$$\begin{aligned}B_{m,q} &= (-1)^m\sum_{n=0}^{m}\widehat{D}_{n,q}S_2(m,n) \\ &= \sum_{n=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{k}(-1)^{l-m}S_1(n,k)S_1(k,l)S_2(m,n)B_{l,q}.\end{aligned}$$

Now, we define the *degenerate λ-q-Daehee polynomials of the second kind with order k* where $k \in \mathbb{N}$:

$$\widehat{D}^{(k)}_{n,\lambda,q}(x|u) = \sum_{m=0}^{n} u^{n-m} S_1(n,m) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-\lambda x_1 - \cdots - \lambda x_k + x)_m d\mu_q(x_1) \cdots d\mu_q(x_k). \tag{3.8}$$

From (3.8), we can derive the generating function of

$$\begin{aligned}\sum_{n=0}^{\infty} \widehat{D}^{(k)}_{n,\lambda,q}(x|u) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-\lambda x_1 - \cdots - \lambda x_k + x)_n d\mu_q(x_1) \cdots d\mu_q(x_k) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(1 + \frac{1}{u} \log(1+ut)\right)^{-\lambda x_1 - \cdots - \lambda x_k + x} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ &= \left(\frac{q - 1 - \frac{q-1}{\log q} \lambda \log\left(1 + \frac{1}{u}\log(1+ut)\right)}{q\left(1 + \frac{1}{u}\log(1+ut)\right)^{-\lambda} - 1} \right)^k \left(1 + \frac{1}{u}\log(1+ut)\right)^x .\end{aligned} \tag{3.9}$$

Replacing t by $\frac{1}{u}\left(e^t - 1\right)$ in (3.9), we get

$$\begin{aligned}\left(\frac{q - 1 - \frac{q-1}{\log q}\lambda(1+t)}{q(1+t)^{-\lambda} - 1} \right)^k (1+t)^x &= \sum_{n=0}^{\infty} \widehat{D}^{(k)}_{n,\lambda,q}(x|u) \frac{\left(\frac{1}{u}(e^t-1)\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} \widehat{D}^{(k)}_{n,\lambda,q}(x|u) \frac{1}{n!} u^{-n} \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \left(u^{-n} \sum_{m=0}^{n} \widehat{D}^{(k)}_{m,\lambda,q}(x|u) S_2(n,m) \right) \frac{t^n}{n!}.\end{aligned} \tag{3.10}$$

On the other hand, by replacing t by $\frac{1}{u}\left(e^{u(e^t-1)} - 1\right)$ in (3.9), we have

$$\begin{aligned}\left(\frac{q - 1 - \frac{q-1}{\log q}\lambda t}{q e^{-\lambda t} - 1} \right)^k e^{tx} &= \sum_{n=0}^{\infty} \widehat{D}^{(k)}_{n,\lambda,q}(x|u) \frac{u^{-n}}{n!} \left(e^{u(e^t-1)} - 1\right)^n \\ &= \sum_{n=0}^{\infty} \widehat{D}^{(k)}_{n,\lambda,q}(x|u) \frac{u^{-n}}{n!} n! \sum_{l=n}^{\infty} S_2(l,n) \frac{u^l}{l!} \left(e^t - 1\right)^l \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \widehat{D}^{(k)}_{m,\lambda,q}(x|u) S_2(n,m) \frac{u^{n-m}}{n!} \left(e^t - 1\right)^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{l=0}^{m} \widehat{D}^{(k)}_{l,\lambda,q}(x|u) u^{m-l} S_2(m,l) S_2(n,m) \right) \frac{t^n}{n!},\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\left(\frac{q - 1 - \frac{q-1}{\log q}\lambda t}{q e^{-\lambda t} - 1} \right)^k e^{xt} &= \left(\frac{q - 1 + \frac{q-1}{\log q}(-\lambda) t}{q e^{-\lambda t} - 1} \right)^k e^{-\lambda t\left(-\frac{\lambda}{x}\right)} \\ &= \sum_{n=0}^{\infty} (-\lambda)^n B^{(k)}_{n,q}\left(-\frac{x}{\lambda}\right) \frac{t^n}{n!}.\end{aligned} \tag{3.12}$$

By (1.1) and (3.8), we get

$$\begin{aligned}\widehat{D}^{(k)}_{n,\lambda,q}(x|u) &= \sum_{m=0}^{n} u^{n-m} S_1(n,m) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-\lambda x_1 - \cdots - \lambda x_k + x)_m d\mu_q(x_1) \cdots d\mu_q(x_k) \\ &= \sum_{m=0}^{n} \sum_{l=0}^{m} u^{n-m} S_1(n,m) S_1(m,l) B^{(k)}_{l,q}\left(-\frac{x}{\lambda}\right).\end{aligned} \tag{3.13}$$

Hence, by (3.10), (3.11), (3.12) and (3.13), we obtain the following theorem.

Theorem 3.3. *For $n \geq 0$, we have*

$$\widehat{D}^{(k)}_{n,\lambda,q}(x|u) = \sum_{m=0}^{n}\sum_{l=0}^{m} u^{n-m} S_1(n,m) S_1(m,l) B^{(k)}_{l,q}\left(-\frac{x}{\lambda}\right)$$

and

$$D^{(k)}_{n,-\lambda,q}(x) = u^{-n}\sum_{m=0}^{n} \widehat{D}^{(k)}_{m,\lambda,q}(x|u) S_2(n,m).$$

In addition,

$$\begin{aligned}(-\lambda)^n B^{(k)}_{n,q}\left(-\frac{x}{\lambda}\right) &= \sum_{m=0}^{n}\sum_{l=0}^{m} \widehat{D}^{(k)}_{l,\lambda,q}(x|u) u^{m-l} S_2(m,l) S_2(n,m) \\ &= \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{r=0}^{l}\sum_{s=0}^{r} u^{m-r} S_1(l,r) S_1(r,s) S_2(m,l) S_2(n,m) B^{(k)}_{s,q}\left(-\frac{x}{\lambda}\right).\end{aligned}$$

As a special case of Theorem 3.3, if we put $x = 0$, then

$$\widehat{D}^{(k)}_{n,\lambda,q}(u) = \sum_{m=0}^{n}\sum_{l=0}^{m} u^{n-m} S_1(n,m) S_1(m,l) B^{(k)}_{l,q},$$

and

$$\begin{aligned}(-\lambda)^n B^{(k)}_{n,q} &= \sum_{m=0}^{n}\sum_{l=0}^{m} \widehat{D}^{(k)}_{l,\lambda,q}(u) u^{l} S_2(m,l) S_2(n,m) \\ &= \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{r=0}^{l}\sum_{s=0}^{r} u^{2l-r} S_1(l,r) S_1(r,s) S_2(m,l) S_2(n,m) B^{(k)}_{s,q}.\end{aligned}$$

In particular, if we put $\lambda = 1$ and $u = 1$, then

$$\widehat{D}^{(k)}_{n,q} = \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m) S_1(m,l) B^{(k)}_{l,q},$$

and

$$\begin{aligned}B^{(k)}_{n,q} &= (-1)^n \sum_{m=0}^{n}\sum_{l=0}^{m} \widehat{D}^{(k)}_{l,\lambda,q} S_2(m,l) S_2(n,m) \\ &= (-1)^n \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{r=0}^{l}\sum_{s=0}^{r} S_1(l,r) S_1(r,s) S_2(m,l) S_2(n,m) B^{(k)}_{s,q}.\end{aligned}$$

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