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Positive solutions for an impulsive boundary value problem with Caputo fractional derivative

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Abstract

In this work we use fixed point theorem method to discuss the existence of positive solutions for the impulsive boundary value problem with Caputo fractional derivative

$$\begin{cases} {}^cD_t^q u(t) = f(t,u(t)), \quad \text{a.e. } t\in[0,1];\\ \Delta u(t_k) = I_k(u(t_k)), \quad \Delta u'(t_k) = J_k(u(t_k)), \quad k=1,2,\dots,m;\\ au(0)-bu(1)=0, \quad au'(0)-bu'(1)=0, \end{cases}$$

where $q\in(1,2)$ is a real number, a,b are real constants with $a>b>0$, and ${}^cD_t^q$ is the Caputo's fractional derivative of order q, $f:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ and $I_k, J_k:\mathbb{R}^+\to\mathbb{R}^+$ are continuous functions, $k=1,2,\dots,m$, $\mathbb{R}^+ := [0,+\infty)$.

Keywords: Caputo fractional derivative, impulsive boundary value problem, fixed point theorem, positive solution.
2010 MSC: 34B10, 34B15, 34B37.

1. Introduction

In this work we study the impulsive boundary value problem with Caputo fractional derivative

$$\begin{cases} {}^cD_t^q u(t) = f(t,u(t)), \quad \text{a.e. } t\in[0,1];\\ \Delta u(t_k) = I_k(u(t_k)), \quad \Delta u'(t_k) = J_k(u(t_k)), \quad k=1,2,\dots,m;\\ au(0)-bu(1)=0, \quad au'(0)-bu'(1)=0, \end{cases} \tag{1.1}$$

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where $q \in (1,2)$ is a real number, a, b are real constants with $a > b > 0$, and ${}^cD_t^q$ is the Caputo's fractional derivative of order q; $t_k (k = 1, 2, \ldots, m,\ m \geq 1$ is a fixed integer) are constants with $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $u(t_k^+) = \lim_{h\to 0} u(t_k + h)$ and $u(t_k^-) = \lim_{h\to 0} u(t_k - h)$ represent the right-hand and left-hand limits of $u(t)$ at $t = t_k$, respectively. Moreover, $f, I_k,$ and J_k satisfy the condition:

(H1). $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ and $I_k, J_k : \mathbb{R}^+ \to \mathbb{R}^+ (k = 1, 2, \ldots, m)$ are continuous functions.

Denote $J = [0,1]$, $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}] (k = 1, 2, \ldots, m)$. Furthermore, we define

$$PC(J) = \{u | u : J \to \mathbb{R} \text{is continuous at} t \neq t_k, \text{and } u(t_k^+), u(t_k^-) \text{ exist, } u(t_k^-) = u(t_k), k = 1, 2, \ldots, m\}.$$

Clearly, $PC(J)$ is a Banach space with the norm $\|u\| = \sup_{t\in J} |u(t)|$ for $u \in PC(J)$. Note that $C(J)$, which represents the set of all continuous functions on J, is also a Banach space with $\|u\|$.

As is well known, it is an important method to express the solutions of differential equations by Green's function. However, for impulsive differential equations of fractional order (see [1–5, 9, 11, 14, 15, 17–19] and the references therein), their integral forms are very complicated, and cannot be formulated by virtue of some suitable Green's functions. For example, in [14], Wang, Ahmad and Zhang investigated the existence and uniqueness of solutions for a mixed boundary value problem of fractional differential equations with impulses

$$\begin{cases} {}^cD^\alpha u(t) = f(t, u(t)), \quad 1 < \alpha \leq 2, \quad t \in J'; \\ \Delta u(t_k) = I_k(u(t_k)), \quad \Delta u'(t_k) = I_k^*(u(t_k)), \quad k = 1, 2, \ldots, p; \\ Tu'(0) = -au(0) - bu(T), \quad Tu'(T) = cu(0) + du(T), \end{cases}$$

which can be written in the form

$$\begin{aligned} u(t) = & \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) \mathrm{d}s + \lambda_1(t) \int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) \mathrm{d}s \\ & - \lambda_2(t) \int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, u(s)) \mathrm{d}s + \sum_{i=1}^{k} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) \mathrm{d}s + I_i(u(t_i)) \right] \\ & + \sum_{i=1}^{k-1} (t_k - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, u(s)) \mathrm{d}s + I_i^*(u(t_i)) \right] \\ & + \sum_{i=1}^{k} (t - t_k) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, u(s)) \mathrm{d}s + I_i^*(u(t_i)) \right] \\ & + \lambda_1(t) \sum_{i=1}^{p} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) \mathrm{d}s + I_i(u(t_i)) \right] \\ & + \lambda_1(t) \sum_{i=1}^{p} (t_p - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, u(s)) \mathrm{d}s + I_i^*(u(t_i)) \right] \\ & - \sum_{i=1}^{p} [\lambda_3(t) + \lambda_1(t) t_p] \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, u(s)) \mathrm{d}s + I_i^*(u(t_i)) \right]. \end{aligned}$$

We all know that impulsive differential equations with integer order can be expressed by Green's function (see for example [7, 16, 20]), therefore, it is a natural problem whether or not the same result holds for the fractional order case. To the best of our knowledge, only [10, 21–23] are devoted to this direction. In [10], Liu and Jia considered the fractional impulsive differential equations:

$$\begin{cases} {}^cD_{0+}^\alpha u(t) = f(t, u(t), \quad {}^cD_{0+}^\beta u(t)), \quad t \in J'; \\ \Delta u(t_k) = I_k(u(t_k), \quad {}^cD_{0+}^\beta u(t_k)); \\ \Delta {}^cD_{0+}^\beta u(t_k) = Q_k(u(t_k), \quad {}^cD_{0+}^\beta u(t_k)), \quad k = 1, 2, \ldots, m; \\ u(0) = 0, \quad u(1) = \displaystyle\int_0^1 u(t) g(t) \mathrm{d}t, \end{cases}$$

which can be expressed by

$$u(t) = \int_0^1 G(t,s)f(s,u(s),{}^cD_{0+}^{\beta}u(s))\mathrm{d}s + \sum_{i=1}^{m} H(t,t_i)I_i(u(t_i),{}^cD_{0+}^{\beta}u(t_i)) + \sum_{i=1}^{m} K(t,t_i)Q_i(u(t_i),{}^cD_{0+}^{\beta}u(t_i)).$$

By Schauder fixed point theorem and Krasnoselskii fixed point theorem, they established some existence theorems for the above problem.

Inspired by the above mentioned works, in this paper by Green's function and fixed point theorem method, we obtain the existence of (positive) solutions for (1.1) with the assumptions that the growth of f is superlinear, asymptotically linear and sublinear.

2. Preliminaries

Let us recall some notations and preliminary lemmas of fractional calculus, for more details, see [12, 13].

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f : (0,+\infty) \to (-\infty,+\infty)$ is defined as

$$I_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\mathrm{d}s,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. The fractional derivative of f in the Caputo sense is defined as

$${}^cD_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)\mathrm{d}s, \quad n-1<\alpha<n,$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of the number α.

Lemma 2.3. *Let $\alpha > 0$. Then the differential equation ${}^cD_t^{\alpha}u(t) = 0$ has a unique solution*

$$u(t) = c_0 + c_1t + \cdots + c_{n-1}t^{n-1}$$

for some $c_i \in \mathbb{R}(i = 0,1,\dots,n-1)$, where $n = [\alpha]+1$.

Lemma 2.4. *Assume that $u \in C(0,1) \cap L(0,1)$ with a derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha}\,{}^cD_t^{\alpha}u(t) = u(t) + c_0 + c_1t + \cdots + c_{n-1}t^{n-1}$$

for some $c_i \in \mathbb{R}(i = 0,1,\dots,n-1)$, where $n = [\alpha]+1$.

Lemma 2.5 ([22, Lemma 2.5])**.** *Let $y \in C(J)$. Then the unique solution of the boundary value problem*

$$\begin{cases} {}^cD_t^{q}u(t) = y(t), & a.e.\ t \in [0,1]; \\ \Delta u(t_k) = I_k(u(t_k)), \quad \Delta u'(t_k) = J_k(u(t_k)), & k = 1,2,\dots,m; \\ au(0) - bu(1) = 0, \quad au'(0) - bu'(1) = 0, \end{cases} \tag{2.1}$$

is given by

$$u(t) = \int_0^1 G_1(t,s)y(s)\mathrm{d}s + \sum_{i=1}^{m} G_2(t,t_i)J_i(u(t_i)) + \sum_{i=1}^{m} G_3(t,t_i)I_i(u(t_i)), \tag{2.2}$$

where

$$G_1(t,s)=\begin{cases}\dfrac{(t-s)^{q-1}}{\Gamma(q)}+\dfrac{b(1-s)^{q-1}}{(a-b)\Gamma(q)}+\dfrac{b(q-1)t(1-s)^{q-2}}{(a-b)\Gamma(q)}+\dfrac{b^2(q-1)(1-s)^{q-2}}{(a-b)^2\Gamma(q)}, & 0\le s\le t\le 1;\\ \dfrac{b(1-s)^{q-1}}{(a-b)\Gamma(q)}+\dfrac{b(q-1)t(1-s)^{q-2}}{(a-b)\Gamma(q)}+\dfrac{b^2(q-1)(1-s)^{q-2}}{(a-b)^2\Gamma(q)}, & 0\le t\le s\le 1,\end{cases} \tag{2.3}$$

$$G_2(t,s)=\begin{cases}\dfrac{ab}{(a-b)^2}+\dfrac{a(t-t_i)}{a-b}, & 0\le t_i<t\le 1, \quad i=1,2,\dots,m;\\ \dfrac{ab}{(a-b)^2}+\dfrac{b(t-t_i)}{a-b}, & 0\le t\le t_i\le 1, \quad i=1,2,\dots,m,\end{cases} \tag{2.4}$$

$$G_3(t,s)=\begin{cases}\dfrac{a}{a-b}, & 0\le t_i<t\le 1, \quad i=1,2,\dots,m;\\ \dfrac{b}{a-b}, & 0\le t\le t_i\le 1, \quad i=1,2,\dots,m.\end{cases} \tag{2.5}$$

Lemma 2.6 ([22, Lemma 2.6])**.** *Let a,b be real constants with $a>b>0$. Then $G_i(i=1,2,3)$ have the following properties*

(i) $G_1(t,s)\in C(J\times J,\mathbb{R}^+)$ *and* $G_1(t,s)>0, G_2(t,t_i)>0, G_3(t,t_i)>0$ *for all* $t,t_i,s\in(0,1)$,

(ii) *there exists a negative function* $M(s), s\in[0,1]$ *such that*

$$\frac{b}{a}M(s)\le G_1(t,s)\le M(s),$$

where

$$M(s)=\frac{a[(1-s)a-(2-s-q)b](1-s)^{q-2}}{(a-b)^2\Gamma(q)}, \quad s\in[0,1],$$

(iii)

$$\frac{b^2}{(a-b)^2}\le G_2(t,t_i)\le\frac{a^2}{(a-b)^2}, \quad \frac{b}{a-b}\le G_3(t,t_i)\le\frac{a}{a-b}, \quad \forall t,t_i\in[0,1].$$

For convenience, we need to calculate the following integral

$$\kappa_1:=\int_0^1 M(s)\mathrm{d}s=\frac{a^2(q-1)+abq(q-2)+ab}{q(q-1)(a-b)^2\Gamma(q)}.$$

We define the operator $\mathcal{A}:PC(J)\to PC(J)$ by

$$(\mathcal{A}u)(t):=\int_0^1 G_1(t,s)f(s,u(s))\mathrm{d}s+\sum_{i=1}^m G_2(t,t_i)J_i(u(t_i))+\sum_{i=1}^m G_3(t,t_i)I_i(u(t_i)),$$

where $G_i(i=1,2,3)$ are defined in (2.3), (2.4) and (2.5). Then from Lemma 2.5, solving the solutions of (1.1) reduces to solve the fixed points of the operator equation $u=\mathcal{A}u$. Furthermore, we can adopt the Ascoli-Arzela theorem to prove $\mathcal{A}$ is a completely continuous operator.

Define $P=\{u\in PC(J):u(t)\ge 0, t\in[0,1]\}$, and $P_0=\{u\in PC(J):u(t)\ge\frac{b^2}{a^2}\|u\|, t\in[0,1]\}$. Then P,P_0 are cone on $PC(J)$. Moreover, we easily obtain the following lemma.

Lemma 2.7. $\mathcal{A}(P)\subset P_0$.

Let E be a Banach space, P be a cone on E, and $B_R:=\{u\in E:\|u\|<R\}$ for $R>0$ in the sequel.

Lemma 2.8 ([6])**.** *Let $\mathcal{A}:\overline{B}_R\cap P\to P$ be a completely continuous operator. If there exists $v_0\in P\setminus\{0\}$ such that $v-\mathcal{A}v\ne\lambda v_0$ for all $v\in\partial B_R\cap P$ and $\lambda\ge 0$, then $i(\mathcal{A},B_R\cap P,P)=0$, where i is the fixed point index on P.*

Lemma 2.9 ([6])**.** *Let $\mathcal{A} : \overline{B}_R \cap P \to P$ be a completely continuous operator. If $v \neq \lambda \mathcal{A} v$ for all $v \in \partial B_R \cap P$ and $0 \leq \lambda \leq 1$, then $i(\mathcal{A}, B_R \cap P, P) = 1$.*

Lemma 2.10 ([8])**.** *Let $\mathcal{A} : E \to E$ be a completely continuous operator. Assume that $T : E \to E$ is a bounded linear operator such that 1 is not an eigenvalue of T and*

$$\lim_{\|u\| \to \infty} \frac{\|\mathcal{A}u - Tu\|}{\|u\|} = 0.$$

Then $\mathcal{A}$ has a fixed point in E.

3. Main results

Theorem 3.1. *Assume that*
(H2). *$f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $I_k, J_k : \mathbb{R} \to \mathbb{R} (k = 1, 2, \ldots, m)$ are continuous functions, moreover,*

$$\lim_{u \to \infty} \frac{f(t, u)}{u} = \lambda, \quad \textit{uniformly int} \in [0, 1],$$

and

$$\lim_{u \to \infty} \frac{I_k(u)}{u} = \lambda, \ \lim_{u \to \infty} \frac{J_k(u)}{u} = \lambda, \ k = 1, 2, \ldots, m.$$

If

$$|\lambda| < \left[\kappa_1 + m\left(\frac{a^2}{(a-b)^2} + \frac{a}{a-b}\right)\right]^{-1},$$

then (1.1) has a nontrivial solution when $f(t, 0) \not\equiv 0$ for $t \in [0, 1]$.

Proof. Define $T : PC(J) \to PC(J)$ by

$$(Tu)(t) := \lambda \left[\int_0^1 G_1(t, s)u(s)\mathrm{d}s + \sum_{i=1}^m G_2(t, t_i)u(t_i) + \sum_{i=1}^m G_3(t, t_i)u(t_i)\right]. \tag{3.1}$$

Then T is a bounded linear operator. From Lemma 2.5, equation (3.1) is equivalent to

$$\begin{cases} {}^cD_t^q u(t) = \lambda u(t), & \text{a.e. } t \in [0, 1]; \\ \Delta u(t_k) = \lambda u(t_k), \quad \Delta u'(t_k) = \lambda u(t_k), & k = 1, 2, \ldots, m; \\ au(0) - bu(1) = 0, \quad au'(0) - bu'(1) = 0. \end{cases} \tag{3.2}$$

Next, we consider the following two cases.
Case 1. $\lambda = 0$. Equation (3.2) is a problem without impulse, and from Lemma 2.3 we have

$$u(t) = c_0 + c_1 t$$

for some $c_i \in \mathbb{R}, i = 0, 1$. In view of the boundary conditions (3.2), we have $c_0 = c_1 = 0$ and thus $u(t) \equiv 0$ for $t \in [0, 1]$. This shows (3.2) has only a trivial solution.
Case 2. $\lambda \neq 0$. From Case 1 we see (3.2) has nontrivial solutions. Let u be a nontrivial solution for (3.2) and then $\|u\| > 0$. Suppose that 1 is an eigenvalue of T. Then we have

$$\begin{aligned} \|u\| = \|Tu\| &\leq |\lambda| \|u\| \left[\int_0^1 G_1(t, s)\mathrm{d}s + \sum_{i=1}^m G_2(t, t_i) + \sum_{i=1}^m G_3(t, t_i)\right] \\ &\leq |\lambda| \left[\kappa_1 + m\left(\frac{a^2}{(a-b)^2} + \frac{a}{a-b}\right)\right] \|u\| < \|u\|. \end{aligned}$$

This is impossible.

To sum up, 1 is not an eigenvalue of T.

From (H2), for all $\varepsilon > 0$, there exists $M > 0$ such that

$$|f(t,u) - \lambda u| \le \varepsilon |u|,\ |I_k(u) - \lambda u| \le \varepsilon |u|,\ |J_k(u) - \lambda u| \le \varepsilon |u|,\ \text{for } t \in [0,1],\ |u| \ge M.$$

Moreover, if $|u| \le M$, then $|f(t,u) - \lambda u|$, $|I_k(u) - \lambda u|$ and $|J_k(u) - \lambda u|$ are bounded. Hence, there exists $M_1 > 0$ such that

$$|f(t,u) - \lambda u| \le \varepsilon |u| + M_1,\ |I_k(u) - \lambda u| \le \varepsilon |u| + M_1,\ |J_k(u) - \lambda u| \le \varepsilon |u| + M_1,\ \text{for } t \in [0,1],\ u \in \mathbb{R}.$$

Hence

$$\begin{aligned}
\|\mathcal{A}u - Tu\| &= \sup_{t\in[0,1]} \Bigg| \int_0^1 G_1(t,s)\,[f(s,u(s)) - \lambda u(s)]\,\mathrm{d}s \\
&\quad + \sum_{i=1}^m G_2(t,t_i)\,[J_i(u(t_i)) - \lambda u(t_i)] + \sum_{i=1}^m G_3(t,t_i)\,[I_i(u(t_i)) - \lambda u(t_i)] \Bigg| \\
&\le \sup_{t\in[0,1]} \int_0^1 G_1(t,s)\,|f(s,u(s)) - \lambda u(s)|\,\mathrm{d}s \\
&\quad + \sup_{t\in[0,1]} \sum_{i=1}^m G_2(t,t_i)\,|J_i(u(t_i)) - \lambda u(t_i)| + \sup_{t\in[0,1]} \sum_{i=1}^m G_3(t,t_i)\,|I_i(u(t_i)) - \lambda u(t_i)| \\
&\le (\varepsilon\|u\| + M_1)\left[\kappa_1 + m\left(\frac{a^2}{(a-b)^2} + \frac{a}{a-b}\right)\right],
\end{aligned}$$

which implies that

$$\lim_{\|u\|\to\infty} \frac{\|\mathcal{A}u - Tu\|}{\|u\|} \le \lim_{\|u\|\to\infty} \frac{(\varepsilon\|u\| + M_1)\left[\kappa_1 + m\left(\frac{a^2}{(a-b)^2} + \frac{a}{a-b}\right)\right]}{\|u\|} = \varepsilon\left[\kappa_1 + m\left(\frac{a^2}{(a-b)^2} + \frac{a}{a-b}\right)\right].$$

Note that the arbitrariness of ε, so

$$\lim_{\|u\|\to\infty} \frac{\|\mathcal{A}u - Tu\|}{\|u\|} = 0.$$

Therefore, from Lemma 2.10, $\mathcal{A}$ has a fixed point in $PC(J)$, that is, (1.1) has at least one solution u. Further, we can assert that u is nontrivial when $f(t,0) \not\equiv 0$ for $t \in [0,1]$. This completes the proof. □

In order to establish the following two theorems, we need some conditions as follows:

(H3). There exist $c > 0$ and $a_1 \ge 0, a_2 \ge 0, a_3 \ge 0$ satisfying

$$a_3 b^4 m + a_2 b^3 (a-b) m > (a-b)^2 (a^2 - aba_1\kappa_1)$$

such that

$$f(t,u) \ge a_1 u - c,\ I_k(u) \ge a_2 u - c,\ J_k(u) \ge a_3 u - c,\quad \text{for all } t \in [0,1], u \in \mathbb{R}^+.$$

(H4). There exist $r > 0$ and $b_1 \ge 0, b_2 \ge 0, b_3 \ge 0$ satisfying

$$(1 - \kappa_1 b_1) b^2 (a-b)^2 > m\left[a^4 b_3 + a^3 (a-b) b_2\right]$$

such that

$$f(t,u) \le b_1 u,\ I_k(u) \le b_2 u,\ J_k(u) \le b_3 u,\quad \text{for all } t \in [0,1],\ u \in [0,r].$$

(H5). There exist $r > 0$ and $a_4 \geq 0, a_5 \geq 0, a_6 \geq 0$ satisfying

$$a_6 b^4 m + a_5 b^3 (a-b) m > (a-b)^2 (a^2 - aba_4\kappa_1)$$

such that

$$f(t,u) \geq a_4 u,\ I_k(u) \geq a_5 u,\ J_k(u) \geq a_6 u, \quad \text{for all } t \in [0,1], u \in [0,r].$$

(H6). There exist $c > 0$ and $b_4 \geq 0, b_5 \geq 0, b_6 \geq 0$ satisfying

$$(1-\kappa_1 b_4) b^2 (a-b)^2 > m\left[a^4 b_6 + a^3 (a-b) b_5\right]$$

such that

$$f(t,u) \leq b_4 u + c,\ I_k(u) \leq b_5 u + c,\ J_k(u) \leq b_6 u + c, \quad \text{for all } t \in [0,1], u \in \mathbb{R}^+.$$

Theorem 3.2. *Suppose that* (H1)*,* (H3) *and* (H4) *hold. Then* (1.1) *has at least one positive solution.*

Proof. Let $\mathcal{M}_1 = \{u \in P : u = \mathcal{A}u + \lambda\psi, \lambda \geq 0\}$, where $\psi \in P_0$ is a given element. From Lemma 2.7, $u \in \mathcal{M}_1$ implies that $u \in P_0$. We shall prove that $\mathcal{M}_1$ is bounded. If $u \in \mathcal{M}_1$, then $u \geq \mathcal{A}u$. This shows

$$u(t) \geq \int_0^1 G_1(t,s) f(s,u(s))\mathrm{d}s + \sum_{i=1}^m G_2(t,t_i) J_i(u(t_i)) + \sum_{i=1}^m G_3(t,t_i) I_i(u(t_i)). \tag{3.3}$$

Multiplying by $M(t)$ on both sides of the above and integrating over $[0,1]$, we obtain

$$\begin{aligned}\int_0^1 u(t)M(t)\mathrm{d}t &\geq \int_0^1 M(t)\left[\int_0^1 G_1(t,s) f(s,u(s))\mathrm{d}s + \sum_{i=1}^m G_2(t,t_i) J_i(u(t_i)) + \sum_{i=1}^m G_3(t,t_i) I_i(u(t_i))\right]\mathrm{d}t \\ &\geq \frac{b}{a}\kappa_1 \int_0^1 f(t,u(t))M(t)\mathrm{d}t + \frac{b^2}{(a-b)^2}\kappa_1 \sum_{i=1}^m J_i(u(t_i)) + \frac{b}{a-b}\kappa_1 \sum_{i=1}^m I_i(u(t_i)).\end{aligned} \tag{3.4}$$

Combining this and (H3), we find

$$\begin{aligned}\int_0^1 u(t)M(t)\mathrm{d}t &\geq \frac{b}{a}\kappa_1 \int_0^1 M(t)(a_1 u(t) - c)\mathrm{d}t + \frac{b^2}{(a-b)^2}\kappa_1 \sum_{i=1}^m (a_3 u(t_i) - c) + \frac{b}{a-b}\kappa_1 \sum_{i=1}^m (a_2 u(t_i) - c) \\ &= \frac{b}{a} a_1 \kappa_1 \int_0^1 u(t)M(t)\mathrm{d}t + \frac{b^2}{(a-b)^2} a_3\kappa_1 \sum_{i=1}^m u(t_i) + \frac{b}{a-b} a_2 \kappa_1 \sum_{i=1}^m u(t_i) - c_1,\end{aligned} \tag{3.5}$$

where $c_1 = \frac{bc}{a}\kappa_1^2 + \frac{b^2 cm}{(a-b)^2}\kappa_1 + \frac{bcm}{a-b}\kappa_1$. Next we consider the following two cases.

Case 1. $\frac{b}{a} a_1 \kappa_1 \geq 1$. From (3.5) and $u \in P_0$, we obtain

$$\begin{aligned}c_1 &\geq (\frac{b}{a} a_1\kappa_1 - 1) \int_0^1 u(t)M(t)\mathrm{d}t + \frac{b^2}{(a-b)^2} a_3 \kappa_1 \sum_{i=1}^m u(t_i) + \frac{b}{a-b} a_2 \kappa_1 \sum_{i=1}^m u(t_i) \\ &\geq (\frac{b}{a} a_1\kappa_1 - 1) \int_0^1 \frac{b^2}{a^2}\|u\| M(t)\mathrm{d}t + \frac{b^2}{(a-b)^2} a_3\kappa_1 \sum_{i=1}^m \frac{b^2}{a^2}\|u\| + \frac{b}{a-b} a_2\kappa_1 \sum_{i=1}^m \frac{b^2}{a^2}\|u\|.\end{aligned} \tag{3.6}$$

This shows that there exists $M_2 > 0$ such that

$$\|u\| \leq \frac{a^2(a-b)^2}{b^2} \cdot \frac{c_1}{a_3 b^2 \kappa_1 m + a_2 b(a-b)\kappa_1 m + \kappa_1 (a-b)^2 (\frac{b}{a} a_1 \kappa_1 - 1)} := M_2, \text{ for all } u \in \mathcal{M}_1.$$

Case 2. $\frac{b}{a}a_1\kappa_1 < 1$. From (3.5), we have

$$c_1 + (1 - \frac{b}{a}a_1\kappa_1)\int_0^1 u(t)M(t)\mathrm{d}t \geq \frac{b^2}{(a-b)^2}a_3\kappa_1\sum_{i=1}^m u(t_i) + \frac{b}{a-b}a_2\kappa_1\sum_{i=1}^m u(t_i). \tag{3.7}$$

Note that $u \in P_0$, we have

$$c_1 + (1 - \frac{b}{a}a_1\kappa_1)\kappa_1\|u\| \geq \frac{b^2}{(a-b)^2}a_3\kappa_1\sum_{i=1}^m \frac{b^2}{a^2}\|u\| + \frac{b}{a-b}a_2\kappa_1\sum_{i=1}^m \frac{b^2}{a^2}\|u\|. \tag{3.8}$$

Therefore,

$$\|u\| \leq \frac{c_1a^2(a-b)^2}{a_3b^4\kappa_1 m + a_2b^3(a-b)\kappa_1 m - (a-b)^2(a^2 - aba_1\kappa_1)\kappa_1} =: M_3, \quad \text{for all } u \in \mathcal{M}_1.$$

To sum up, $\mathcal{M}_1$ is a bounded set, as required. Taking $R > \max\{M_2, M_3\}$, we obtain

$$u \neq \mathcal{A}u + \lambda\psi, \quad \text{for all } u \in \partial B_R \cap P, \quad \lambda \geq 0.$$

Lemma 2.8 yields

$$i(\mathcal{A}, B_R \cap P, P) = 0. \tag{3.9}$$

Let $\mathcal{M}_2 := \{u \in \overline{B}_r \cap P : u = \lambda\mathcal{A}u, \lambda \in [0,1]\}$. We shall prove $\mathcal{M}_2 = \{0\}$. Indeed, if $u \in \mathcal{M}_2$, we have $u \in P_0$ and

$$u(t) \leq \int_0^1 G_1(t,s)f(s,u(s))\mathrm{d}s + \sum_{i=1}^m G_2(t,t_i)J_i(u(t_i)) + \sum_{i=1}^m G_3(t,t_i)I_i(u(t_i)), \text{ for all } u \in \overline{B}_r \cap P.$$

Similar to (3.4), multiplying by $M(t)$ on both sides of the above and integrating over $[0,1]$, we obtain

$$\begin{aligned}\int_0^1 u(t)M(t)\mathrm{d}t \leq & \kappa_1\int_0^1 M(t)f(t,u(t))\mathrm{d}t + \frac{a^2}{(a-b)^2}\kappa_1\sum_{i=1}^m J_i(u(t_i)) \\ & + \frac{a}{a-b}\kappa_1\sum_{i=1}^m I_i(u(t_i)), \text{ for all } u \in \overline{B}_r \cap P.\end{aligned} \tag{3.10}$$

This, together with (H4), implies that

$$\int_0^1 u(t)M(t)\mathrm{d}t \leq \kappa_1b_1\int_0^1 u(t)M(t)\mathrm{d}t + \frac{a^2}{(a-b)^2}b_3\kappa_1\sum_{i=1}^m u(t_i) + \frac{a}{a-b}b_2\kappa_1\sum_{i=1}^m u(t_i). \tag{3.11}$$

From $u \in P_0$ we have

$$\frac{(1-\kappa_1b_1)b^2}{a^2}\kappa_1\|u\| \leq (1-\kappa_1b_1)\int_0^1 u(t)M(t)\mathrm{d}t \leq \frac{a^2}{(a-b)^2}b_3\kappa_1\sum_{i=1}^m \|u\| + \frac{a}{a-b}b_2\kappa_1\sum_{i=1}^m \|u\|,$$

which contradicts the condition $\frac{(1-\kappa_1b_1)b^2}{a^2}\kappa_1 > m\left[\frac{a^2}{(a-b)^2}b_3\kappa_1 + \frac{a}{a-b}b_2\kappa_1\right]$. This implies $\mathcal{M}_2 = \{0\}$ and thus $u \neq \lambda\mathcal{A}u$ for all $u \in \partial B_r \cap P$ and $\lambda \in [0,1]$. Lemma 2.9 yields

$$i(\mathcal{A}, B_r \cap P, P) = 1. \tag{3.12}$$

Equations (3.9) and (3.12) imply that

$$i(\mathcal{A}, (B_R\backslash\overline{B}_r) \cap P, P) = 0 - 1 = -1.$$

Hence the operator $\mathcal{A}$ has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$ and therefore (1.1) has at least one positive solution. This completes the proof. □

Theorem 3.3. *Suppose that* (H1), (H5) *and* (H6) *hold. Then* (1.1) *has at least one positive solution.*

Proof. Let $\mathcal{M}_3 := \{u \in \overline{B}_r \cap P : u = \mathcal{A}u + \lambda\psi, \lambda \geq 0\}$, where $\psi \in P_0$ is a given element. We claim $\mathcal{M}_3 \subset \{0\}$. Indeed, if $u \in \mathcal{M}_3$, then $u \in P_0$ and $u \geq \mathcal{A}u$. By (H5) and (3.5), we have

$$\int_0^1 u(t)M(t)\mathrm{d}t \geq \frac{b}{a}a_4\kappa_1 \int_0^1 u(t)M(t)\mathrm{d}t + \frac{b^2}{(a-b)^2}a_6\kappa_1 \sum_{i=1}^m u(t_i) + \frac{b}{a-b}a_5\kappa_1 \sum_{i=1}^m u(t_i). \tag{3.13}$$

If $\frac{b}{a}a_4\kappa_1 \geq 1$, note that $u \in P_0$, then

$$\begin{aligned} 0 &\geq (\frac{b}{a}a_4\kappa_1 - 1) \int_0^1 u(t)M(t)\mathrm{d}t + \frac{b^2}{(a-b)^2}a_6\kappa_1 \sum_{i=1}^m u(t_i) + \frac{b}{a-b}a_5\kappa_1 \sum_{i=1}^m u(t_i) \\ &\geq (\frac{b}{a}a_4\kappa_1 - 1) \int_0^1 \frac{b^2}{a^2}\|u\|M(t)\mathrm{d}t + \frac{b^2}{(a-b)^2}a_6\kappa_1 \sum_{i=1}^m \frac{b^2}{a^2}\|u\| + \frac{b}{a-b}a_5\kappa_1 \sum_{i=1}^m \frac{b^2}{a^2}\|u\|. \end{aligned}$$

This shows $\|u\| \equiv 0, \forall u \in \mathcal{M}_3$.

If $\frac{b}{a}a_4\kappa_1 < 1$, then

$$\begin{aligned} (1 - \frac{b}{a}a_4\kappa_1)\kappa_1\|u\| \geq (1 - \frac{b}{a}a_4\kappa_1) \int_0^1 u(t)M(t)\mathrm{d}t &\geq \frac{b^2}{(a-b)^2}a_6\kappa_1 \sum_{i=1}^m u(t_i) + \frac{b}{a-b}a_5\kappa_1 \sum_{i=1}^m u(t_i) \\ &\geq \frac{b^2}{(a-b)^2}a_6\kappa_1 \sum_{i=1}^m \frac{b^2}{a^2}\|u\| + \frac{b}{a-b}a_5\kappa_1 \sum_{i=1}^m \frac{b^2}{a^2}\|u\|, \end{aligned}$$

which contradicts the property $(1 - \frac{b}{a}a_4\kappa_1)\kappa_1 < \kappa_1 m \left[\frac{b^2}{(a-b)^2}a_6\frac{b^2}{a^2} + \frac{b}{a-b}a_5\frac{b^2}{a^2}\right]$. This also verify $\|u\| \equiv 0, \forall u \in \mathcal{M}_3$.

Hence $\mathcal{M}_3 \subset \{0\}$, as claimed. As a result, we have $u - \mathcal{A}u \neq \lambda\psi$ for all $u \in \partial B_r \cap P$ and $\lambda \geq 0$. Lemma 2.8 gives

$$i(\mathcal{A}, B_r \cap P, P) = 0. \tag{3.14}$$

Let $\mathcal{M}_4 := \{u \in P : u = \lambda\mathcal{A}u, \lambda \in [0,1]\}$. We assert $\mathcal{M}_4$ is bounded. Indeed, if $u \in \mathcal{M}_4$, then we have $u \in P_0$ and $u \leq \mathcal{A}u$, which can be written in the form

$$u(t) \leq \int_0^1 G_1(t,s)f(s,u(s))\mathrm{d}s + \sum_{i=1}^m G_2(t,t_i)J_i(u(t_i)) + \sum_{i=1}^m G_3(t,t_i)I_i(u(t_i)).$$

By (H6) and (3.10), we obtain

$$\int_0^1 u(t)M(t)\mathrm{d}t \leq b_4\kappa_1 \int_0^1 u(t)M(t)\mathrm{d}t + \frac{a^2}{(a-b)^2}b_6\kappa_1 \sum_{i=1}^m u(t_i) + \frac{a}{a-b}b_5\kappa_1 \sum_{i=1}^m u(t_i) + c_2,$$

where $c_2 = \kappa_1^2 c + \frac{a^2cm}{(a-b)^2}\kappa_1 + \frac{acm}{a-b}\kappa_1$.

From $u \in P_0$, we get

$$\begin{aligned} \frac{(1-b_4\kappa_1)b^2}{a^2}\kappa_1\|u\| \leq (1-b_4\kappa_1) \int_0^1 u(t)M(t)\mathrm{d}t &\leq \frac{a^2}{(a-b)^2}b_6\kappa_1 \sum_{i=1}^m u(t_i) + \frac{a}{a-b}b_5\kappa_1 \sum_{i=1}^m u(t_i) + c_2 \\ &\leq \frac{a^2}{(a-b)^2}b_6\kappa_1 \sum_{i=1}^m \|u\| + \frac{a}{a-b}b_5\kappa_1 \sum_{i=1}^m \|u\| + c_2. \end{aligned}$$

Consequently, we see

$$\|u\| \leq \frac{c_2 a^2(a-b)^2}{(1-\kappa_1 b_4)b^2(a-b)^2\kappa_1 - \kappa_1 m\left[a^4 b_6 + a^3(a-b)b_5\right]} := M_4.$$

Now $\mathcal{M}_4$ is a bounded set, as asserted. Taking $R > M_4$, we have $u \neq \lambda \mathcal{A} u$ for all $u \in \partial B_R \cap P$ and $\lambda \in [0,1]$. Lemma 2.9 yields

$$i(\mathcal{A}, B_R \cap P, P) = 1. \tag{3.15}$$

Equations (3.14) and (3.15) imply that

$$i(\mathcal{A}, (B_R \backslash \overline{B}_r) \cap P, P) = 1 - 0 = 1.$$

Hence the operator $\mathcal{A}$ has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$ and therefore, (1.1) has at least one positive solution. This completes the proof. □

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