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On fixed points of (η, θ)-quasicontraction mappings in generalized metric spaces

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Abstract

We establish some fixed point results for mappings satisfying (η, θ)-quasicontraction condition in complete generalized metric spaces. Our results generalize many others. An example is provided to support our work.

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1. Introduction and Preliminaries

Fixed point theory is one of the crucial methods in applied mathematics that ensure the existence and uniqueness of the solutions to many application problems of the theory of ordinary differential equations, partial differential equations and integral equations. The first fixed point theorem is the Banach contraction principle [7]. For modifications of Banach contraction principle, we refer the reader to [4, 5, 14, 16–18].

In 1989, Bakhtin [6] presented b-metric spaces as a generalization of metric spaces. After that, several authors have studied fixed point theory or the variational principle for single-valued and multivalued mappings in b-metric spaces (see [1–9] and the references therein). In 2000, Hitzler and Seda [10] introduced dislocated metric spaces also as a generalization of metric spaces. The theory of modular spaces was initiated

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by Nakano [15] in connection with the theory of order spaces. Those spaces were redefined and generalized in [13]. By defining a norm, particular Banach spaces of functions can be considered, and metric fixed theory for such spaces has been widely studied. Even without metric, many problems in fixed point theory can be formulated in modular spaces. The generalized metric spaces [11], initiated by Jleli and Samet in 2015, generalized many spaces: ordinary metric spaces, b-metric spaces, modular metric spaces and dislocated metric spaces.

In 1994, Khan et al. [12] presented the notation of an altering distance function as follows.

Definition 1.1 ([12])**.** A function $\eta : [0,\infty) \to [0,\infty)$ is called an altering distance function if η satisfies the following conditions:

η_1: η is continuous and nondecreasing;
η_2: $\eta(t) = 0$ iff $t = 0$.

We start with the following notations that we need in our work.

Definition 1.2 ([11])**.** Let Y be a nonempty set and $\mathcal{D} : Y \times Y \to [0,\infty)$ a given mapping. For every $y \in Y$, let us define the set

$$K(\mathcal{D}, Y, y) = \left\{ \{y_n\} \subset Y : \lim_{n\to\infty} \mathcal{D}(y_n, y) = 0 \right\}.$$

Definition 1.3 ([11])**.** $\mathcal{D}$ is a generalized metric on Y if it satisfies the following conditions:

$\mathcal{D}_1$: for every $(y,z) \in Y \times Y$, we have $\mathcal{D}(y,z) = 0 \Rightarrow y = z$;
$\mathcal{D}_2$: for every $(y,z) \in Y \times Y$, we have $\mathcal{D}(y,z) = \mathcal{D}(z,y)$;
$\mathcal{D}_3$: there exists a $k > 0$ such that

if $(y,z) \in Y \times Y$, $\{y_n\} \in K(\mathcal{D}, Y, y)$, then $\mathcal{D}(y,z) \le k \limsup_{n\to\infty} \mathcal{D}(y_n, z)$.

Then $(Y, \mathcal{D})$ is a generalized metric space.

Remark 1.4. If $K(\mathcal{D}, Y, y)$ is empty set for every $y \in Y$, then $(Y, \mathcal{D})$ is a generalized metric space if and only if $(\mathcal{D}_1)$ and $(\mathcal{D}_2)$ are satisfied.

Definition 1.5 ([11])**.** Let $(Y, \mathcal{D})$ be a generalized metric space. Then

1. $\{y_n\}$ is a $\mathcal{D}$-Cauchy sequence if $\lim_{n,m\to\infty} \mathcal{D}(y_n, y_{n+m}) = 0$;
2. $\{y_n\}$ is a $\mathcal{D}$-convergent sequence if $\{y_n\} \in K(\mathcal{D}, Y, y)$.

Definition 1.6 ([11])**.** Let $(Y, \mathcal{D})$ be a generalized metric space. It is termed to be complete if every $\mathcal{D}$-Cauchy sequence in Y is $\mathcal{D}$-convergent to an element in Y.

Proposition 1.7 ([11])**.** *Let $(Y, \mathcal{D})$ be a generalized metric space, $\{y_n\}$ a sequence in it, and $y, z \in Y$. If $\{y_n\}$ $\mathcal{D}$-converges to y and z, then $y = z$.*

Definition 1.8 ([11])**.** A mapping $g : Y \to Y$ is said to be weak continuous if for each sequence $\{y_n\}$ of Y that is $\mathcal{D}$-convergent to an $y \subset Y$, there exists a subsequence $\{y_{n_s}\}$ of $\{y_n\}$ such that $\{g(y_{n_s})\}$ is $\mathcal{D}$-convergent to $g(y)$ (as $s \to \infty$).

Definition 1.9 ([11])**.** Suppose that $(Y, \mathcal{D})$ is a generalized metric space and let $\preceq$ be a partial order on Y. Define

$$\Gamma_{\preceq} = \{(y,z) \in Y \times Y : y \preceq z\}.$$

Definition 1.10 ([11])**.** The mapping $g : Y \to Y$ is said to be monotone if

$$(y,z) \in \Gamma_{\preceq} \Rightarrow (g(y), g(z)) \in \Gamma_{\preceq}.$$

Definition 1.11 ([11])**.** The couple $(Y, \preceq)$ is $\mathcal{D}$-regular if the following condition is satisfied:

For every $\{y_n\} \subset Y$ satisfying $(y_n, y_{n+1}) \in \Gamma_{\preceq}$ and for every n large enough, if $\{y_n\}$ is $\mathcal{D}$-convergent to an $y \in Y$, then there exists a subsequence $\{y_{n_s}\}$ of $\{y_n\}$ such that $(y_{n_s}, y) \in \Gamma_{\preceq}$, for every s large enough.

Definition 1.12 ([11])**.** Let $(Y, \mathcal{D})$ be a generalized metric space. Let $g : Y \to Y$ be a mapping. For $y \in Y$, the number $\delta(\mathcal{D}, g, y)$ is defined by

$$\delta(\mathcal{D}, g, y) = \sup\{\mathcal{D}(g^i(y), g^j(y)) : i, j \in \mathbb{N}\},$$

where $g^i(y) = g(g^{i-1}(y))$.

In this paper, we introduce notions of (η, θ)-contraction and (η, θ)-quasicontraction mappings in generalized metric spaces and prove some fixed point results for them.

2. Main result

In the rest of this paper, we denote by Θ the family of all functions $\theta : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

- θ_1: θ is a continuous;
- θ_2: θ is a continuous increasing;
- θ_3: $\theta(0) = 0$;
- θ_4; $\lim_{n\to\infty} \theta^n(t) = 0$ for all $t \in [0, \infty)$.

Definition 2.1. Let $(Y, \mathcal{D})$ be a generalized metric space. Let η be an altering distance function and $\theta \in \Theta$. We say that $g : Y \to Y$ is an (η, θ)-contraction mapping if

$$\eta(\mathcal{D}(gy, gz)) \le \theta(\eta(\mathcal{D}(y, z))) \tag{2.1}$$

holds for every $(y, z) \in Y \times Y$.

Theorem 2.2. *Suppose that $(Y, \mathcal{D})$ is a complete generalized metric space. Let $g : Y \to Y$ be an (η, θ)-contraction. Suppose that there exists a $y_0 \in Y$ such that $\delta(\mathcal{D}, g, y_0) < \infty$. Then $\{g^n(y_0)\}$ converges to a fixed point of $u \in Y$ of g. Moreover, if $u' \in Y$ is another fixed point of g such that $\mathcal{D}(u, u') < +\infty$, then $u = u'$.*

Proof. Let $n \in \mathbb{N}$. Since g is an (η, θ)-contraction, for all $i, j \in \mathbb{N}$, we have

$$\eta(\mathcal{D}(g^{n+i}(y_0), g^{n+j}(y_0))) \le \theta(\eta(\mathcal{D}(g^{n-1+i}(y_0), g^{n-1+j}(y_0)))).$$

Which implies that

$$\eta(\mathcal{D}(g^{n+i}(y_0), g^{n+j}(y_0))) \le \theta^n(\eta(\mathcal{D}(g^i(y_0), g^j(y_0)))).$$

By using the definition of $\delta(\mathcal{D}, g, y_0)$, we get that

$$\eta(\mathcal{D}(g^{n+i}(y_0), g^{n+j}(y_0))) \le \theta^n(\eta(\delta(\mathcal{D}, g, y_0))). \tag{2.2}$$

By the properties of θ and since $\delta(\mathcal{D}, g, y_0) < +\infty$, we get

$$\lim_{n\to+\infty} \eta(\mathcal{D}(g^{n+i}(y_0), g^{n+j}(y_0))) = 0,$$

and consequently

$$\lim_{n\to+\infty} \mathcal{D}(g^{n+i}(y_0), g^n(y_0)) = 0.$$

By using (2.2), for every $n, m \in \mathbb{N}$, we obtain

$$\eta(\mathcal{D}(g^n(y_0), g^{n+m}(y_0)) \leq \theta^n(\eta(\delta(\mathcal{D}, g, y_0))).$$

Since $\delta(\mathcal{D}, g, y_0) < +\infty$ and $\theta \in \Theta$, we get

$$\lim_{n,m\to+\infty} \eta(\mathcal{D}(g^n(y_0), g^{n+m}(y_0))) = 0.$$

Thus, $\{g^n(y_0)\}$ is a $\mathcal{D}$-Cauchy sequence. Since Y is a complete generalized metric space, there exists a $u \in Y$ such that $\{g^n(y_0)\}$ is $\mathcal{D}$-convergent to u. Since g is an (η, θ)-contraction, for all $n \geq 1$, we get

$$\eta(\mathcal{D}(g^{n+1}(y_0), g(u)) \leq \theta(\eta(\mathcal{D}(g^n(y_0), u))),$$

by taking limit in above inequality and using the property of η, we obtain

$$\lim_{n\to+\infty} \mathcal{D}(g^{n+1}(y_0), g(u)) = 0,$$

so $\{g^n(y_0)\}$ is $\mathcal{D}$-convergent to $g(u)$. Proposition 1.7 implies the uniqueness of the limit, so we have $u = gu$. Now, assume that $u^{'} \in Y$ is a fixed point of g such that $\mathcal{D}(u, u^{'}) < +\infty$. Since g is an (η, θ)-contraction mapping, we have

$$\begin{aligned}\eta(\mathcal{D}(u, u^{'})) &= \eta(\mathcal{D}(g(u), g(u^{'})))\\ &\leq \theta(\eta(\mathcal{D}(u, u^{'}))).\end{aligned}$$

Thus for $n \in \mathbb{N}$, we have $\eta(\mathcal{D}(u, u^{'})) \leq \theta^n(\eta(\mathcal{D}(u, u^{'})))$. Letting $n \to +\infty$, we get

$$\eta(\mathcal{D}(u, u^{'})) = 0$$

which implies by condition $\mathcal{D}_1$ that $u = u^{'}$. □

Definition 2.3. Let $(Y, \mathcal{D})$ be a generalized metric space. We say that $g : Y \to Y$ is an (η, θ)-quasicontraction mapping if

$$\eta(\mathcal{D}(gy, gz)) \leq \theta(\eta v(y, z)), \tag{2.3}$$

where

$$v(y, z) \in \{\mathcal{D}(y, z), \mathcal{D}(y, gy), \mathcal{D}(z, gz), \mathcal{D}(gy, z), \mathcal{D}(y, gz)\}$$

holds for every $(y, z) \in Y \times Y$.

Proposition 2.4. *Suppose that g is an (η, θ)-quasicontraction mapping. If $u \in Y$ is a fixed point of g with $\mathcal{D}(u, u)) < \infty$, then $\mathcal{D}(u, u) = 0$.*

Proof. Let $u \in Y$ be a fixed point of g such that $\theta(\eta(\mathcal{D}(u, u)) < \infty$. Since g is an (η, θ)-quasicontraction, we have

$$\eta(\mathcal{D}(u, u)) = \eta(\mathcal{D}(gu, gu)) \leq \theta(\eta(\mathcal{D}(u, u))).$$

By using the properties of (η, θ), we get $\mathcal{D}(u, u) = 0$. □

Theorem 2.5. *Suppose $(Y, \mathcal{D})$ is a complete generalized metric space. Let $g : Y \to Y$ be (η, θ)-quasicont -raction mapping. If there exists $y_0 \in Y$ such that $\delta(\mathcal{D}, g, y_0) < \infty$, then $\{g^n(y_0)\}$ converges to some $u \in Y$. If $\mathcal{D}(y_0, g(u)) < \infty$ and $\mathcal{D}(u, g(u)) < \infty$, then u is a fixed point of g. Moreover, if $u' \in Y$ is another fixed point of g such that $\mathcal{D}(u, u') < \infty$ and $\mathcal{D}(u', u') < \infty$, then $u = u'$.*

Proof. From (η,θ)-quasicontraction mapping for all $i,j\in\mathbb{N}$, we have

$$\begin{aligned}\eta(\mathcal{D}(g^{n+i}(y_0),g^{n+j}(y_0)))\leq \theta(\eta(&\mathcal{D}(g^{n-1+i}(y_0),g^{n-1+j}(y_0)),\mathcal{D}(g^{n-1+i}(y_0),g^{n+i}(y_0)),\\ &\mathcal{D}(g^{n-1+j}(y_0),g^{n+j}(y_0)),\mathcal{D}(g^{n-1+i}(y_0),g^{n+j}(y_0))),\\ &\mathcal{D}(g^{n-1+j}(y_0),g^{n+i}(y_0))),\end{aligned}$$

which implies

$$\eta(\delta(\mathcal{D},g,g^n(y_0))\leq\theta(\eta(\delta(\mathcal{D},g,g^{n-1}(y_0))),\delta(\mathcal{D},g,g^n(y_0))). \tag{2.4}$$

If $v(g^{n+i}(y_0),g^{n+j}(y_0))=\delta(\mathcal{D},g,g^n(y_0))$, then by (2.4) and using the properties of (η,θ), we have

$$\begin{aligned}\eta(\delta(\mathcal{D},g,g^n(y_0))&\leq\theta(\eta\delta(\mathcal{D},g,g^n(y_0)))\\ &<\eta(\delta(\mathcal{D},g,g^n(y_0))),\end{aligned}$$

a contradiction. Hence

$$v(g^{n+i}(y_0),g^{n+j}(y_0))=\delta(\mathcal{D},g,g^{n-1}(y_0)),$$

and so

$$\eta(\delta(\mathcal{D},g,g^n(y_0))\leq\theta(\eta(\delta(\mathcal{D},g,g^{n-1}(y_0)))<\eta(\delta(\mathcal{D},g,g^{n-1}(y_0)))$$

for all $n\in\mathbb{N}$. We get

$$\begin{aligned}\eta(\delta(\mathcal{D},g,g^n(y_0)))&\leq\theta(\eta(\delta(\mathcal{D},g,g^{n-1}(y_0))))\\ &\leq\theta^2(\eta(\delta(\mathcal{D},g,g^{n-2}(y_0))))\\ &\;\;\vdots\\ &\leq\theta^n(\eta(\delta(\mathcal{D},g,y_0))),\end{aligned}$$

and by using the above inequality for every $n,m\in\mathbb{N}$, we obtain

$$\begin{aligned}\eta(\mathcal{D}(g^n(y_0),g^{n+m}(y_0))&\leq\theta(\eta(\delta(\mathcal{D},g^ny_0)))\\ &\leq\theta^n(\eta(\delta(\mathcal{D},g,y_0))).\end{aligned}$$

Since $\delta(\mathcal{D},g,y_0)<\infty$ and by the property of η, we get $\lim_{n,m\to\infty}\eta(\mathcal{D}(g^n(y_0),g^{n+m}(y_0)))=0$, and consequently

$$\lim_{n,m\to\infty}\mathcal{D}(g^n(y_0),g^{n+m}(y_0))=0,$$

thus, $\{g^n(y_0)\}$ is a $\mathcal{D}$-Cauchy sequence. Since Y is a complete generalized metric space, there exists a $u\in Y$ such that $\{g^n(y_0)\}$ is $\mathcal{D}$-convergent to u.

We assume that $\mathcal{D}(y_0,g(u))<\infty$. By the inequality

$$\eta(\mathcal{D}(g^n(y_0),g^{n+m}(y_0))\leq\theta^n(\eta(\delta(\mathcal{D},g,y_0))) \tag{2.5}$$

for every $n,m\in\mathbb{N}$, and by the condition $(\mathcal{D}_3)$, there exists a $k>0$ such that

$$\psi(\mathcal{D}(u,g^n(y_0)))\leq k\limsup_{n\to\infty}\eta(\mathcal{D}(g^n(y_0),g^{n+m}(y_0))\leq k\theta^n(\eta(\delta(\mathcal{D},g,y_0))) \tag{2.6}$$

for every $n\geq 1$.

Now, we show that u is a fixed point of g. We have

$$\eta(\mathcal{D}(g(y_0), g(u))) \leq \theta(\eta\{\mathcal{D}(y_0, u), \mathcal{D}(y_0, g(y_0)), \mathcal{D}(u, g(u)), \mathcal{D}(g(y_0), u), \mathcal{D}(y_0, g(u))\}.$$

Using (2.5) and (2.6), we obtain

$$\eta(\mathcal{D}(g(y_0), g(u))) \leq \{k\theta(\eta\delta(\mathcal{D}, g, y_0)), \theta(\eta\delta(\mathcal{D}, g, y_0)), \theta(\eta(\mathcal{D}(u, g(u)))), \theta(\eta(\mathcal{D}(y_0, g(u))))\}.$$

Again, using the previous inequality, we get

$$\eta(\mathcal{D}(g^2(y_0), g(u))) \leq \{k\theta^2(\eta\delta(\mathcal{D}, g, y_0)), \theta^2(\eta\delta(\mathcal{D}, g, y_0)), \theta^2(\eta(\mathcal{D}(u, g(u)))), \theta^2(\eta(\mathcal{D}(y_0, g(u))))\}.$$

Continuing this process, by induction, we obtain

$$\eta(\mathcal{D}(g^n(y_0), g(u))) \leq \{k\theta^n(\eta\delta(\mathcal{D}, g, y_0)), \theta^n(\eta\delta(\mathcal{D}, g, y_0)), \theta^n(\eta(\mathcal{D}(u, g(u)))), \theta^n(\eta(\mathcal{D}(y_0, g(u))))\}$$

for every $n \in \mathbb{N}$, and therefore

$$\lim_{n,m\to\infty} \eta(\mathcal{D}(g^n(y_0), g(u))) \leq \theta(\eta(\mathcal{D}(u, g(u))).$$

Since $\mathcal{D}(y_0, g(u)) < \infty$ and $\delta(\mathcal{D}, g, y_0) < \infty$. By using the property($\mathcal{D}_3$), we get

$$\eta(\mathcal{D}(g(u), u))) \leq \limsup_{n,m\to\infty} \mathcal{D}(g^n(y_0), g(u)) \leq \theta(\eta(\mathcal{D}(u, g(u))),$$

which implies that $\theta((\mathcal{D}(u, g(u)))) = 0$. From the properties of (η, θ), we get $(\mathcal{D}(u, g(u))) = 0$ and since $\mathcal{D}(u, g(u)) < \infty$, then u is a fixed point of g. By Proposition 2.4, we have $\mathcal{D}(u, u) = 0$.

Finally, assume that $u^{'} \in Y$ is another fixed point of g such that $\mathcal{D}(u, u^{'}) < \infty$ and $\mathcal{D}(u^{'}, u^{'}) < \infty$. By Proposition 2.4, we have $\mathcal{D}(u^{'}, u^{'}) = 0$. Since g is an (η, θ)-quasicontraction, we have

$$\eta(\mathcal{D}(u, u^{'}) = \eta(\mathcal{D}(g(u), g(u^{'})) \leq \theta(\eta(\mathcal{D}(u, u^{'})),$$

which implies that $u = u^{'}$ □

Corollary 2.6. *Suppose $(Y, \mathcal{D})$ is a complete generalized metric space. Let $g : Y \to Y$ be a mapping satisfying*

$$\eta(\mathcal{D}(gy, gz)) \leq \theta(\eta(M(y, z))), \tag{2.7}$$

where

$$M(y, z) = \max\{\mathcal{D}(y, z), \mathcal{D}(y, gy), \mathcal{D}(z, gz), \mathcal{D}(gy, z), \mathcal{D}(y, gz)\}$$

for every $(y, z) \in Y \times Y$. If there exists a $y_0 \in Y$ such that $\delta(\mathcal{D}, g, y_0) < \infty$, then $\{g^n(y_0)\}$ converges to a $u \in Y$. If $\mathcal{D}(y_0, g(u)) < \infty$ and $\mathcal{D}(u, g(u)) < \infty$, then u is a fixed point of g. Moreover, if a $u' \in Y$ is another fixed point of g such that $\mathcal{D}(u, u') < \infty$ and $\mathcal{D}(u', u') < \infty$, then $u = u'$.

Proof. Since $v(y, z) = \max\{\mathcal{D}(y, z), \mathcal{D}(y, gy), \mathcal{D}(z, gz), \mathcal{D}(gy, z), \mathcal{D}(y, gz)\}$, the result follows from Theorem 2.5. □

In the following theorems, we extend the Banach contraction principle to (η, θ)-contraction mappings in complete generalized metric spaces with partial orders.

Theorem 2.7. *Suppose $(Y, \mathcal{D})$ is a complete generalized metric space. Let $g : Y \to Y$ be an (η, θ)-contraction mapping, weak continuous and monotone. If there exists a $y_0 \in Y$ such that $\delta(\mathcal{D}, g, y_0) < \infty$ and $(g(y), g(z)) \in \Gamma_{\preceq}$, then $\{g^n(y_0)\}$ converges to a $u \in Y$ such that u is a fixed point of g. Moreover, if $\mathcal{D}(u, u) < \infty$, then $\mathcal{D}(u, u) = 0$.*

Proof. Since g is $\preceq$-monotone and $(y_0, g(y_0)) \in \Gamma_{\preceq}$, then

$$(g^n(y_0, g^{n+1}(y_0))) \in \Gamma_{\preceq}$$

for every $n \in \mathbb{N}$. The relation $\preceq$ is a partial order (hence it is transitive), so

$$(s,r) \in \mathbb{N} \times \mathbb{N},\ s \leq r \Rightarrow g^s(y_0) \preceq g^r(y_0).$$

Assume $n \in \mathbb{N}$. Since g is an (η, θ)-contraction and by condition $(\mathcal{D}_2)$ in Definition 1.3, for all $i, j \in \mathbb{N}$, we get

$$\eta(\mathcal{D}(g^{n+i}(y_0), g^{n+j}(y_0))) \leq \theta(\eta(\mathcal{D}(g^{n-1+i}(y_0), g^{n-1+j}(y_0)))),$$

which implies that

$$\eta(\delta(\mathcal{D}, g, g^n(y_0)) \leq \theta(\eta(\delta(\mathcal{D}, g, g^{n-1}(y_0)))).$$

Thus, for every $n \geq 1$, we get

$$\eta(\delta(\mathcal{D}, g, g^n(y_0)) \leq \theta^n(\eta(\delta(\mathcal{D}, g, y_0))).$$

By using the above inequality, for every $n, m \in \mathbb{N}$, we obtain

$$\eta(\mathcal{D}(g^n(y_0), g^{n+m}(y_0)) \leq \eta(\delta(\mathcal{D}, g, g^n(y_0))) \leq \theta^n(\eta(\delta(\mathcal{D}, g, y_0))).$$

Since $\delta(\mathcal{D}, g, y_0) < \infty$ and by the property of η, we get $\lim_{n,m\to\infty} \eta(\mathcal{D}(g^n(y_0), g^{n+m}(y_0))) = 0$, and consequently

$$\lim_{n,m\to\infty} \mathcal{D}(g^n(y_0), g^{n+m}(y_0)) = 0,$$

thus, $\{g^n(y_0)\}$ is a $\mathcal{D}$-Cauchy sequence. Since $(Y, \preceq)$ is a complete generalized metric space, there exists a $u \in Y$ such that $\{g^n(y_0)\}$ is $\mathcal{D}$-convergent to u. Since g is an (η, θ)-contraction, for all $n \geq 1$, there exists a subsequence $\{g^{n_s+1}(y_0)\}$ of $\{g^n(y_0)\}$ such that $\{g^{n_s}(y_0)\}$ is $\mathcal{D}$-convergent to $g(u)$ as $(s \to \infty)$. From the uniqueness of the limit, we have $gu = u$. By the assumption $\mathcal{D}(u,u) < \infty$ and assume $\mathcal{D}(u,u) \neq 0$. Since $(u,u) \in E_{\preceq}$, we have

$$\mathcal{D}(u,u) = \mathcal{D}(g(u), g(u)) \leq \theta(\eta(\mathcal{D}(u,u)).$$

By using the properties of (η, θ), we have $\mathcal{D}(u,u) \leq \mathcal{D}(u,u))$, which is a contradiction. Hence$\mathcal{D}(u,u) = 0$. □

In the following theorem, we use the $\mathcal{D}$-regularity of $(Y, \preceq)$ instead of the weak continuity assumption.

Theorem 2.8. *Suppose $(Y, \preceq, \mathcal{D})$ is a $\mathcal{D}$-regular complete generalized metric space. Let $g : Y \to Y$ be an (η, θ)-contraction mapping. If there exists a $y_0 \in Y$ such that $\delta(\mathcal{D}, g, y_0) < \infty$ and $(y_0, g(y_0)) \in E_{\preceq}$, then $\{g^n(y_0)\}$ converges to some $u \in Y$ such that u is a fixed point of g. Moreover, if $\mathcal{D}(u,u) < \infty$, then $\mathcal{D}(u,u) = 0$.*

Proof. Following the proof of Theorem 2.7, we realize that $\{g^n(y_0)\}$ is $\mathcal{D}$-convergent to a $u \in Y$ and

$$(g^n(y_0), g^{n+1}(y_0)) \in \Gamma_{\preceq}$$

for every $n \in \mathbb{N}$. Since $(Y, \preceq)$ is $\mathcal{D}$-regular, there exists a subsequence $\{g^{n_s}(y_0)\}$ of $\{g^n(y_0)\}$ such that $(g^{n_s}(y_0), u) \in \Gamma_{\preceq}$. Furthermore, g is an (η, θ)-contraction, so we obtain

$$\eta\mathcal{D}(g^{n_s+1}, g(u)) \leq \theta(\eta\mathcal{D}(g^{n_s}(y_0)), g(u))$$

for every s large enough. Taking $s \to \infty$ in the previous inequality and by using the properties of η, we get

$$\lim_{n,m\to\infty} \mathcal{D}(g^{n_s+1}, g(u)) = 0,$$

which implies that $\{g^{n_s+1}\}$ is $\mathcal{D}$-convergent to $g(u)$. From the uniqueness of the limit, we get $gu = u$.

Similar to the proof of Theorem 2.7, we get $\mathcal{D}(u,u) = 0$. □

Remark 2.9. Theorems (3.3), (4.3), (5.5) and (5.7) in [11] are a special case of Theorems 2.2,2.5, 2.7 and 2.8 respectively.

Example 2.10. Let $Y = [0, \infty]$ and $\mathcal{D}(y, z) = |y - z|$ for each $y, z \in [0, \infty]$, $\mathcal{D}(y, \infty) = \infty$ for each $y \in [0, \infty]$ and assume $\mathcal{D}(y, \infty) = 0$. Then $(Y, \mathcal{D})$ is a complete generalized metric space. Let $g : Y \to Y$ be given by $gy = 2y$ for each $y \in [0, \infty]$ and $g\infty = \infty$. Take $\eta(t) = t$ and $\theta(t) = \frac{1}{2}t$; then we get

$$\eta(|gy - gz|) \leq \theta(\eta\{|y - z|, |y - gy|, |z - gz|, |y - gz|, |z - gy|\}),$$

and $\mathcal{D}(y, gy) < \infty$ for each $y, z \in Y$. Hence, all the hypothesis of Theorem 2.5 are satisfied, thus g has a unique fixed point $y = \infty$.

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