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Generalized contraction mapping principle in locally convex topological vector spaces

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Abstract

The purpose of this paper is to present the concept of contraction mapping in a locally convex topological vector spaces and to prove the generalized contraction mapping principle in such spaces. The neighborhood-type error estimate formula was also established. The results of this paper improve and extend Banach contraction mapping principle in new idea. ©2016 All rights reserved.

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1. Introduction and Preliminaries

Banach contraction mapping principle is one of the important tool (or method) in nonlinear analysis and other mathematical field. Weak contractions are generalizations of Banach contraction mappings which have been studied by several authors. Let (X, d) be a metric space and $\phi : [0, +\infty) \to [0, +\infty)$ be a function. We say that $T : X \to X$ is a ϕ-contraction if

$$d(Tx, Ty) \leq \phi(d(x, y)), \quad \forall\ x, y \in X.$$

In 1968, Browder [3] proved that if ϕ is non-decreasing and right continuous and (X, d) is complete, then T has a unique fixed point x^* and $\lim_{n\to\infty} T^n x_0 = x^*$ for any given $x_0 \in X$. Subsequently, this result was

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extended in 1969 by Boyd and Wong [2] by weakening the hypothesis on ϕ, in the sense that it is sufficient to assume that ϕ is right upper semi-continuous. For a comprehensive study of relations between several such contraction type conditions, see [4, 8, 9, 15].

In 1973, Geraghty [4] introduced the Geraghty-contraction and obtained the fixed point theorem. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a Geraghty-contraction if there exists $\beta \in \Gamma$ such that for any $x, y \in X$

$$d(Tx, Ty) \leq \beta\left(d(x, y)\right) d(x, y),$$

where the class Γ denotes those functions $\beta : [0, +\infty) \to [0, +\infty)$ satisfying the following condition: $\beta(t_n) \to 1 \Rightarrow t_n \to 0$.

On the other hand, in 2015, Su and Yao [17] proved the following generalized contraction mapping principle.

Theorem 1.1. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a mapping such that*

$$\psi(d(Tx, Ty)) \leq \phi(d(x, y)),\ \forall\ x, y \in X, \tag{1.1}$$

where $\psi, \phi : [0, +\infty) \to [0, +\infty)$ are two functions satisfying the conditions:

$$(1).\quad \psi(a) \leq \phi(b) \Rightarrow a \leq b;$$

$$(2).\quad \begin{cases} \psi(a_n) \leq \phi(b_n) \\ a_n \to \varepsilon,\ b_n \to \varepsilon \end{cases} \Rightarrow \varepsilon = 0.$$

Then, T has a unique fixed point and, for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to this fixed point.

Definition 1.2. Let (X, d) be a metric space, $T : X^N \to X$ be a N-variables mapping, an element $p \in X$ is called a multivariate fixed point (or a fixed point of order N, see [16]) of T if

$$p = T(p, p, \ldots, p).$$

Recently [16], Yongfu Su, A. Petruşel and Jen-Chih Yao proved a multivariate fixed point theorem for the N-variables contraction mappings which further generalizes Banach Contraction Principle.

In particular, the study of the fixed points for weak contractions and generalized contractions was extended to partially ordered metric spaces in [1, 5–7, 11–14, 18]. Among them, some results involve altering distance functions. Such functions were introduced by Khan et al. in [10], where some fixed point theorems are presented.

The purpose of this paper is to present the concept of contraction mapping in a locally convex topological vector spaces and to prove the generalized contraction mapping principle in such spaces. The results of this paper improve Banach contraction mapping principle.

2. Contraction mapping principle in locally convex spaces

Let us recall some concepts and results on the topological vector spaces.

Definition 2.1. A Hausdorff topology τ on a real vector space X over R is said to be a vector space topology for X if addition and scalar-multiplication are continuous, that is, the mappings

$$(x, y) \mapsto x + y \quad from \quad X \times X \quad into \quad X$$

and

$$(\alpha, x) \mapsto \alpha x \quad from \quad R \times X \quad into \quad X$$

are continuous, where $X \times X$ and $R \times X$ are equipped with the respective product topologies. X itself, or more precisely (X, τ) is then called a topological vector space.

Remark 2.2. Continuity of addition means: For every neighborhood W of x_0+y_0 there exist neighborhood U of x_0 and V of y_0 such that $U+V\subset W$. Continuity of scalar-multiplication means: For every neighborhood W of α_0x_0 there exist a $\delta>0$ and a neighborhood U of x_0 such that

$$\alpha U\subset W,\quad \forall\quad |\alpha-\alpha_0|<\delta.$$

Definition 2.3. A topological vector space (X,τ) is said to be locally convex, if there exists a basis of neighborhood of zero Ω such that every $U\in\Omega$ is convex set.

Conclusion 2.4. Let (X,τ) be a locally convex topological vector space. For any convex neighborhood of zero $U\in\Omega$, there exists a balanced convex neighborhood of zero V such that $V\subset U$.

Proof. For any convex neighborhood of zero $U\in\Omega$, there exists a balanced neighborhood of zero W such that $W\subset U$. Let

$$A=\bigcap_{|\alpha|=1}\alpha U,$$

then A and A^0 are convex. Since W is balanced, we have

$$W=\alpha W\subset\alpha U,\ \ \forall\ |\alpha|=1,$$

which implies

$$W\subset A,\ \ W\subset A^0.$$

Hence A^0 is a neighborhood of zero. Next, we show A^0 is balanced. In fact that, for any $|\lambda|\le 1$, we have

$$\lambda A=\bigcap_{|\alpha|=1}\lambda\alpha U=\bigcap_{|\alpha|=1}|\lambda|\alpha U\subset\bigcap_{|\alpha|=1}\alpha U=A,$$

which implies A is balanced, so is A^0. Let $V=A^0$, we have V is a balanced convex neighborhood of zero such that $V\subset U$. This completes the proof. □

From Conclusion 2.4, we can get the following result.

Conclusion 2.5. Let (X,τ) be a locally convex topological vector space. Then for (X,τ), there exists a basis of balanced convex neighborhood of zero Ω. Furthermore, each $U\in\Omega$ is absorbing, balanced and convex.

Definition 2.6. Let (X,τ) be a locally convex topological vector space with a basis of balanced convex neighborhood of zero Ω.

(1). A mapping $T:X\to X$ is said to be contractive, if there exists a constant $h\in(0,1)$ such that for any $U\in\Omega$ and any $x,y\in X$

$$x-y\in tU\quad implies\quad Tx-Ty\in htU$$

for any $t>0$.

(2). A mapping $T:X\to X$ is said to be (ψ,ϕ)- contractive, if there exist two functions $\psi:[0,+\infty)\to[0,+\infty)$, $\phi:[0,+\infty)\to[0,+\infty)$ such that for any $U\in\Omega$ and any $x,y\in X$

$$x-y\in\phi(t)U\quad implies\quad Tx-Ty\in\psi(t)U$$

for any $t>0$.

Definition 2.7. Let (X,τ) be a topological vector space with a basis of balanced convex neighborhood of zero Ω, a net net $\{x_\lambda\}_{\lambda\in I}\subset X$ is said to be Cauchy, if for any $U\in\Omega$, there exists a $\lambda_0\in I$ such that

$$x_{\lambda_1}-y_{\lambda_2}\in U,\quad \forall\ \lambda_1,\lambda_2\ge\lambda_0.$$

The topological vector space (X,τ) is said to be complete, if every Cauchy net is convergent.

The following results are well-known in the theory of topological vector space.

Conclusion 2.8. Let (X,τ) be a locally convex topological vector space with a basis of balanced convex neighborhood of zero Ω. For any $U \in \Omega$, the Minkowski functional of U is defined by

$$M_U(x) = \inf\{t > 0 : x \in tU\}, \quad \forall\, x \in X.$$

Then the following hold:

(1). $M_U(x) \geq 0$, for any $x \in X$, and $x = 0$ implies $M_U(x) = 0$;
(2). $M_U(\lambda x) = |\lambda| M_U(x)$ for any $x \in X, \lambda \in R$;
(3). $M_U(x+y) \leq M_U(x) + M_U(y)$ for any $x, y \in X$;
(4). net $\{x_\lambda\}_{\lambda\in I} \subset X$ convergent to $x_0 \in X$ if and only if $\lim_{\lambda\in I} M_U(x_\lambda - x_0) = 0$;
(5). net $\{x_\lambda\}_{\lambda\in I} \subset X$ is a Cauchy net if and only if for any $U \in \Omega$

$$\lim_{\lambda_1,\lambda_2\in I} M_U(x_{\lambda_1} - x_{\lambda_2}) = 0.$$

Remark 2.9. In fact, for any $U \in \Omega$, the Minkowski functional $M_U(\cdot)$ is a semi-norm on the X.

Theorem 2.10 (Generalized contraction mapping principle)**.** *Let (X,τ) be a complete locally convex topological vector space with a basis of balanced convex neighborhood of zero Ω. Let $T : X \to X$ be a (ψ,ϕ)-contractive mapping satisfying the following conditions:*

(1). $\psi(t), \phi(t)$ are continuous and strictly increasing;
(2). $\psi(0) = \phi(0)$ and $\psi(t) < \phi(t)$ for all $t > 0$.

Then T has a unique fixed point and for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to this fixed point.

Proof. Since T is a (ψ,ϕ)-contractive mapping, we have, for any $U \in \Omega$, that

$$\begin{aligned}
\psi^{-1}(M_U(Tx-Ty)) &= \psi^{-1}\left(\inf\{t>0 : Tx-Ty \in tU\}\right)\\
&= \psi^{-1}\left(\inf\{\psi(t)>0 : Tx-Ty \in \psi(t)U\}\right)\\
&= \psi^{-1}\left(\psi(\inf\{t>0 : Tx-Ty \in \psi(t)U\})\right)\\
&= \inf\{t>0 : Tx-Ty \in \psi(t)U\}\\
&\leq \inf\{t>0 : x-y \in \phi(t)U\}\\
&= \phi^{-1}\left(\phi(\inf\{t>0 : x-y \in \phi(t)U\})\right)\\
&= \phi^{-1}\left(\phi(\phi^{-1}\inf\{\phi(t)>0 : x-y \in \phi(t)U\})\right)\\
&= \phi^{-1}\left(\inf\{\phi(t)>0 : x-y \in \phi(t)U\}\right)\\
&= \phi^{-1}\left(\inf\{t>0 : x-y \in tU\}\right)\\
&= \phi^{-1}\left(M_U(x-y)\right), \quad \forall\, x,y \in X.
\end{aligned} \tag{2.1}$$

For any given $x_0 \in X$, we define an iterative sequence as follows

$$x_1 = Tx_0,\ x_2 = Tx_1, \ldots, x_{n+1} = Tx_n, \ldots. \tag{2.2}$$

Then, for each integer $n \geq 1$, from (2.1) and (2.2) we get

$$\psi^{-1}\left(M_U(x_{n+1}-x_n)\right) = \psi^{-1}\left(M_U(Tx_n - Tx_{n-1})\right) \leq \phi^{-1}\left(M_U(x_n - x_{n-1})\right). \tag{2.3}$$

Using the condition (2) we have, $\psi^{-1}(t) \geq \phi^{-1}(t)$, therefore

$$M_U(x_{n+1}-x_n) \leq M_U(x_n - x_{n-1})$$

for all $n \geq 1$. Hence the sequence $M_U(x_{n+1} - x_n)$ is non-increasing and, consequently, there exists $r \geq 0$ such that

$$M_U(x_{n+1} - x_n) \to r,$$

as $n \to \infty$. By using the condition (2) and (2.1) we know $r = 0$.

In what follows, we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there exists a $V \in \Omega$ such that

$$\lim_{m,n\to\infty} M_V(x_m - x_n) = 0$$

does not holds. Therefore, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ with $n_k > m_k > k$ such that

$$M_V(x_{n_k} - x_{m_k}) \geq \varepsilon \tag{2.4}$$

for all $k \geq 1$. Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (2.4). Then

$$M_V(x_{n_k-1} - x_{m_k}) < \varepsilon. \tag{2.5}$$

From (2.4) and (2.5), we have

$$\varepsilon \leq M_V(x_{n_k} - x_{m_k}) \leq M_V(x_{n_k} - x_{n_k-1}) + M_V(x_{n_k-1} - x_{m_k}) < M_V(x_{n_k} - x_{n_k-1}) + \varepsilon.$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} M_V(x_{n_k} - x_{m_k}) = \varepsilon. \tag{2.6}$$

By using the triangular inequality we have

$$M_V(x_{n_k} - x_{m_k}) \leq M_V(x_{n_k} - x_{n_k-1}) + M_V(x_{n_k-1} - x_{m_k-1}) + M_V(x_{m_k-1} - x_{m_k})$$
$$M_V(x_{n_k-1} - x_{m_k-1}) \leq M_V(x_{n_k-1} - x_{n_k}) + M_V(x_{n_k} - x_{m_k}) + M_V(x_{m_k} - x_{m_k-1}).$$

Letting $k \to \infty$ in the above two inequalities and applying (2.6), we have

$$\lim_{k\to\infty} M_V(x_{n_k-1} - x_{m_k-1}) = \varepsilon.$$

Since

$$\psi(M_V(x_{n_k} - x_{m_k})) \leq \phi(M_V(x_{n_k-1} - x_{m_k-1})).$$

By using the condition (2) we know $\varepsilon = 0$, this is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and, since (X, τ) is a complete locally convex topological vector space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. From (2.1) and (2.2) we have

$$\psi^{-1}(M_U(x_n - Tz)) \leq \phi^{-1}(M_U(x_{n-1} - z)).$$

By using the condition (1) we get

$$M_U(x_n - Tz) \leq M_U(x_{n+1} - z),$$

so that $M_U(x_n - Tz) \to 0$, as $n \to +\infty$. Therefore

$$M_U(z - Tz) \leq M_U(x_n - z) + M_U(x_n - Tz) \to 0,$$

as $n \to +\infty$. This implies $M_U(z - Tz) = 0$. From the arbitrariness of $U \in \Omega$, we know that, $z - Tz = 0$. Hence z is a fixed point of T. Next we prove the uniqueness of the fixed point. Assume there exist two fixed point z and w. Then from (2.1) we have that

$$psi^{-1}(M_U(z, w)) = \psi^{-1}(M_U(Tz, Tw)) \leq \phi^{-1}(M_U(z, w)),$$

by using the condition (2) we know $M_U(z, w) = 0$, from the arbitrariness of $U \in \Omega$, we know that, $z = w$. This completes the proof. □

Corollary 2.11 (Contraction mapping principle and the error estimate formula)**.** *Let (X,τ) be a complete locally convex topological vector space with a basis of balanced convex neighborhood of zero Ω. Let $T : X \to X$ be a contractive mapping. Then*

(1). *T has a unique fixed point x^* and for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to this fixed point;*

(2). *the following error estimate formula holds: for any $U \in \Omega$, if take sufficiently large n such that $x_1 - x_0 \in \frac{1-h}{h^n} U$, then $x_n - x^* \in U$.*

Proof. By using Theorem 2.10, we know that, for any initial element $x_0 \in X$, the iterative sequence $x_n = T^n x_0$ converges to a unique fixed point x^*. We also have, for any $U \in \Omega$, that

$$\begin{aligned} M_U(x_n - x_{n+m}) &\leq M_U(x_n - x_{n+1}) + M_U(x_{n+1} - x_{n+2}) + \cdots + M_U(x_{n+m-1} - x_{n+m}) \\ &\leq (h^n + h^{n+1} + \cdots + h^{n+m}) M_U(Tx_0 - x_0) \\ &= \frac{h^n(1-h^{m-1})}{1-h} M_U(Tx_0 - x_0), \quad \forall\, n \geq 1,\ m \geq 1. \end{aligned}$$

Let $m \to \infty$, from above inequality, we get that

$$M_U(x_n - x^*) \leq \frac{h^n}{1-h} M_U(Tx_0 - x_0), \quad \forall\, n \geq 1. \tag{2.7}$$

Next, we write the following equivalent inequalities with (2.7):

$$(1-h) M_U(x_n - x^*) \leq h^n M_U(Tx_0 - x_0), \quad \forall\, n \geq 1.$$

$\Leftrightarrow$

$$(1-h)\inf\{t > 0 : x_n - x^* \in tU\} \leq h^n \inf\{t > 0 : Tx_0 - x_0 \in tU\}, \quad \forall\, n \geq 1.$$

$\Leftrightarrow$

$$\inf\{(1-h)t > 0 : x_n - x^* \in tU\} \leq \inf\{h^n t > 0 : Tx_0 - x_0 \in tU\}, \quad \forall\, n \geq 1.$$

$\Leftrightarrow$

$$\inf\{t > 0 : x_n - x^* \in \frac{t}{1-h} U\} \leq \inf\{t > 0 : Tx_0 - x_0 \in \frac{t}{h^n} U\}, \quad \forall\, n \geq 1.$$

$\Leftrightarrow$ (since U is is absorbing, balanced and convex)

$$\{t > 0 : x_n - x^* \in \frac{t}{1-h} U\} \supset \{t > 0 : Tx_0 - x_0 \in \frac{t}{h^n} U\}, \quad \forall\, n \geq 1. \tag{2.8}$$

From the (2.8), we know that

$$Tx_0 - x_0 \in \frac{t}{h^n} U \ \Rightarrow\ x_n - x^* \in \frac{t}{1-h} U \tag{2.9}$$

for $n \geq 1$ and $t > 0$. Let t be replaced by $(1-h)t$ in the (2.9), we get that

$$Tx_0 - x_0 \in \frac{1-h}{h^n} tU \ \Rightarrow\ x_n - x^* \in tU. \tag{2.10}$$

Let $t = 1$ in the (2.10), we get the following conclusion

$$Tx_0 - x_0 \in \frac{1-h}{h^n} U \ \Rightarrow\ x_n - x^* \in U$$

for $n \geq 1$. This completes the proof. □

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