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Dual synchronization of chaotic and hyperchaotic systems

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Abstract

The existence of the dual synchronization behavior between a pair of chaotic and hyperchaotic systems is investigated via a nonlinear controller, in which the nonlinear functions of the system are used as a nonlinear feedback term. The sufficient conditions for achieving the dual synchronization behavior between a pair of chaotic and hyperchaotic systems using a nonlinear feedback controller are derived by using the Lyapunov stability theorem. The dual synchronization behavior between a pair of chaotic systems (Chen and Lorenz system) and a pair of hyperchaotic systems hyperchaotic Chen system and hyperchaotic Lü system are taken as two illustrative examples to show the effectiveness of the proposed method. Theoretical analysis and numerical simulations are performed to verify the results.

Keywords: Dual synchronization, chaos, hyperchaos, lyapunov stability theory.
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1. Introduction

Several chaotic and hyperchaotic systems have been discovered and thoroughly analyzed over the past decades. These systems are interesting as its study links between the sciences and nature. Scientists who understand its existence have been struggling to control these systems to our benefit. There is a great need to control the chaotic and hyperchaotic systems, as they play an important role in industrial applications particularly in chemical reactions, biological systems, information processing and secure communications

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[1]. A very important aspect in chaos theory is the synchronization of chaotic systems. The concept of synchronization chaos is to make two chaotic systems oscillate in a synchronized manner by using the output of the drive system to control the response system so that the output of the response system follows the output of the drive system. Over the past decades, much attention has been devoted to the search for better and more efficient methods to synchronize chaotic and hyperchaotic systems. Up to now, various methods have been developed to design controllers in the chaotic and hyperchaotic systems, such as adaptive synchronization, active synchronization, linear and nonlinear feedback [3, 5, 9, 10, 16, 19, 21, 22] etc. However all of the aforementioned methods are mainly concerned with the synchronization of one drive system and one response system, so these methods cannot be applied for multiuser communication systems [20].

Recently, the concept of dual synchronization of two different pairs of chaotic dynamical systems has been investigated and used experimentally in communication applications. Dual synchronization of chaos is a technique to separate two mixed chaotic signals by using synchronization. In dual synchronization technique, there is a pair of response systems that must be synchronized with pairs of drive systems by using a signal generated through linear combination of the drive systems states. Dual synchronization in colpitts electronic oscillators is studied in [18]. Dual and cross dual synchronization of chaotic external cavity laser diodes is investigated in [15]. Experimental and numerical dual synchronization of chaos in two pairs of one-way coupled microchip lasers using only one transmission channel is studied in [20]. Dual synchronization of the Lorenz and Rössler systems is studied in [13], where the output signal from the drive systems is a scalar signal, constructed by a linear combination of their states. Dual synchronization in modulated time delayed systems is discussed in [7]. Projective-dual synchronization in delay dynamical systems with time-varying coupling delay is investigated in [6]. Dual synchronization of chaotic and hyperchaotic systems with fully uncertain parameters via Adaptive control method is discussed in [14].

To the best of our knowledge, there are few theoretical results about dual synchronization of chaotic systems, and on the other hand, all of the aforementioned methods [6, 7, 13, 15, 18] are mainly concerned with the dual synchronization of chaotic systems with low dimensional attractors characterized by one positive Lyapunov exponent and do not consist of the dual synchronization of hyperchaotic systems. This feature limits the complexity of the chaotic dynamics. It is believed that the chaotic systems with higher dimensional attractors have much wider applications. In this work, we investigate the existence of the dual synchronization behavior between a pair of chaotic and hyperchaotic systems via a nonlinear controller, in which the nonlinear functions of the system are used as a nonlinear feedback term. The sufficient conditions for achieving the dual synchronization behavior are derived by using the Lyapunov stability theorem. By this nonlinear feedback controller, one can synchronize a pair of chaotic and hyperchaotic systems effectively. The simulation results demonstrate that this control method is commendable, effective and feasible. The organization of the paper is as follows. In Section 2, the problem statement and dual synchronization scheme are presented for the chaotic and hyperchaotic systems. In Sections 3 and 4, numerical studies are performed to show the effectiveness of proposed method. Finally a concluding remark is given.

2. Problem statement

Consider a pair of chaotic system in the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1), \\ \dot{y}_1 &= g_1(y_1), \end{aligned} \tag{2.1}$$

where $x_1 = [x_{11}, x_{12}, \ldots, x_{1n}]^T$ and $y_1 = [y_{11}, y_{12}, \ldots, y_{1n}]^T$ are the state vectors of the two master systems, $f_1 \in C[R^n \times R^n, R^n]$ and $g_1 \in C[R^n \times R^n, R^n]$ are two known functions. The corresponding two slave systems are defined by

$$\begin{aligned} \dot{x}_2 &= f_2(x_2) + u_1, \\ \dot{y}_2 &= g_2(y_2) + u_2, \end{aligned} \tag{2.2}$$

where $x_2 = [x_{21}, x_{22}, \ldots, x_{2n}]^T$ and $y_2 = [y_{21}, y_{22}, \ldots, y_{2n}]^T$ are the state vectors of the two slave systems, $f_2 \in C[R^n \times R^n, R^n]$ and $g_2 \in C[R^n \times R^n, R^n]$ are two known functions and $u = (u_1^T \quad u_2^T)^T \in R^{2n}$, is a controller. Our goal is to design an appropriate controller $u = (u_1^T \quad u_2^T)^T$ such that the trajectory of the pair of the response system (2.2) could be synchronized with the pair of the drive system (2.1) where the errors between systems (2.1) and (2.2) should satisfy

$$\lim_{t\to\infty} \|x_2(t) - x_1(t)\| = 0, \quad \lim_{t\to\infty} \|y_2(t) - y_1(t)\| = 0, \tag{2.3}$$

where $\|\cdot\|$ is the Euclidean norm.

2.1. Dual Synchronization

System (2.1) can be rewritten in the form

$$\begin{bmatrix}\dot{x}_1\\ \dot{y}_1\end{bmatrix} = \begin{bmatrix}f_1(x_1)\\ g_1(y_1)\end{bmatrix}, \quad \dot{x} = f(x), \tag{2.4}$$

where $\dot{x} = \begin{bmatrix}\dot{x}_1\\ \dot{y}_1\end{bmatrix}$, $f(x) = \begin{bmatrix}f_1(x_1)\\ g_1(x_1)\end{bmatrix}$. Similarly, system (2.2) can be rewritten in the form

$$\begin{bmatrix}\dot{x}_2\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}f_2(x_2)\\ g_2(y_2)\end{bmatrix} + \begin{bmatrix}u_1\\ u_2\end{bmatrix}, \quad \dot{y} = g(y) + u, \tag{2.5}$$

where $\dot{y} = \begin{bmatrix}\dot{x}_2\\ \dot{y}_2\end{bmatrix}$, $g(y) = \begin{bmatrix}f_2(x_2)\\ g_2(x_2)\end{bmatrix}$, and $u = \begin{bmatrix}u_1\\ u_2\end{bmatrix}$. Let

$$\varepsilon_d = (a_1, a_2, ..., a_n, b_1, b_2, ..., b_n)(x_{11}, x_{12}, \ldots, x_{1n}, y_{11}, y_{12}, \ldots, y_{n1})^T = Cx$$

denote the linear coupling of the two drive systems, and

$$\varepsilon_r = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)(x_{21}, x_{22}, \ldots, x_{2n}, y_{21}, y_{21}, \ldots, y_{2n})^T = Cy$$

denote the linear coupling of the two response systems, let $A = (a_1, a_2, \ldots, a_n)^T$ and $B = (b_1, b_2, \ldots, b_n)^T$ be two known matrices such that $a_i, b_j, i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n$ cannot be zero at the same time. The error for dual synchronization is $e_s = Ce$, where $e = y - x$ and $C = diag(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$.

Theorem 2.1. *If the nonlinear feedback controller U is designed as*

$$U = -F(e, x) + ke_s, \tag{2.6}$$

then the response system (2.5) *can synchronize the drive system* (2.4) *asymptotically, where x is the state variable, e is the error of the state variable of the two systems, e_s is the linear coupling of the master and slave systems, and k is a feedback gain.*

Proof. The drive and the response systems (2.4) and (2.5) are split into linear terms $f_i(x), g_i(y)$ and nonlinear terms $f_j(x), g_j(y)$ where

$$\dot{x} = f_i(x) + f_j(x), \tag{2.7}$$
$$\dot{y} = g_i(y) + g_j(y) + u. \tag{2.8}$$

Hence, the error dynamics system can be written as

$$\dot{e} = g_i(y) + g_j(y) - f_i(x) - f_j(x) + u, \tag{2.9}$$

where $e = y - x$. The difference between the two linear terms $g_i(y), f_i(x)$ can be written as

$$g_i(y) - f_i(x) = Ae + f'(x), \tag{2.10}$$

where A is the coefficient matrix of the error system. Equation (2.9) and $f_i'(x)$ consist of residual terms. The difference between the two nonlinear terms $g_j(y) - f_j(x)$ is then written as

$$g_j(y) - f_j(x) = F(e,x) - f'(x). \tag{2.11}$$

Equation (2.9) becomes

$$\dot{e} = Ae + F(e,x) + u = Ae + ke_s. \tag{2.12}$$

Construct a Lyapunov function in the form

$$V = \frac{1}{2}e^T e. \tag{2.13}$$

Then its time derivative is

$$\dot{V} = e^T \dot{e}. \tag{2.14}$$

Inserting (2.12) into the time derivative of V leads to

$$\dot{V} = -e^T Pe \leq 0. \tag{2.15}$$

Since V is positive definite and $\dot{V}$ is negative definite in the neighborhood of zero solution of system (2.9), it follows that $\lim_{t\to\infty} \|e\| = 0$, based on the Lyapunov stability theorem [8]. Therefore, the response system (2.8) is synchronized with the drive system (2.7). This completes the proof. □

3. Dual synchronization of two chaotic systems

We define the master systems and slave systems as follows.

Master 1. Chen system [4] is given by

$$\begin{aligned}
\dot{x_1} &= \alpha(y_1 - x_1),\\
\dot{y_1} &= (\delta-\alpha)x_1 - x_1 z_1 + \delta y_1,\\
\dot{z_1} &= x_1 y_1 - \beta z_1.
\end{aligned} \tag{3.1}$$

Master 2. Lorenz system [12] is given by

$$\begin{aligned}
\dot{x_2} &= \sigma(y_2 - x_2),\\
\dot{y_2} &= \rho x_2 - x_2 z_2 - y_2,\\
\dot{z_2} &= x_2 y_2 - \gamma z_2.
\end{aligned} \tag{3.2}$$

So the corresponding slave systems are

Slave 1.

$$\begin{aligned}
\dot{x_3} &= \alpha(y_3 - x_3) + u_1,\\
\dot{y_3} &= (\delta-\alpha)x_3 - x_3 z_3 + \delta y_3 + u_2,\\
\dot{z_3} &= x_3 y_3 - \beta z_3 + u_3.
\end{aligned} \tag{3.3}$$

Slave 2.

$$\begin{aligned}
\dot{x_4} &= \sigma(y_4 - x_4) + u_4,\\
\dot{y_4} &= \rho x_4 - x_4 z_4 - y_4 + u_5,\\
\dot{z_4} &= x_4 y_4 - \gamma z_4 + u_6,
\end{aligned} \tag{3.4}$$

where $U = [u_1, u_2, u_3, u_4, u_5, u_6]^T$ is the controller function. Subtracting (3.1) from (3.3) and (3.2) from (3.4) yields the following error dynamical system:

$$\begin{aligned}
\dot{e_1} &= \alpha(e_2 - e_1) + u_1,\\
\dot{e_2} &= (\delta-\alpha)e_1 - e_1 e_3 - z_1 e_1 - x_1 e_3 + \delta e_2 + u_2,\\
\dot{e_3} &= e_1 e_2 + y_1 e_1 + x_1 e_2 - \beta e_3 + u_3,\\
\dot{e_4} &= \sigma(e_5 - e_4) + u_4,\\
\dot{e_5} &= \rho e_4 - e_5 - e_4 e_6 - z_2 e_4 - x_2 e_6 + u_5,\\
\dot{e_6} &= e_4 e_5 + y_2 e_4 + x_2 e_5 - \gamma e_6 + u_6,
\end{aligned} \tag{3.5}$$

where $e_1 = x_3 - x_1$, $e_2 = y_3 - y_1$, $e_3 = z_3 - z_1$, $e_4 = x_4 - x_2$, $e_5 = y_4 - y_2$, $e_6 = z_4 - z_2$. Our goal is to find proper control functions u_i $(i = 1, \ldots, 6)$, such that the pair of the master system equations (3.1) and (3.2) synchronizes the pair of the slave system equations (3.3) and (3.4) asymptotically, that is, $\lim_{t\to\infty} \|e\| = 0$, where $e = [e_1, \ldots, e_6]^T$. For this end, we propose the following corollary.

Corollary 3.1. *The pair of the master system equations* (3.1) *and* (3.2) *can be synchronized the pair of the slave system equations* (3.3) *and* (3.4) *asymptotically for any different initial condition with the following nonlinear controller.*

$$
\begin{aligned}
u_1 &= -\alpha e_2 + k_1 e,\\
u_2 &= -(\delta - \alpha)e_1 + e_1 e_3 + z_1 e_1 + x_1 e_3 - 2\delta e_2 + k_2 e,\\
u_3 &= -e_1 e_2 - y_1 e_1 - x_1 e_2 + k_3 e,\\
u_4 &= -\sigma e_5 + k_4 e,\\
u_5 &= -\rho e_4 + e_4 e_6 + z_2 e_4 + x_2 e_6 + k_5 e,\\
u_6 &= -e_4 e_5 - y_2 e_4 - x_2 e_5 + k_6 e,
\end{aligned}
\tag{3.6}
$$

where $e = a_1 e_1 + a_2 e_2 + a_3 e_3 + b_1 e_4 + b_2 e_5 + b_3 e_6$, is the linear coupling of the masters and slave systems.

Proof. Substituting (3.6) into (3.5) leads to the following error system

$$
\begin{aligned}
\dot{e_1} &= -\alpha e_1 + k_1 e,\\
\dot{e_2} &= -\delta e_2 + k_2 e,\\
\dot{e_3} &= -\beta e_3 + k_3 e,\\
\dot{e_4} &= -\sigma e_4 + k_4 e,\\
\dot{e_5} &= -e_5 + k_5 e,\\
\dot{e_6} &= -\gamma e_6 + k_6 e.
\end{aligned}
\tag{3.7}
$$

Construct a Lyapunov function in the form

$$V = \frac{1}{2} e^T e. \tag{3.8}$$

The time derivative of V along the solution of error dynamical system (3.7) gives

$$
\begin{aligned}
\dot{V} =& e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 + e_5\dot{e}_5 + e_6\dot{e}_6\\
=& e_1(-\alpha e_1 + k_1 e) + e_2(-\delta e_2 + k_2 e) + e_3(-\beta e_3 + k_3 e)\\
&+ e_4(-\sigma e_4 + k_4 e) + e_5(-e_5 + k_5 e) + e_6(-\gamma e_6 + k_6 e)\\
=& (k_1 a_1 - \alpha)e_1^2 + (k_1 a_2 + k_2 a_1)e_1 e_2 + (k_1 a_3 + k_3 a_1)e_1 e_3\\
&+ (k_1 b_1 + k_4 a_1)e_1 e_4 + (k_1 b_2 + k_5 a_1)e_1 e_5 + (k_1 b_3 + k_6 a_1)e_1 e_6\\
&+ (k_2 a_2 - \delta)e_2^2 + (k_2 a_3 + k_3 a_2)e_2 e_3 + (k_2 b_1 + k_4 a_2)e_2 e_4\\
&+ (k_2 b_2 + k_5 a_2)e_2 e_5 + (k_2 b_3 + k_6 a_2)e_2 e_6 + (k_3 a_3 - \beta)e_3^2\\
&+ (k_3 b_1 + k_4 a_3)e_3 e_4 + (k_3 b_2 + k_5 a_3)e_3 e_5 + (k_3 b_3 + k_6 a_3)e_3 e_6\\
&+ (k_4 b_1 - \sigma)e_4^2 + (k_4 b_2 + k_5 b_1)e_4 e_5 + (k_4 b_3 + k_6 b_1)e_4 e_6\\
&+ (k_5 b_2 - 1)e_5^2 + (k_5 b_3 + k_6 b_2)e_5 e_6 + (k_6 b_3 - \gamma)e_6^2\\
=& - e^T P e,
\end{aligned}
\tag{3.9}
$$

where $e = [|e_1|, |e_2|, |e_3|, |e_4|, |e_5|, |e_6|]$ and P is real symmetric. Obviously, P should be positive definite to ensure that the origin of error system (3.5) is asymptotically stable. According to Sylvester's theorem [17], P is positive definite if and only if $\Delta_i > 0, i = 1, 2, \ldots, 6$, where Δ_i represents the ith order sequential subdeterminant of a matrix. That is, we should choose the appropriate parameters. This completes the proof. □

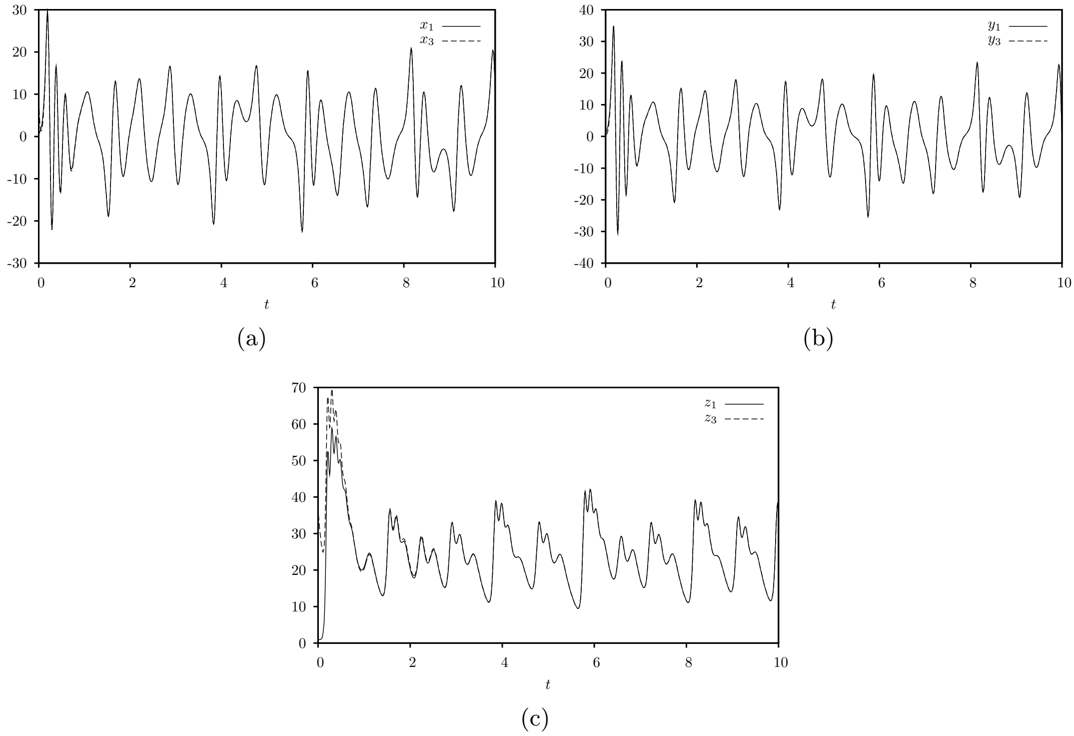


Figure 1: State trajectories between the pair of Chen systems (3.1) and (3.3), **(a)** signals x_1 and x_3; **(b)** signals y_1 and y_3; **(c)** signals z_1 and z_3.

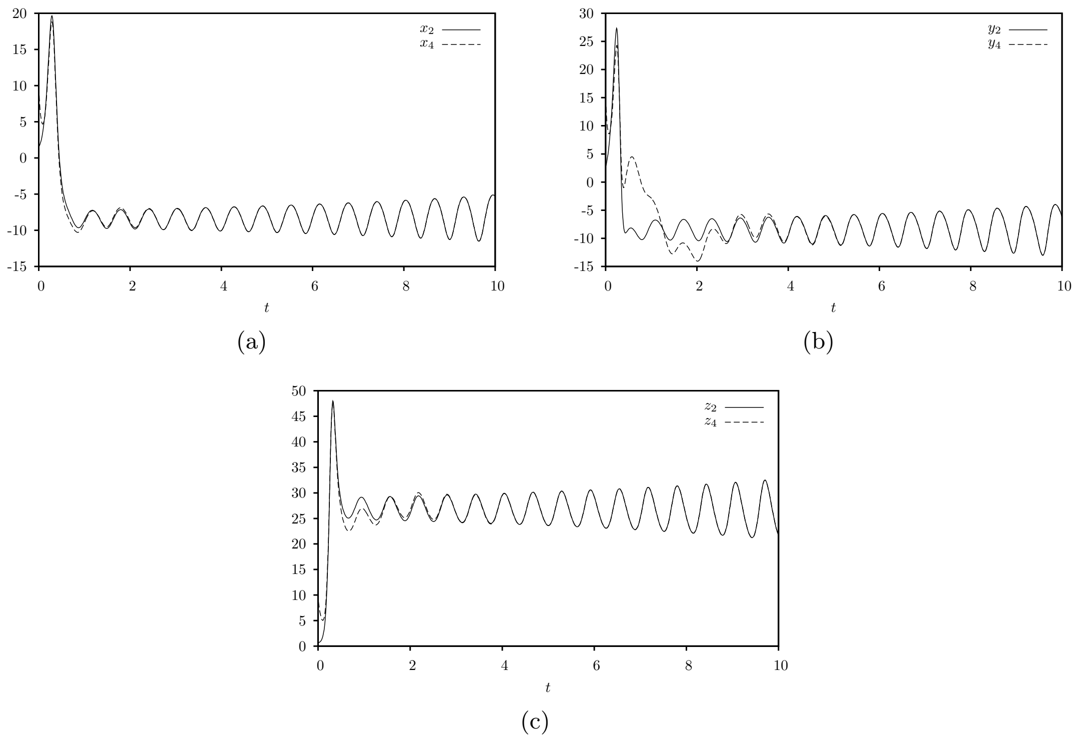
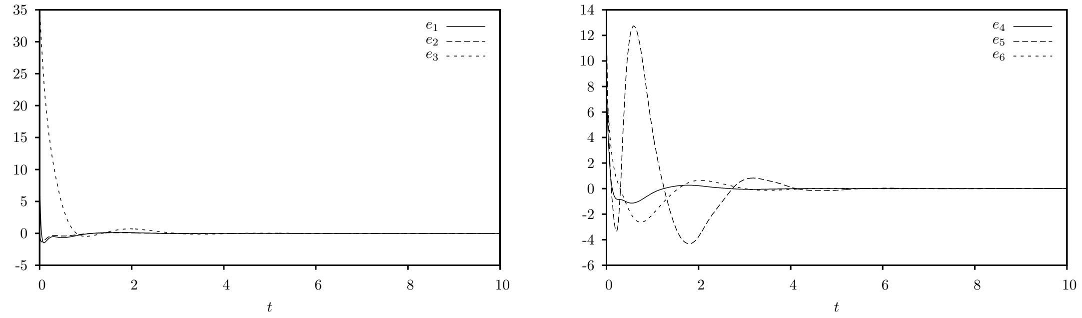


Figure 2: State trajectories between the pair of Lorenz systems (3.2) and (3.4), **(a)** signals x_2 and x_4; **(b)** signals y_2 and y_4; **(c)** signals z_2 and z_4.



(a) (b)

Figure 3: **(a)**The error signals e_1, e_2, e_3 between the pair of Chen systems; **(b)**The error signals e_4, e_5, e_6 between the pair of Lorenz systems.

3.1. Numerical simulations

The problem of dual synchronization of Chen system and Lorenz system is simulated. The system parameters are set to $\alpha = 35, \delta = 28$ and $\beta = 3$ for the pair of Chen systems and $\sigma = 10, \gamma = 8/3$ and $\rho = 28$ for the pair of Lorenz system, so both systems exhibit chaotic behavior. In addition, the coupled parameters are valued as $a_i = (1,1,1), b_i = (1,1,1), i = 1,2,3$ and $k_i = (-2), i = 1,\ldots,6$, so that the condition P is positive definite. The initial conditions of the master systems (3.1) and (3.2) are taken as $x_1(0) = 0.5, y_1(0) = 1, z_1(0) = 1, x_2(0) = 1.5$ and $y_2(0) = 2.5, z_2(0) = 0.65$. The initial conditions of the slave systems (3.3) and (3.4) are taken as $x_3(0) = 10.5, y_3(0) = 1, z_3(0) = 37$ and $x_4(0) = 10, y_4(0) = 15.5, z_4(0) = 9.65$, so the initial conditions of the error system are set to be $e_1(0) = 10, e_2(0) = 0, e_3(0) = 36$ and $e_4(0) = 8.5, e_5(0) = 13, e_6(0) = 9$. Dual synchronizations of Chen system and Lorenz system are shown in Figurs 1, 2 and 3. Figure 1 (**a**)–(**c**) show the state trajectories of pair of Chen systems (3.1) and (3.3). Figure 2 (**a**)–(**c**) show the state trajectories of pair of Lorenz systems (3.2) and (3.4). Figure 3 (**a**)–(**b**) show the error e_1, e_2, e_3 and e_4, e_5, e_6 between the pair of Chen systems and the pair of Lorenz systems, respectively.

4. Dual synchronization of two hyperchaotic systems

We define the master and slave systems as follows:

Master 1. Hyperchaotic Chen system [11] is given by

$$\begin{aligned}
\dot{x_1} &= \alpha(y_1 - x_1) + w_1,\\
\dot{y_1} &= \delta x_1 - x_1 z_1 + \theta y_1,\\
\dot{z_1} &= x_1 y_1 - \beta z_1,\\
\dot{w_1} &= y_1 z_1 + \varrho w_1.
\end{aligned} \tag{4.1}$$

Master 2. Hyperchaotic Lü system [2] is given by

$$\begin{aligned}
\dot{x_2} &= \alpha_1(y_2 - x_2) + w_2,\\
\dot{y_2} &= -x_2 z_2 + \theta_1 y_2,\\
\dot{z_2} &= x_2 y_2 - \beta_1 z_2,\\
\dot{w_2} &= x_2 z_2 + \varrho_1 w_2.
\end{aligned} \tag{4.2}$$

So, the corresponding slave systems are:

Slave 1.

$$\begin{aligned} \dot{x_3} &= \alpha(y_3 - x_3) + w_3 + u_1, \\ \dot{y_3} &= \delta x_3 - x_3 z_3 + \theta y_3 + u_2, \\ \dot{z_3} &= x_3 y_3 - \beta z_3 + u_3, \\ \dot{w_3} &= y_3 z_3 + \varrho w_3 + u_4. \end{aligned} \tag{4.3}$$

Slave 2.

$$\begin{aligned} \dot{x_4} &= \alpha_1(y_4 - x_4) + w_4 + u_5, \\ \dot{y_4} &= -x_4 z_4 + \theta_1 y_4 + u_6, \\ \dot{z_4} &= x_4 y_4 - \beta_1 z_4 + u_7, \\ \dot{w_4} &= x_4 z_4 + \varrho_1 w_4 + u_8, \end{aligned} \tag{4.4}$$

where $U = [u_1, \ldots, u_8]^T$ is the controller function. Subtracting (4.3) from (4.1), and (4.4) from (4.2), yields the following error dynamical system:

$$\begin{aligned} \dot{e_1} &= \alpha(e_2 - e_1) + e_4 + u_1, \\ \dot{e_2} &= \delta e_1 - e_1 e_3 - z_1 e_1 - x_1 e_3 + \theta e_2 + u_2, \\ \dot{e_3} &= e_1 e_2 + y_1 e_1 + x_1 e_2 - \beta e_3 + u_3, \\ \dot{e_4} &= z_1 e_2 + y_1 e_3 + e_2 e_3 + \varrho e_4 + u_4, \\ \dot{e_5} &= \alpha_1(e_6 - e_5) + e_8 + u_5, \\ \dot{e_6} &= -x_2 e_7 - z_2 e_5 - e_5 e_7 + \theta_1 e_6 + u_6, \\ \dot{e_7} &= y_2 e_5 + x_2 e_6 + e_5 e_6 - \beta_1 e_7 + u_7, \\ \dot{e_8} &= z_2 e_5 + x_2 e_7 + e_5 e_7 + \varrho_1 e_8 + u_8, \end{aligned} \tag{4.5}$$

where $e_1 = x_3 - x_1$, $e_2 = y_3 - y_1$, $e_3 = z_3 - z_1$, $e_4 = w_3 - w_1$, $e_5 = x_4 - x_2$, $e_6 = y_4 - y_2$, $e_7 = z_4 - z_2$, $e_8 = w_4 - w_2$. Our goal is to find proper control functions u_i $(i = 1, \ldots, 8)$, such that the pair of the master systems (4.1) and (4.2) synchronizes the pair of the slave systems (4.3) and (4.4) asymptotically, that is, $\lim_{t\to\infty} \|e\| = 0$, where $e = [e_1, \ldots, e_8]^T$.

For this end, we propose the following corollary.

Corollary 4.1. *The pair of the master system equations* (4.1) *and* (4.2) *can be synchronized the pair of the slave system equations* (4.3) *and* (4.4) *asymptotically for any different initial condition with following nonlinear controller.*

$$\begin{aligned} u_1 &= -\alpha e_2 - e_4 + k_1 e, \\ u_2 &= -\delta e_1 + e_1 e_3 + z_1 e_1 + x_1 e_3 - 2\theta e_2 + k_2 e, \\ u_3 &= -e_1 e_2 - y_1 e_1 - x_1 e_2 + k_3 e, \\ u_4 &= -z_1 e_2 - y_1 e_3 - e_2 e_3 - 2\varrho e_4 + k_5 e, \\ u_5 &= -\alpha_1 e_6 - e_8 + k_5 e, \\ u_6 &= x_2 e_7 + z_2 e_5 + e_5 e_7 - 2\theta_1 e_6 + k_6 e, \\ u_7 &= -y_2 e_5 - x_2 e_6 - e_5 e_6 + k_7 e, \\ u_8 &= -z_2 e_5 - x_2 e_7 - e_5 e_7 - 2\varrho_1 e_8 + k_8 e, \end{aligned} \tag{4.6}$$

where $e = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + b_1 e_5 + b_2 e_6 + b_3 e_7 + b_4 e_8$ is the linear coupling of the masters and slave systems.

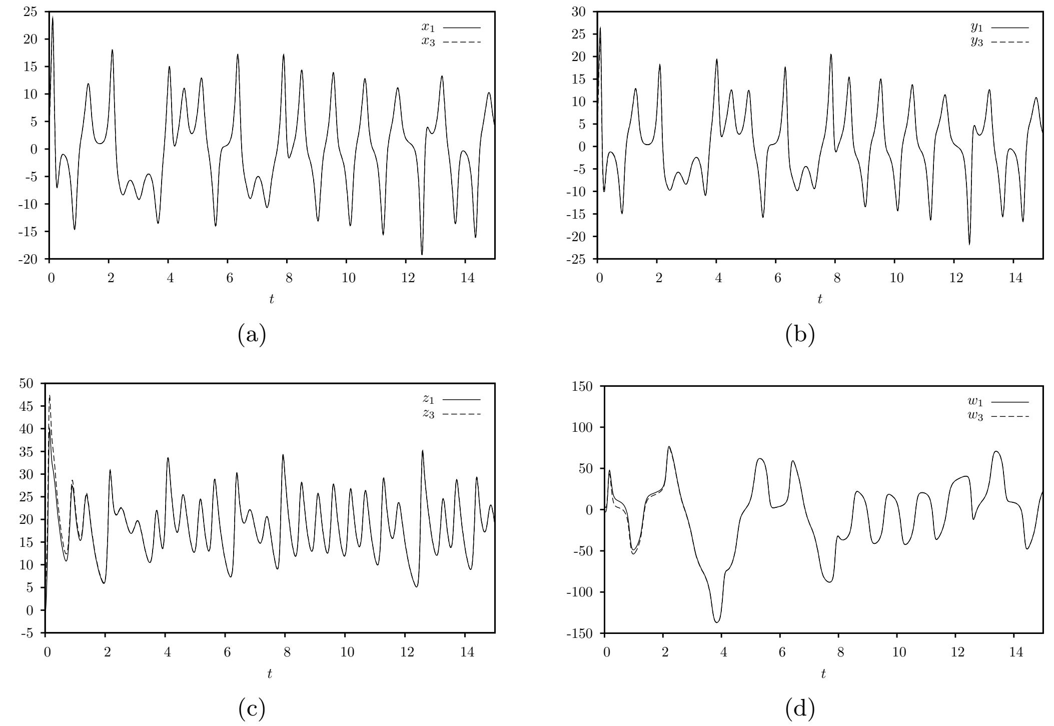


Figure 4: State trajectories between the pair of hyperchaotic Chen systems (4.1) and (4.3), **(a)** signals x_1 and x_3; **(b)** signals y_1 and y_3; **(c)** signals z_1 and z_3; **(d)** signals w_1 and w_3.

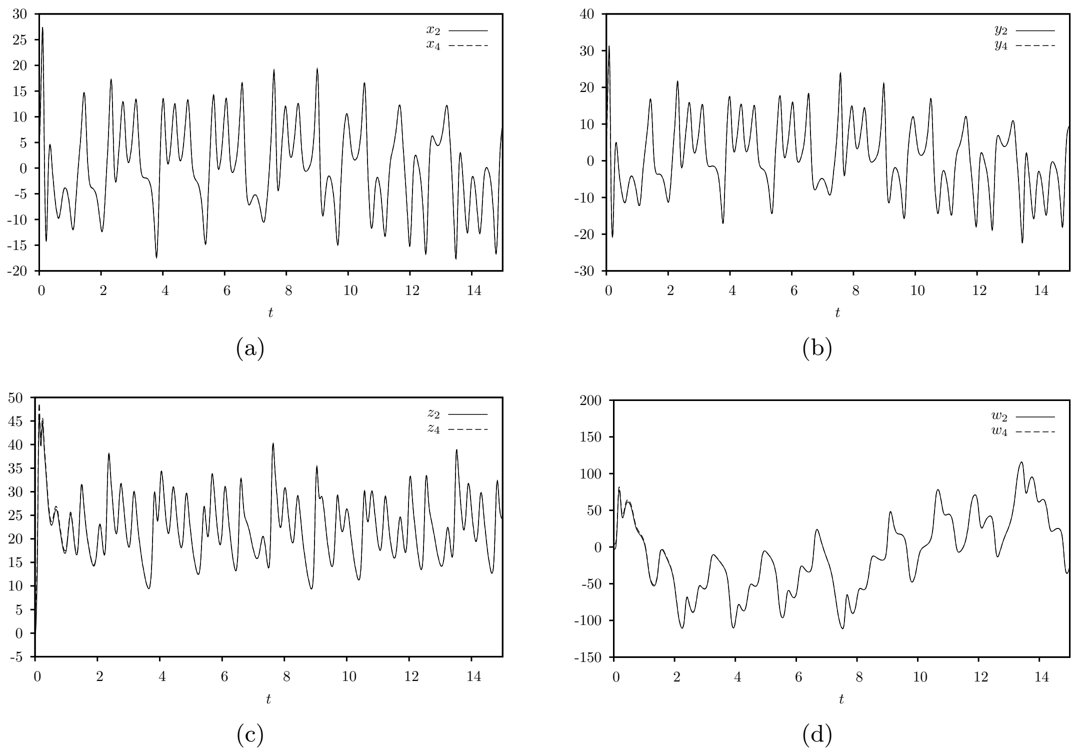
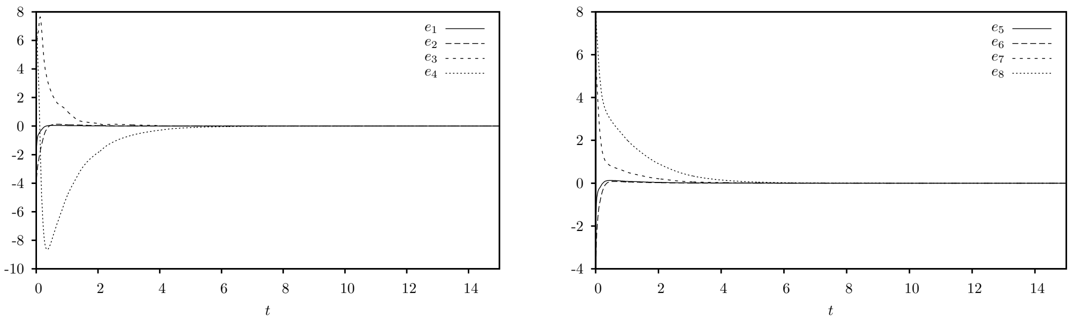


Figure 5: State trajectories between the pair of hyperchaotic Lü systems (4.2) and (4.4), **(a)** signals x_2 and x_4; **(b)** signals y_2 and y_4; **(c)** signals z_2 and z_4; **(d)** signals w_2 and w_4.



(a) (b)

Figure 6: **(a)** The error signals e_1, e_2, e_3, e_4 between the pair of hyperchaotic Chen systems; **(b)** The error signals e_5, e_6, e_7, e_8 between the pair of hyperchaotic Lü systems.

Proof. Substituting (4.6) into (3.5) leads to the following error system

$$
\begin{aligned}
\dot{e_1} &= -\alpha e_1 + k_1 e,\\
\dot{e_2} &= -\theta e_2 + k_2 e,\\
\dot{e_3} &= -\beta e_3 + k_3 e,\\
\dot{e_4} &= -\varrho e_4 + k_4 e,\\
\dot{e_5} &= -\alpha_1 e_5 + k_5 e,\\
\dot{e_6} &= -\theta_1 e_6 + k_6 e,\\
\dot{e_7} &= -\beta_1 e_7 + k_7 e,\\
\dot{e_8} &= -\varrho_1 e_8 + k_8 e.
\end{aligned}
\tag{4.7}
$$

Construct a Lyapunov function in the form

$$V = \frac{1}{2} e^T e. \tag{4.8}$$

The time derivative of V along the solution of error dynamical system (4.7) gives

$$
\begin{aligned}
\dot{V} =& e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 + e_5\dot{e}_5 + e_6\dot{e}_6 + e_7\dot{e}_7 + e_8\dot{e}_8\\
=&(a_1k_1 - \alpha)e_1^2 + (a_2k_1 + a_1k_2)e_1e_2 + (a_3k_1 + a_1k_3)e_1e_3 + (a_4k_1 + a_1k_4)e_1e_4\\
&+ (b_1k_1 + a_1k_5)e_1e_5 + (b_2k_1 + a_1k_6)e_1e_6 + (b_3k_1 + a_1k_7)e_1e_7 + (b_4k_1 + a_1k_8)e_1e_8\\
&+ (a_2k_2 - \theta)e_2^2 + (a_3k_2 + a_2k_3)e_2e_3 + (a_4k_2 + a_2k_4)e_2e_4 + (b_1k_2 + a_2k_4)e_2e_5\\
&+ (b_2k_2 + a_2k_6)e_2e_6 + (b_3k_2 + a_2k_7)e_2e_7 + (b_4k_2 + a_2k_8)e_2e_8 + (a_3k_3 - \beta)e_3^2\\
&+ (a_4k_3 + a_3k_4)e_3e_4 + (b_1k_3 + a_3k_5)e_3e_5 + (b_2k_3 + a_3k_6)e_3e_6 + (b_3k_3 + a_3k_7)e_3e_7\\
&+ (b_4k_3 + a_3k_8)e_3e_8 + (a_4k_4 - \varrho)e_4^2 + (b_1k_4 + a_4k_5)e_4e_5 + (b_2k_4 + a_4k_6)e_4e_6\\
&+ (b_3k_4 + a_4k_7)e_4e_7 + (b_4k_4 + a_4k_8)e_4e_8 + (b_1k_5 - \alpha_1)e_5^2 + (b_2k_5 + b_1k_6)e_5e_6\\
&+ (b_3k_5 + b_1k_7)e_5e_7 + (b_4k_5 + b_1k_8)e_5e_8 + (b_2k_6 - \theta_1)e_6^2 + (b_3k_6 + b_2k_7)e_6e_7\\
&+ (b_4k_6 + b_2k_8)e_6e_8 + (b_3k_7 - \beta_1)e_7^2 + (b_3k_8 + b_4k_7)e_7e_8 + (b_4k_8 - \varrho_1)e_8^2\\
=& - e^T P e,
\end{aligned}
\tag{4.9}
$$

where $e = [|e_1|, |e_2|, |e_3|, |e_4|, |e_5|, |e_6|, |e_7|, |e_8|]$ and P is real symmetric. Obviously, P should be positive definite to ensure that the origin of error system (4.5) is asymptotically stable. According to Sylvester's theorem [17], P is positive definite if and only if $\Delta_i > 0, i = 1, 2, ..., 8$, where Δ_i represents the ith order sequential subdeterminant of matrix. That is, we should choose the appropriate parameters. This completes the proof. □

4.1. Numerical simulations

The dual synchronization problem of the hyperchaotic Chen system and hyperchaotic Lü system is simulated. The system parameters are set to $\alpha = 35, \theta = 12$, $\beta = 3$, $\delta = 7$ and $\varrho = 0.5$ for the pair of the hyperchaotic Chen systems and $\alpha_1 = 36, \theta_1 = 20$, $\beta_1 = 3$ and $\varrho_1 = 1.3$ for the pair of hyperchaotic Lorenz system, so both systems exhibits hyperchaotic behavior. In addition, the coupled parameters are valued as $a_i = (1,1,1,1), b_i = (1,1,1,1), i = 1,2,3,4$ and $k_i = (-2), i = 1,...,8$ so that the condition P is positive definite. The initial conditions of the master system (3.1) and the master system (3.2) are taken as $x_1(0) = 5, y_1(0) = 8, z_1(0) = -1, w_1(0) = -3$, and $x_2(0) = 5, y_2(0) = 8, z_2(0) = -1, w_2(0) = -3$, the initial conditions of the slave system (3.3) and the slave system (3.4) are taken as $x_3(0) = 3, y_3(0) = 4, z_3(0) = 5, w_3(0) = 5$ and $x_4(0) = 3, y_4(0) = 4, z_4(0) = 5, w_4(0) = 5$, so the initial conditions of the error system are set to be $e_1(0) = -2, e_2(0) = -4, e_3(0) = 6, e_4(0) = 8, e_5(0) = -2, e_6(0) = -4, e_7(0) = 6, e_8(0) = 8$. Dual synchronization of pair hyperchaotic Chen system and pair hyperchaotic Lü system are shown in Figures 4, 5 and 6. Figure 4 (**a**)–(**d**) show the state trajectories of pair of hyperchaotic Chen systems (4.1) and (4.3). Figure 5 (**a**)–(**d**) show the state trajectories of pair of hyperchaotic Lü systems (4.2) and (4.4). Figure 6 (**a**)–(**b**) show the errors e_1, e_2, e_3, e_4 and e_5, e_6, e_7, e_8 between the pair of the hyperchaotic Chen systems and the pair of the hyperchaotic Lü systems, respectively.

5. Concluding remark

We investigate the dual synchronization behavior of a pair of chaotic systems and extend the dual synchronization behavior for a pair of hyperchaotic systems. We proposed a novel nonlinear feedback control scheme for chaos and hyperchaos dual synchronization according to the Lyapunov method. The dual synchronization behavior between a pair of chaotic systems (Chen and Lorenz systems) and a pair of hyperchaotic systems (hyperchaotic Chen and hyperchaotic Lü systems) are illustrated by two examples to show the effectiveness of the proposed method. Theoretical analysis and numerical simulations verified the results.

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