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Lag synchronization of complex dynamical networks with hybrid coupling via adaptive pinning control

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Abstract

In this paper, the problem of the lag synchronization between two general complex dynamical networks with mixed coupling by pinning control is studied. Based on the Lyaponov functional theory and mathematical analysis method, less conservative conditions of lag synchronization are obtained by adding the controllers to part of nodes. Moreover, the coupling configuration matrices are not required to be symmetric or irreducible. It is shown that the lag synchronization of the drive and response systems can be realized via the linear feedback pinning control and adaptive feedback pinning control. These results remove some restrictions on the node dynamics and the number of the pinned nodes. Numerical examples are presented to illustrate the effectiveness of the theoretical results.

Keywords: Lag synchronization, complex dynamical systems, pinning control, hybrid time-varying coupling.
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1. Introduction

A complex dynamical network is a set of coupled nodes interconnected by edges, in which each node represents a dynamical system. Many real systems in nature can be described as complex dynamical networks such as social organizations, Internet, communication networks, food webs, disease transmission networks, the World Wide Web, power grids, and so on [1, 30, 31]. This has led to much interest to the studies of the complex networks. In particular, with the wide applications of the complex networks in fields of neural networks [27], biological systems [23], information science [14], and secure communication [3, 19],

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synchronization of the complex network has become an important topic due to its realistic significance and study value.

In recent years, synchronization and its control of complex dynamical networks have been widely studied. Many synchronization methods have been proposed including linear state feedback control [25], pinning control [8, 16, 33, 37, 38], state observer based control [11, 32], impulsive control [20, 39], and adaptive control [9] and so on. However, most of them focus on the inner synchronization, in which all nodes in a network achieve a coherent behavior. Different from the inner synchronization [12], there is another kind of synchronization namely outer synchronization [28, 36], which has quickly caught much attention since Li first proposed in 2007 [18]. In general, there are several kinds of synchronization, such as, complete synchronization [34], phase synchronization [2], lag synchronization [21], generalized synchronization [24], and projective synchronization [13]. Among them, lag synchronization, which requires the states of response system to synchronize with the past states of the drive system, has been widely observed in many practical systems like electronic circuits, lasers and neural systems [35]. It has been proved to be a reasonable scheme from the viewpoint of engineering applications and the characteristics of channel in secure communication, parallel image processing, and pattern storage [17]. Therefore, lag synchronization has become a hot topic in many fields [10, 22, 29]. For example, in [6], the author investigated the issue of the lag synchronization between two coupled networks by adding the controllers to part of nodes. Zhao et al. [40] considered the lag synchronization problem of two different complex networks based on the approach of state observer.

However, although the approach realized the lag synchronization for complex dynamical networks, there are still some problems which need to be studied. These include: (1) the coupling configuration matrices are always assumed to be irreducible and their off-diagonal entries are nonnegative, and the inner connecting matrices are diagonal positive define; (2) it is very expensive and even impractical to apply the controllers to all or many nodes, especially for the engineering applications. For this reason, as described in [38], to achieve low cost and easy implementation, it is significant to investigate how the drive and response networks are synchronized by pinning only a small portion of nodes in a network; (3) in a real network, since the speed of signal travel between nodes is limited and the network nodes may be required to have non-local interconnections like telecommunications [15, 41], the discrete delay coupling and distributed time coupling are inevitable in the network. Thus, the synchronization of complex networks with delayed coupling including discrete and distributed delay coupling should be considered. Sufficient conditions for adaptive lag synchronization of complex dynamical network with discrete delayed coupling have been provided in [10]. To the best of our knowledge, up to now, there has been no literature concerning the problems of lag synchronization for complex dynamical networks with mixed coupling.

Inspired by the above mentioned discussions, in this paper, a lag synchronization method between two general complex dynamical networks with hybrid coupling by pinning control a small portion of nodes of the network has been proposed. The main contributions of this paper are listed as follows: first, the hybrid coupling, which is made up of non-delay coupling, discrete delay coupling and distributed delay coupling is considered; second, by applying the Lyaponov functional theory and mathematical analysis method, sufficient verifiable conditions are constructed for the lag synchronization of the drive and response networks. These results are less conservative and easy to verify through the numerical simulation. Moreover, the coupling matrices are not necessary to be symmetric and irreducible, and without assuming diagonal or positive define of the inner linking matrices; third, in numerical simulation section, we verify that pinning only one node can realize lag synchronization of the networks adequately and the node can be chosen according to the high-degree of vertex or the maximum norm of synchronization error. The rest of this paper is organized as follows: in Section 2, the complex dynamical network is introduced and some related definitions and lemmas are given; then in Section 3, the linear feedback pinning control and the adaptive feedback pinning control are designed and the corresponding lag synchronization theorems are derived respectively; in Section 4, two illustrative examples are provided to examine the effectiveness of the theoretical results; finally Section 5 concludes this paper.

From now on, throughout this paper, I_n denotes an n-dimensional identity matrix, $\Re^n$ indicates the n-dimensional Euclidean space and $\Re^{n\times n}$ is the set of all $n \times n$ real matrices. For symmetric matrices X

and Y, the notation $X > Y (X \geq Y)$ means that the matrix $X - Y$ is positive definite (nonnegative). The symbol $diag\{\ldots\}$ denotes the block diagonal matrix. For a real symmetric matrix P, $\lambda_{min}(P)$ and $\lambda_{max}(P)$ denote the minimum and maximum eigenvalues of P. Besides, $\|\cdot\|$ and $|\cdot|$ indicate the Euclidean vector norm and the absolute value, respectively. The superscript T denotes matrix or vector transposition. The symmetric terms in a symmetric matrix are denoted by $*$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2. Problem formulation and preliminaries

Consider the following complex dynamical networks with hybrid time-varying delays coupling:

$$\dot{x}_i(t) = f(x_i(t), x_i(t-\sigma(t))) + \sum_{j=1}^{N} \bar{c}_{ij}^{(1)} \Gamma_1 x_j(t) + \sum_{j=1}^{N} \bar{c}_{ij}^{(2)} \Gamma_2 x_j(t-\sigma(t)) + \sum_{j=1}^{N} \bar{c}_{ij}^{(3)} \Gamma_3 \int_{t-d(t)}^{t} x_j(s)ds, \quad i = 1, 2, \ldots, N, \tag{2.1}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^T \in \Re^n$ stands for the drive state of the ith node, $f : \Re^n \times \Re^n \to \Re^n$ is a continuous nonlinear vector-valued function, $\sigma(t)$ is discrete time-varying delay, and $d(t)$ is distributed time-varying delay. Here, Γ_1, Γ_2 and $\Gamma_3 \in \Re^n \times \Re^n$ represent the inner connecting matrix, the discrete-delay inner connecting matrix and the distributed-delay inner connecting matrix, respectively; $\bar{C}^{(k)} = (\bar{c}_{ij}^{(k)}) \in \Re^{N\times N}$, $k = 1, 2, 3$, represent the coupling configuration of the drive networks and satisfy the diffusive coupling connections:

$$\bar{c}_{ii}^{(k)} = -\sum_{j=1, j\neq i} \bar{c}_{ij}^{(k)}, \quad i = 1, 2, \ldots, N, \quad k = 1, 2, 3, \tag{2.2}$$

where $\bar{c}_{ij}^{(1)}$ are defined as follows: $\bar{c}_{ij}^{(1)} \geq 0$ for $j \neq i$, that is, $\bar{C}^{(1)}$ is nonnegative diffusive.

Remark 2.1. In this paper, the coupling configuration matrices are not required to be identical, symmetric or irreducible. Moreover, different from [6, 10, 40], in our paper the non-delayed inner connecting matrix, the discrete-delay inner connecting matrix and the distributed-delay inner connecting matrix are arbitrary real matrices.

Throughout this paper, we make the following assumptions on time-varying delays and nonlinear function f.

Assumption 2.2. $0 \leq \sigma(t) \leq \sigma$, $0 \leq d(t) \leq d$, and $\dot{\sigma}(t) \leq \bar{\sigma} < 1$, $\dot{d}(t) \leq \mu < 1$, where σ, d, $\bar{\sigma}$ and μ are constants.

Assumption 2.3. The nonlinear function f satisfies uniform semi-Lipschitz condition, that is, there exists positive constants α_1 and α_2 such that

$$(x - y)^T (f(x, \tilde{x}) - f(y, \tilde{y})) \leq \alpha_1 (x - y)^T (x - y) + \alpha_2 (\tilde{x} - \tilde{y})^T (\tilde{x} - \tilde{y}) \tag{2.3}$$

for any $x \in \Re^n$, $y \in \Re^n$, $\tilde{x} \in \Re^n$, $\tilde{y} \in \Re^n$.

It has been verified that many typical benchmark chaotic systems such as the Lorenz system, Chua's system and the unified chaotic system satisfy Assumption 2.3. Correspondingly, the response system is designed by

$$\begin{aligned}\dot{y}_i(t) =& f(y_i(t), y_i(t-\sigma(t))) + \sum_{j=1}^{N} \bar{c}_{ij}^{(1)} \Gamma_1 y_j(t) + \sum_{j=1}^{N} \bar{c}_{ij}^{(2)} \Gamma_2 y_j(t-\sigma(t)) + \sum_{j=1}^{N} \bar{c}_{ij}^{(3)} \Gamma_3 \int_{t-d(t)}^{t} y_j(s)ds \\ &+ u_i, \quad i = 1, 2, \ldots, N,\end{aligned} \tag{2.4}$$

where $y_i(t) = [y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t)]^T \in \Re^n$ is the response state of the ith node, $u_i (i = 1, 2, \ldots, N)$ are the controllers to be designed later, and other notations are the same as above. The following definition and lemmas are useful in deriving our main results:

Definition 2.4 ([6])**.** The drive system (2.1) is said to be a lag synchronization with the response system (2.4) at time τ if satisfies the following property:

$$\lim_{t\to\infty} \|y_i(t) - x_i(t-\tau)\| = 0, \quad i = 1, 2, \dots, N, \tag{2.5}$$

where τ is a given positive time delay.

Lemma 2.5 ([5])**.** *For any constant matrix $W \in \Re^{n\times n}$, $W^T = W > 0$, scalar $d > 0$, and vector function $\omega : [0, d] \to \Re^n$ such that the integrations concerned are well defined, then*

$$d\int_0^d \omega^T(s)W\omega(s)ds \geq (\int_0^d \omega(s)ds)^T W(\int_0^d \omega(s)ds).$$

Lemma 2.6 ([16])**.** *For an $n \times n$ matrix A, the following inequality holds:*

$$AA^T \leq \|A\|^2 I.$$

Lemma 2.7 ([7])**.** *Assume that A and B are $n \times n$ Hermitian matrices. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, and $\varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_n$ be eigenvalues of A, B and $A + B$, respectively. Then one has $\lambda_i + \mu_n \leq \varepsilon_i \leq \lambda_i + \mu_1$, $i = 1, 2, \dots, n$.*

Lemma 2.8 ([26])**.** *Assume that $Q = (q_{ij})_{N\times N}$ is symmetric. Let*

$$D = diag(d_1, d_2, \dots, d_m, 0, 0, \dots, 0), \quad Q - D = \begin{pmatrix} Q_{11} - D^* & Q_{12} \\ Q_{12}^T & Q_m \end{pmatrix}, and \quad d = \min_{1\leq i\leq m} d_i,$$

where $1 \leq m \leq N$, $d_i > 0$, $i = 1, 2, \dots, m$, Q_m is the minor matrix of Q by removing its first row-column pairs, Q_{11} and Q_{12} are matrices with appropriate dimensions and $D^* = diag(d_1, d_2, \dots, d_m)$. When $d > \lambda_{max}(Q_{11} - Q_{12}Q_m^{-1}Q_{12}^T)$, $Q - D < 0$ is equivalent to $Q_m < 0$.

3. Main results

3.1. Lag synchronization via the linear feedback pinning control

In this subsection, we use the linear feedback control to pin the lag synchronization. Without loss of generality, we assume that the first m $(1 \leq m \leq N)$ nodes are selected and pinned with the linear controllers, which are described as

$$\begin{cases} u_i = -\gamma_1 k_i e_i(t), & 1 \leq i \leq m, \\ u_i = 0, & 1 + m \leq i \leq N, \end{cases} \tag{3.1}$$

where $\gamma_1 = ||\Gamma_1||$, $e_i(t) = y_i(t) - x_i(t-\tau)$, $k_i(i = 1, 2, \dots, m) > 0$ are feedback gains.

According to (3.1), we obtain the following lag synchronization error system,

$$\begin{cases} \dot{e}_i(t) = f(y_i(t), y_i(t-\sigma(t))) - f(x_i(t-\tau), x_i(t-\sigma(t)-\tau)) + \sum_{j=1}^N \bar{c}_{ij}^{(1)}\Gamma_1 e_j(t) + \sum_{j=1}^N \bar{c}_{ij}^{(2)}\Gamma_2 e_j(t-\sigma(t)) \\ \qquad + \sum_{j=1}^N \bar{c}_{ij}^{(3)}\Gamma_3 \int_{t-d(t)}^t e_j(s)ds - \gamma_1 k_i e_i(t), \quad 1 \leq i \leq m, \\ \dot{e}_i(t) = f(y_i(t), y_i(t-\sigma(t))) - f(x_i(t-\tau), x_i(t-\sigma(t)-\tau)) + \sum_{j=1}^N \bar{c}_{ij}^{(1)}\Gamma_1 e_j(t) + \sum_{j=1}^N \bar{c}_{ij}^{(2)}\Gamma_2 e_j(t-\sigma(t)) \\ \qquad + \sum_{j=1}^N \bar{c}_{ij}^{(3)}\Gamma_3 \int_{t-d(t)}^t e_j(s)ds, \quad m+1 \leq i \leq N. \end{cases} \tag{3.2}$$

Let $||\Gamma_2|| = \gamma_2$, $||\Gamma_3|| = \gamma_3$, $|\bar{C}^{(k)}| = (|\bar{c}_{ij}^{(k)}|)_{N\times N}$, $(k = 1, 2, 3)$, $\rho_{min} = \lambda_{min}((\Gamma_1 + \Gamma_1^T)/2)$, $\hat{C}^{(1)} = diag(\bar{c}_{11}^{(1)}, \bar{c}_{22}^{(1)}, \dots, \bar{c}_{NN}^{(1)})$, $K = diag(k_1, \dots, k_m, 0, \dots, 0)$, where k_i $(1 \leq i \leq m)$ are positive constants to be determined later. Then we have the following result.

Theorem 3.1. *Suppose that the Assumptions 2.2 and 2.3 hold. The drive system (2.1) and the response system (2.4) with linear controllers (3.1) can realize the lag synchronization if there exist matrices* $H_i = diag(h_1^{(i)}, h_2^{(i)}, \cdots, h_N^{(i)}) \geq 0, (i = 1, 2)$ *such that the following LMI holds:*

$$\Omega = \begin{pmatrix} \Omega_{11} & \frac{1}{2}\gamma_2|\bar{C}^{(2)}| & \frac{1}{2}\gamma_3|\bar{C}^{(3)}| \\ * & \alpha_2 I_N - (1-\bar{\sigma})H_1 & 0 \\ * & * & -\frac{1-\mu}{d}H_2 \end{pmatrix} < 0, \tag{3.3}$$

where

$$\Omega_{11} = \alpha_1 I_N + (\rho_{min} - \gamma_1)\hat{C}^{(1)} + \gamma_1 \frac{\bar{C}^{(1)} + (\bar{C}^{(1)})^T}{2} - \gamma_1 K + H_1 + dH_2.$$

Proof. Choose the following Lyapunov-Krasovskii functional candidate:

$$V(t) = V_1(t) + V_2(t), \tag{3.4}$$

where

$$\begin{aligned} V_1(t) =& \frac{1}{2}\sum_{i=1}^N e_i^T(t)e_i(t), \\ V_2(t) =& \sum_{i=1}^N h_i^{(1)} \int_{t-\sigma(t)}^t e_i^T(s)e_i(s)ds + \sum_{i=1}^N h_i^{(2)} \int_{-d(t)}^0 \int_{t+\theta}^t e_i^T(s)e_i(s)dsd\theta. \end{aligned}$$

Differentiating $V_1(t)$ along the trajectory of the error system (3.2), we have

$$\begin{aligned} \dot{V}_1(t) =& \sum_{i=1}^N e_i^T(t)[f(y_i(t), y_i(t-\sigma(t))) - f(x_i(t-\tau), x_i(t-\sigma(t)-\tau)) + \sum_{j=1}^N \bar{c}_{ij}^{(1)}\Gamma_1 e_j(t) \\ &+ \sum_{j=1}^N \bar{c}_{ij}^{(2)}\Gamma_2 e_j(t-\sigma(t)) + \sum_{j=1}^N \bar{c}_{ij}^{(3)}\Gamma_3 \int_{t-d(t)}^t e_j(s)ds] - \sum_{i=1}^m \gamma_1 k_i e_i^T(t)e_i(t). \end{aligned} \tag{3.5}$$

Then from Assumption 2.3, we have the following estimations:

$$\begin{aligned} \dot{V}_1(t) \leq& \sum_{i=1}^N (\alpha_1||e_i(t)||^2 + \alpha_2||e_i(t-\sigma(t))||^2) + \sum_{i=1}^N\sum_{j=1}^N e_i^T(t)\bar{c}_{ij}^{(1)}\Gamma_1 e_j(t) \\ &+ \sum_{i=1}^N\sum_{j=1}^N e_i^T(t)\bar{c}_{ij}^{(2)}\Gamma_2 e_j(t-\sigma(t)) + \sum_{i=1}^N\sum_{j=1}^N e_i^T(t)\bar{c}_{ij}^{(3)}\Gamma_3 \int_{t-d(t)}^t e_j(s)ds - \sum_{i=1}^m \gamma_1 k_i e_i^T(t)e_i(t) \\ \leq& \sum_{i=1}^N (\alpha_1||e_i(t)||^2 + \alpha_2||e_i(t-\sigma(t))||^2) + \sum_{i=1}^N \sum_{j=1, j\neq i}^N e_i^T(t)\bar{c}_{ij}^{(1)}\Gamma_1 e_j(t) + \sum_{i=1}^N e_i^T(t)\bar{c}_{ii}^{(1)}\Gamma_1 e_i(t) \\ &+ \sum_{i=1}^N\sum_{j=1}^N ||e_i(t)|||\bar{c}_{ij}^{(2)}|||\Gamma_2||||e_j(t-\sigma(t))|| + \sum_{i=1}^N\sum_{j=1}^N ||e_i(t)|||\bar{c}_{ij}^{(3)}|||\Gamma_3|||| \int_{t-d(t)}^t e_j(s)ds|| \\ &- \sum_{i=1}^m \gamma_1 k_i e_i^T(t)e_i(t) \\ \leq& \sum_{i=1}^N (\alpha_1||e_i(t)||^2 + \alpha_2||e_i(t-\sigma(t))||^2) + \gamma_1 \sum_{i=1}^N \sum_{j=1, j\neq i}^N ||e_i(t)||\bar{c}_{ij}^{(1)}||e_j(t)|| + \sum_{i=1}^N \rho_{min}\bar{c}_{ii}^{(1)} e_i^T(t)e_i(t) \end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&+\gamma_2\sum_{i=1}^{N}\sum_{j=1}^{N}||e_i(t)|||\bar{c}_{ij}^{(2)}|||e_j(t-\sigma(t))||+\gamma_3\sum_{i=1}^{N}\sum_{j=1}^{N}||e_i(t)|||\bar{c}_{ij}^{(3)}|||\int_{t-d(t)}^{t}e_j(s)ds||\\
&-\sum_{i=1}^{m}\gamma_1 k_i e_i^T(t)e_i(t)\\
=&e^T(t)(\alpha_1 I_N+(\rho_{min}-\gamma_1)\hat{C}^{(1)}+\gamma_1\frac{\bar{C}^{(1)}+(\bar{C}^{(1)})^T}{2}-\gamma_1 K)e(t)+e^T(t)(\gamma_2|\bar{C}^{(2)}|)e(t-\sigma(t))\\
&+e^T(t-\sigma(t))(\alpha_2 I_N)e(t-\sigma(t))+e^T(t)(\gamma_3|\bar{C}^{(3)}|)\tilde{e}(t),
\end{aligned}
$$

where

$$
e(t)=(||e_1(t)||,||e_2(t)||,\dots,||e_N(t)||)^T, e(t-\sigma(t))=(||e_1(t-\sigma(t))||,||e_2(t-\sigma(t))||,\dots,||e_N(t-\sigma(t))||)^T,
$$
$$
\tilde{e}(t)=(||\int_{t-d(t)}^{t}e_1(s)ds||,||\int_{t-d(t)}^{t}e_2(s)ds||,\dots,||\int_{t-d(t)}^{t}e_N(s)ds||)^T.
$$

By Assumption 2.2 and Lemma 2.5, calculating the time derivation of $V_2(t)$ along the trajectories of system (3.2), we get

$$
\begin{aligned}
V_2(t)\le&\sum_{i=1}^{N}h_i^{(1)}[e_i^T(t)e_i(t)-(1-\bar{\sigma})e_i^T(t-\sigma(t))e_i(t-\sigma(t))]\\
&+\sum_{i=1}^{N}h_i^{(2)}[d(t)e_i^T(t)e_i(t)-(1-\mu)(\int_{t-d(t)}^{t}e_i^T(s)e_i(s)ds)]\\
\le&\sum_{i=1}^{N}h_i^{(1)}[e_i^T(t)e_i(t)-(1-\bar{\sigma})e_i^T(t-\sigma(t))e_i(t-\sigma(t))]\\
&+\sum_{i=1}^{N}h_i^{(2)}[de_i^T(t)e_i(t)-\frac{1-\mu}{d}(\int_{t-d(t)}^{t}e_i(s)ds)^T(\int_{t-d(t)}^{t}e_i(s)ds)]\\
=&e^T(t)(H_1+dH_2)e(t)-e^T(t-\sigma(t))(1-\bar{\sigma})H_1e(t-\sigma(t))-\tilde{e}(t)\frac{1-\mu}{d}H_2\tilde{e}(t).
\end{aligned}
\tag{3.7}
$$

Let $\xi(t)=(e^T(t),e^T(t-\sigma(t)),\tilde{e}^T(t))^T$, $\Xi=-\Omega$.

According to (3.3) and (3.5), (3.6), (3.7), it follows that

$$
\dot{V}(t)=\dot{V}_1(t)+\dot{V}_2(t)\le-\xi^T(t)\Xi\xi(t)\le 0. \tag{3.8}
$$

From (3.8), we get

$$
0\le\lambda_{min}(\Xi)||\xi(t)||^2\le\xi^T(t)\Xi\xi(t)\le-\dot{V}(t). \tag{3.9}
$$

Integrating (3.9) from 0 to t, in view of $V(t)>0$, we obtain

$$
\int_0^t\lambda_{min}(\Xi)||\xi(s)||^2ds\le-\int_0^t\dot{V}(s)ds=V(0)-V(t)\le V(0)<+\infty.
$$

By Barbalat's lemma [4], we have

$$
\lambda_{min}(\Xi)||e||^2\le\lambda_{min}(\Xi)||\xi(t)||^2\to 0,
$$

which implies that $\lim_{t\to\infty}||e(t)||=0$, then we can get $\lim_{t\to\infty}(y_i(t)-x_i(t-\tau))=0$, $(i=1,2,\dots,N)$. That is to say the drive system (2.1) lag synchronization with the response system (2.4) at time τ. This completes the proof. □

Remark 3.2. For any given dynamical network with node dynamics $f(\cdot,\cdot)$, the coupling matrices $\bar{C}^{(1)}$, $\bar{C}^{(2)}$, $\bar{C}^{(3)}$ and Γ_1, Γ_2, Γ_3 are known, so the positive constants α_1, α_2 in Assumption 2.3 and γ_1, γ_2, γ_3, ρ_{min} can be estimated by simple calculations. Thus, from condition (3.3), if the matrices H_1, H_2 and the pinned nodes m are fixed, the feedback gains k_i can be estimated. However, the node dynamics and the coupling matrices are usually nonidentical for different dynamical systems. Therefore, the proposed pinning controllers with fixed feedback gains are not universal.

In the following section, an adaptive pinning strategy will be adopted to design universal controllers.

3.2. Synchronization via the adaptive feedback pinning control

In this subsection, we use the adaptive feedback control to pin the lag synchronization. Without loss of generality, assume that the first $m(1 \le m \le N)$ nodes are selected and pinned with the adaptive controllers, which are described as

$$\begin{cases} u_i(t) = -\gamma_1 k_i(t) e_i(t), \quad 1 \le i \le m, \\ \dot{k}_i(t) = \delta_i e_i^T(t) e_i(t), \quad k_i(0) = 0, \quad \delta_i > 0, \quad 1 \le i \le m, \\ u_i(t) = 0, \quad m+1 \le i \le N, \end{cases} \tag{3.10}$$

where $\gamma_1 = ||\Gamma_1||$, $e_i(t) = y_i(t) - x_i(t-\tau)$, and δ_i are positive constants. According to (3.10), we obtain the following lag synchronization error system,

$$\begin{cases} \dot{e}_i(t) = f(y_i(t), y_i(t-\sigma(t))) - f(x_i(t-\tau), x_i(t-\sigma(t)-\tau)) + \sum_{j=1}^{N} \bar{c}_{ij}^{(1)} \Gamma_1 e_j(t) + \sum_{j=1}^{N} \bar{c}_{ij}^{(2)} \Gamma_2 e_j(t-\sigma(t)) \\ \qquad + \sum_{j=1}^{N} \bar{c}_{ij}^{(3)} \Gamma_3 \int_{t-d(t)}^{t} e_j(s) ds - \gamma_1 k_i(t) e_i(t), \quad 1 \le i \le m, \\ \dot{e}_i(t) = f(y_i(t), y_i(t-\sigma(t))) - f(x_i(t-\tau), x_i(t-\sigma(t)-\tau)) + \sum_{j=1}^{N} \bar{c}_{ij}^{(1)} \Gamma_1 e_j(t) + \sum_{j=1}^{N} \bar{c}_{ij}^{(2)} \Gamma_2 e_j(t-\sigma(t)) \\ \qquad + \sum_{j=1}^{N} \bar{c}_{ij}^{(3)} \Gamma_3 \int_{t-d(t)}^{t} e_j(s) ds, \quad m+1 \le i \le N, \\ \dot{k}_i(t) = \delta_i e_i^T(t) e_i(t), \quad \delta_i > 0, \quad 1 \le i \le m. \end{cases} \tag{3.11}$$

Let $||\Gamma_2|| = \gamma_2$, $||\Gamma_3|| = \gamma_3$, $|\bar{C}^{(k)}| = (|\bar{c}_{ij}^{(k)}|)_{N\times N}$, $(k = 1,2,3)$, $\rho_{min} = \lambda_{min}((\Gamma_1 + \Gamma_1^T)/2)$, $\hat{C}^{(1)} = diag(\bar{c}_{11}^{(1)}, \bar{c}_{22}^{(1)}, \ldots, \bar{c}_{NN}^{(1)})$, $K^* = diag(k_1^*, \ldots, k_m^*, 0, \ldots, 0)$, where k_i^* $(1 \le i \le m)$ are positive constants to be determined later. Then we have the following result:

Theorem 3.3. *Suppose that the Assumption* 2.2 *and* 2.3 *hold. The drive system* (2.1) *and the response system* (2.4) *with adaptive controllers* (3.10) *can realize the lag synchronization if there exist matrices* $R_i = diag(r_1^{(i)}, r_2^{(i)}, \cdots, r_N^{(i)}) \ge 0, (i = 1, 2)$ *such that the following LMI holds:*

$$\Phi = \begin{pmatrix} \Phi_{11} & \frac{1}{2}\gamma_2 |\bar{C}^{(2)}| & \frac{1}{2}\gamma_3 |\bar{C}^{(3)}| \\ * & \alpha_2 I_N - (1-\bar{\sigma}) R_1 & 0 \\ * & * & -\frac{1-\mu}{d} R_2 \end{pmatrix} < 0, \tag{3.12}$$

where $\Phi_{11} = \alpha_1 I_N + (\rho_{min} - \gamma_1)\hat{C}^{(1)} + \gamma_1 \frac{\bar{C}^{(1)} + (\bar{C}^{(1)})^T}{2} - \gamma_1 K^* + R_1 + d R_2$.

Proof. Choose the following Lyaponov-Krasovskii functional candidate as follows:

$$V(t) = V_1(t) + V_2(t), \tag{3.13}$$

where

$$V_1(t) =\frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t) + \gamma_1 \sum_{i=1}^{m} \frac{1}{2\delta_i}(k_i(t) - k_i^*)^2,$$
$$V_2(t) = \sum_{i=1}^{N} r_i^{(1)} \int_{t-\sigma(t)}^{t} e_i^T(s)e_i(s)ds + \sum_{i=1}^{N} r_i^{(2)} \int_{-d(t)}^{0}\int_{t+\theta}^{t} e_i^T(s)e_i(s)dsd\theta.$$

Calculating $V_1(t)$ along the trajectory of the error system (3.11), we have

$$\begin{aligned}&\dot{V}_1(t)\\ =&\sum_{i=1}^{N} e_i^T(t)[f(y_i(t), y_i(t-\sigma(t))) - f(x_i(t-\tau), x_i(t-\sigma(t)-\tau)) + \sum_{j=1}^{N} \bar{c}_{ij}^{(1)}\Gamma_1 e_j(t)\\ &+ \sum_{j=1}^{N} \bar{c}_{ij}^{(2)}\Gamma_2 e_j(t-\sigma(t)) + \sum_{j=1}^{N} \bar{c}_{ij}^{(3)}\Gamma_3 \int_{t-d(t)}^{t} e_j(s)ds] - \sum_{i=1}^{m} \gamma_1 k_i(t)e_i^T(t)e_i(t)\\ &+ 2\gamma_1 \sum_{i=1}^{m} \frac{1}{2\delta_i}(k_i(t) - k_i^*)\dot{k}_i(t).\end{aligned} \tag{3.14}$$

From Assumption 2.3, we have

$$\begin{aligned}&\dot{V}_1(t)\\ \leq&\sum_{i=1}^{N}(\alpha_1||e_i(t)||^2 + \alpha_2||e_i(t-\sigma(t))||^2) + \sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)\bar{c}_{ij}^{(1)}\Gamma_1 e_j(t) + \sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)\bar{c}_{ij}^{(2)}\Gamma_2 e_j(t-\sigma(t))\\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)\bar{c}_{ij}^{(3)}\Gamma_3 \int_{t-d(t)}^{t} e_j(s)ds - \sum_{i=1}^{m}\gamma_1 k_i(t)e_i^T(t)e_i(t) + 2\gamma_1 \sum_{i=1}^{m} \frac{1}{2\delta_i}(k_i(t) - k_i^*)\dot{k}_i(t)\\ \leq&\sum_{i=1}^{N}(\alpha_1||e_i(t)||^2 + \alpha_2||e_i(t-\sigma(t))||^2) + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} e_i^T(t)\bar{c}_{ij}^{(1)}\Gamma_1 e_j(t) + \sum_{i=1}^{N} e_i^T(t)\bar{c}_{ii}^{(1)}\Gamma_1 e_i(t)\\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N} ||e_i(t)|||\bar{c}_{ij}^{(2)}|||\Gamma_2||||e_j(t-\sigma(t))|| + \sum_{i=1}^{N}\sum_{j=1}^{N} ||e_i(t)|||\bar{c}_{ij}^{(3)}|||\Gamma_3|||| \int_{t-d(t)}^{t} e_j(s)ds||\\ &- \sum_{i=1}^{m}\gamma_1 k_i^* e_i^T(t)e_i(t)\\ \leq&\sum_{i=1}^{N}(\alpha_1||e_i(t)||^2 + \alpha_2||e_i(t-\sigma(t))||^2) + \gamma_1\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} ||e_i^T(t)||\bar{c}_{ij}^{(1)}||e_j(t)|| + \sum_{i=1}^{N} \rho_{min}\bar{c}_{ii}^{(1)} e_i^T(t)e_i(t)\\ &+ \gamma_2\sum_{i=1}^{N}\sum_{j=1}^{N} ||e_i(t)|||\bar{c}_{ij}^{(2)}|||e_j(t-\sigma(t))|| + \gamma_3\sum_{i=1}^{N}\sum_{j=1}^{N} ||e_i(t)|||\bar{c}_{ij}^{(3)}||| \int_{t-d(t)}^{t} e_j(s)ds||\\ &- \sum_{i=1}^{m}\gamma_1 k_i^* e_i^T(t)e_i(t)\\ =&e^T(t)(\alpha_1 I_N + (\rho_{min} - \gamma_1)\hat{C}^{(1)} + \gamma_1\frac{\bar{C}^{(1)} + (\bar{C}^{(1)})^T}{2} - \gamma_1 K^*)e(t) + e^T(t)(\gamma_2|\bar{C}^{(2)}|)e(t-\sigma(t))\\ &+ e^T(t-\sigma(t))(\alpha_2 I_N)e(t-\sigma(t)) + e^T(t)(\gamma_3|\bar{C}^{(3)}|)\tilde{e}(t),\end{aligned} \tag{3.15}$$

where

$$e(t) = (||e_1(t)||, ||e_2(t)||, \dots, ||e_N(t)||)^T, e(t-\sigma(t))$$
$$= (||e_1(t-\sigma(t))||, ||e_2(t-\sigma(t))||, \dots, ||e_N(t-\sigma(t))||)^T,$$

$$\tilde{e}(t) = (||\int_{t-d(t)}^{t} e_1(s)ds||, ||\int_{t-d(t)}^{t} e_2(s)ds||, \dots, ||\int_{t-d(t)}^{t} e_N(s)ds||)^T.$$

By Assumption 2.2 and Lemma 2.5, calculating the time derivative of $V_2(t)$ along the trajectories of system (3.11), we get

$$\begin{aligned} V_2(t) \leq & \sum_{i=1}^{N} h_i^{(1)}[e_i^T(t)e_i(t) - (1-\bar{\sigma})e_i^T(t-\sigma(t))e_i(t-\sigma(t))] \\ & + \sum_{i=1}^{N} h_i^{(2)}[d(t)e_i^T(t)e_i(t) - (1-\mu)(\int_{t-d(t)}^{t} e_i^T(s)e_i(s)ds)] \\ \leq & \sum_{i=1}^{N} h_i^{(1)}[e_i^T(t)e_i(t) - (1-\bar{\sigma})e_i^T(t-\sigma(t))e_i(t-\sigma(t))] \\ & + \sum_{i=1}^{N} h_i^{(2)}[de_i^T(t)e_i(t) - \frac{1-\mu}{d}(\int_{t-d(t)}^{t} e_i(s)ds)^T(\int_{t-d(t)}^{t} e_i(s)ds)] \\ = & e^T(t)(H_1 + dH_2)e(t) - e^T(t-\sigma(t))(1-\bar{\sigma})H_1 e(t-\sigma(t)) - \tilde{e}(t)\frac{1-\mu}{d}H_2\tilde{e}(t). \end{aligned} \tag{3.16}$$

Now, let $\eta(t) = (e^T(t), e^T(t-\sigma(t)), \tilde{e}^T(t))^T$. From (3.12) and (3.14)-(3.16), we can see that

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) \leq \eta^T(t)\Phi\eta(t) \leq 0. \tag{3.17}$$

Then, similar to the proof of Theorem 3.1, we have $\lim_{t\to\infty} ||e(t)|| = 0$. This completes the proof. □

Remark 3.4. In general, the strength of linear feedback must be maximal, which is a kind of waste in practice to some extent. Compared with linear control [6], the control gains of adaptive control increase according to the adaptive laws. Hence, adaptive control is more flexible.

Remark 3.5. To avoid solving the LMI (3.12), we have the following corollary, and the conditions of Corollary 3.6 are more easy to verify.

First, let $R = \alpha_2 I_N - (1-\bar{\sigma})R_1$, then using Schur complement lemma, the condition (3.12) is equivalent to

$$\Phi_{11} - \frac{1}{4}\gamma_2^2|\bar{C}^{(2)}|R^{-1}|\bar{C}^{(2)}|^T + \frac{d}{4(1-\mu)}\gamma_3^2|\bar{C}^{(3)}|R_2^{-1}|\bar{C}^{(3)}|^T < 0. \tag{3.18}$$

Moreover, when $R_1 = \frac{1}{1-\bar{\sigma}}(\alpha_2 + \frac{1}{2}\gamma_2 c_2)I_N$, $R_2 = \frac{1}{2}\gamma_3 c_3 I_N$, where $c_2 = ||(|\bar{C}^{(2)}|)||$, $c_3 = ||(|\bar{C}^{(3)}|)||$. According to Lemma 2.6, we have

$$\begin{aligned} \Phi_{11} - \frac{1}{4}\gamma_2^2|\bar{C}^{(2)}|R^{-1}|\bar{C}^{(2)}|^T + \frac{d}{4(1-\mu)}\gamma_3^2|\bar{C}^{(3)}|R_2^{-1}|\bar{C}^{(3)}|^T & \\ \leq \alpha I_N + (\rho_{min} - \gamma_1)\hat{C}^{(1)} + \frac{1}{2}\gamma_1(\bar{C}^{(1)} + (\bar{C}^{(1)})^T) - \gamma_1 K^* & \\ = Q - \gamma_1 K^*, & \end{aligned} \tag{3.19}$$

where

$$Q = \alpha I_N + (\rho_{min} - \gamma_1)\hat{C}^{(1)} + \frac{1}{2}\gamma_1(\bar{C}^{(1)} + (\bar{C}^{(1)})^T),$$
$$\alpha = \alpha_1 + \frac{1}{2(1-\bar{\sigma})}(2\alpha_2 + (2-\bar{\sigma})\gamma_2 c_2) + \frac{d}{2}\gamma_3 c_3(\frac{2-\mu}{1-\mu}).$$

Then let

$$Q - \gamma_1 K^* = \begin{pmatrix} Q_{11} - \tilde{K}^* & Q_{12} \\ Q_{12}^T & Q_m \end{pmatrix},$$

where Q_m is the minor matrix of Q by removing its first $m(1 \le m \le N)$ row-column pairs, Q_{11} and Q_{12} are matrices with appropriate dimensions, $\tilde{K}^* = diag(k_1^*, k_2^*, \ldots, k_m^*)$. Now we can obtain the following corollary.

Corollary 3.6. *Suppose that Assumption* 2.2 *and* 2.3 *hold. The drive system* (2.1) *and the response system* (2.4) *with adaptive controllers* (3.10) *can realize the lag synchronization if the following two conditions are satisfied:*

$$k_i^* > \frac{1}{\gamma_1}\lambda_{max}(Q_{11} - Q_{12}Q_m^{-1}Q_{12}^T), \quad 1 \le i \le m \tag{3.20}$$

and

$$\lambda_{max}(\frac{\bar{C}^{(1)} + (\bar{C}^{(1)})^T}{2})_m < -\frac{\delta}{\gamma_1}, \tag{3.21}$$

where $\delta = \alpha + \lambda_{max}((\rho_{min} - \gamma_1)\hat{C}^{(1)})_m$.

Proof. From Lemma 2.8 and condition (3.20), we can see that $Q - \gamma_1 K^* < 0$ is equivalent to $Q_m < 0$. So, we only need to prove that $Q_m < 0$. By applying Lemma 2.7, we get

$$\lambda_{max}(Q_m) \le \delta + \gamma_1\lambda_{max}(\frac{\bar{C}^{(1)} + (\bar{C}^{(1)})^T}{2})_m. \tag{3.22}$$

From condition (3.21), it is not difficult to see that $\delta + \gamma_1(\frac{\bar{C}^{(1)}+(\bar{C}^{(1)})^T}{2})_m < 0$. Then, in view of (3.22), we have $\lambda_{max}(Q_m) < 0$. Therefore, along with (3.18), condition (3.12) is satisfied. This completes the proof. □

Remark 3.7. As similar to the proof in Corollary 3.6, our lag synchronization criterion of Theorem 3.1 is also easily verified and does not need to solve any linear matrix inequality. And the corresponding results are verified through a simulation experiment. However, from the magnified inequalities (3.19) and (3.22), we can see that the results of Corollary 3.6 are more conservative than Theorem 3.3.

Remark 3.8. Different from [6, 10, 22, 29, 40], the proposed conditions in this paper depend on the time-varying delays. Moreover, in [16, 35], the time-varying delay meets $\sigma(t) = d(t)$, which is a strong condition, and most of the situations do not have this property. Thus the results in this paper have less conservativeness and expand the results in the existing literatures.

4. Illustrative example

In this section, two numerical examples are given to illustrate the effectiveness of our results. Firstly, we consider the following time-delayed Chua's system,

$$\begin{cases} \dot{x}_1(t) = -m(1+b)x_1(t) + mx_2(t) + \varphi(x_1(t)), \\ \dot{x}_2(t) = x_1(t) - x_2(t) + x_3(t), \\ \dot{x}_3(t) = -\rho x_2(t) - \omega x_3(t) - \rho\omega_0 \sin(v x_1(t - \sigma(t))), \end{cases} \tag{4.1}$$

where $\varphi(x_1(t)) = -\frac{m}{2}(a-b)(|x_1(t)+1| - |x_1(t)-1|)$. When $m = 10$, $b = -0.7831$, $a = -1.4325$, $\rho = 19.53$, $\omega = 0.1636$, $\omega_0 = 0.2$, $v = 0.5$, and $\sigma(t) = 0.02 + 0.01\sin(10t)$, one can easily verify that the nonlinear function $f(\cdot,\cdot)$ satisfies Assumption 2.2 with $\alpha_1 = 12.5008$ and $\alpha_2 = 0.2441$.

Example 4.1. Based on the above Chua's system (4.1), we use the drive-response systems (2.1) and (2.4) consisting of $N = 4$ identical time-delayed Chua systems with mixed delays coupling to verify the correctness of Theorem 3.1. Now we choose the following coupling matrices:

$$\bar{C}^{(1)} = \begin{pmatrix} -13 & 4 & 3 & 6 \\ 6 & -14 & 5 & 3 \\ 4 & 6 & -15 & 5 \\ 8 & 2 & 6 & -16 \end{pmatrix}, \quad \bar{C}^{(2)} = \begin{pmatrix} 0.2 & 0.1 & -0.3 & 0 \\ 0.3 & -0.1 & 0.5 & -0.7 \\ -0.5 & 0.1 & 0.5 & 0.1 \\ 0.3 & -0.3 & -0.1 & 0.1 \end{pmatrix},$$

$$\bar{C}^{(3)} = \begin{pmatrix} 0.1 & -0.2 & 0.6 & -0.5 \\ 0.3 & -0.6 & 0.2 & 0.1 \\ 0 & 0.5 & 0.2 & -0.7 \\ 0.2 & 0.1 & 0 & -0.3 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 3.5 & 0 & -0.1 \\ 0 & 3.5 & -0.5 \\ 0.1 & 0.6 & 3.5 \end{pmatrix},$$

$$\Gamma_2 = \begin{pmatrix} -0.3 & -0.1 & 0.1 \\ 0.1 & 0.2 & 0 \\ 0 & -0.1 & 0.4 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0.1 & -0.3 & 0.1 \\ -0.1 & 0.1 & -0.2 \\ 0 & -0.1 & 0.3 \end{pmatrix}.$$

For $\tau = 0.5$ and $d(t) = 0.3 + 0.5\cos(5t)$, choose the initial conditions $x_i(0) = (-1.8 + 0.5i, -0.9 + 0.5i, -4.7 + 0.5i)^T$ and $y_i(0) = (1.8 + 0.5i, 0.9 + 0.5i, 4.7 + 0.5i)^T$, $1 \le i \le 4$. Fig. 1 shows the curves of error dynamics between the drive-response networks without controllers. It is clear that the complex dynamical networks cannot achieve synchronization.

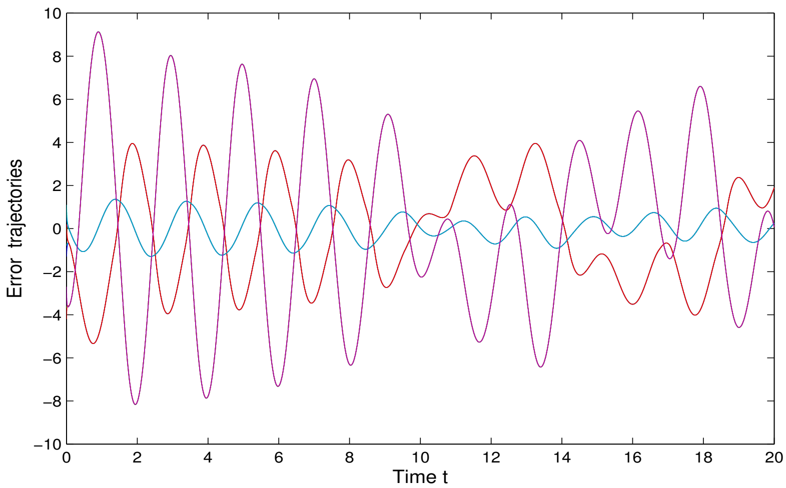

Figure 1: The error-state trajectory without controllers.

However, by applying the linear feedback pinning control, we assume $m = 1$, that is, the number of nodes to be controlled is 1. By simple computation, we can obtain $-\frac{\delta}{\gamma_1} = -4.4549$, $\lambda_{max}(\frac{\bar{C}^{(1)}+(\bar{C}^{(1)})^T}{2})_m = -5.8223$, and $\lambda_{max}(Q_{11} - Q_{12}Q_mQ_{12}^T)/\gamma_1 = 16.4058$. Then, choosing the appropriate feedback gain $k_1 = 20$, the corresponding simulation can be seen in Fig. 2, showing the drive system (2.1) and response system (2.4) can reach synchronization by using the above controllers. Moreover, Figs. 3–5 illustrate that the state trajectory of response network (2.4) and drive network (2.1) with $\tau = 0.5$.

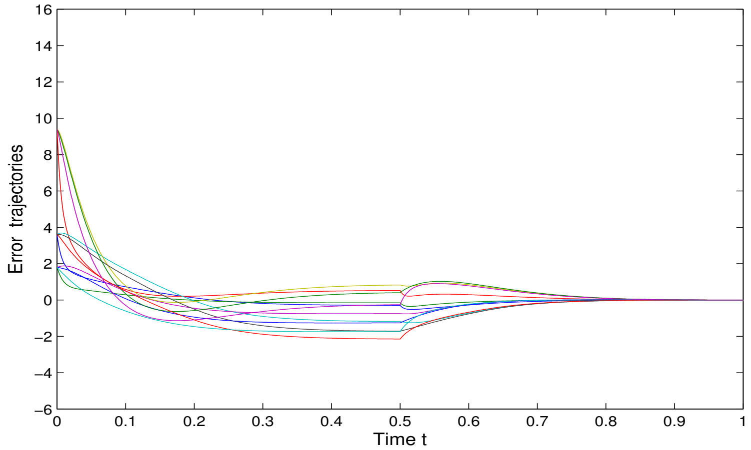

Figure 2: The error-state trajectory by the linear pinning control.

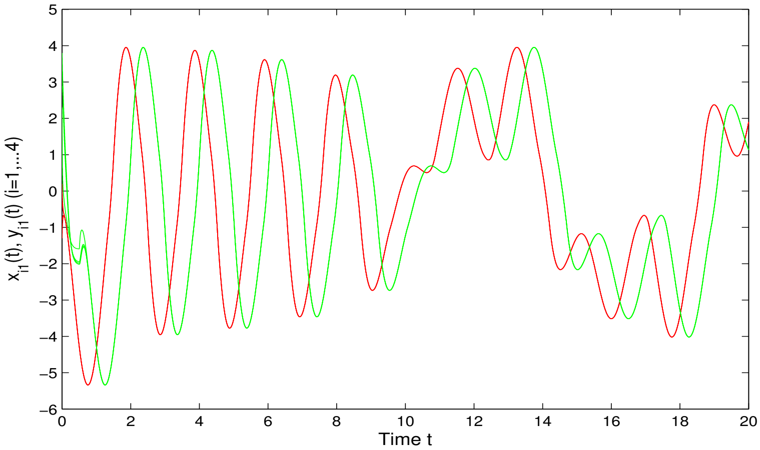

Figure 3: The state trajectories of $x_{i1}(t)$ and $y_{i1}(t)$ $(i = 1, 2, \ldots, 4)$ under the linear pinning control.

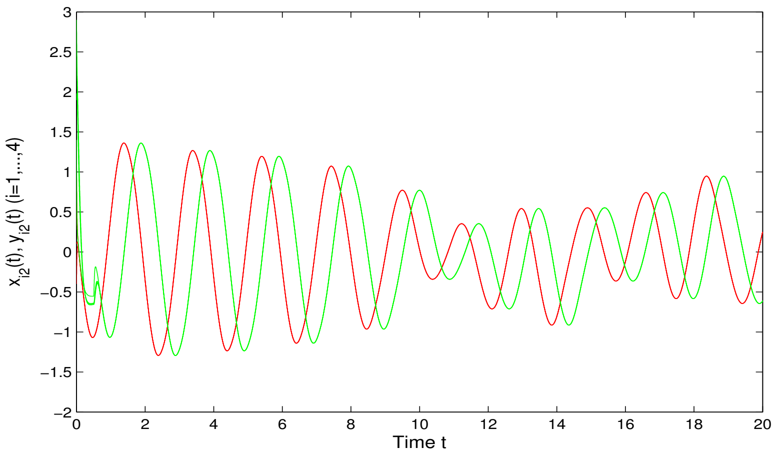

Figure 4: The state trajectory of $x_{i2}(t)$ and $y_{i2}(t)$ $(i = 1, 2, \ldots, 4)$ under the linear pinning control.

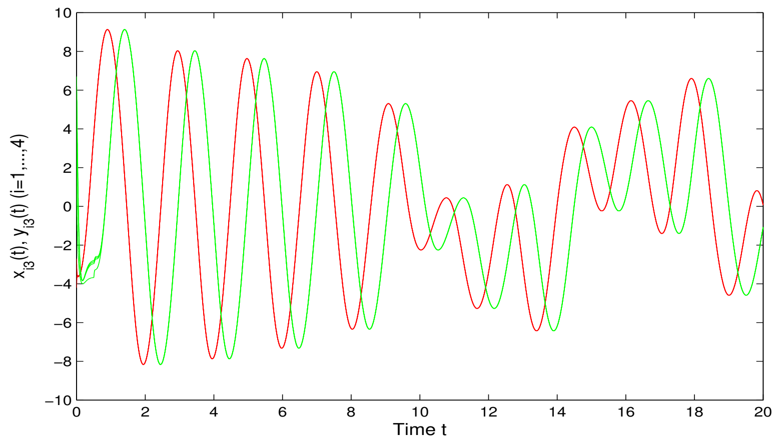

Figure 5: The state trajectory of $x_{i3}(t)$ and $y_{i3}(t)$ $(i = 1, 2, \ldots, 4)$ under the linear pinning control.

From the above example we can see that the strength of linear feedback may be maximum, which is a kind of waste in practice to some extent. Compared with linear control, the gains of the adaptive control increase according to the adaptive laws.

In the following, a numerical example is given to show the application of the adaptive pinning control.

Example 4.2. We consider the drive-response systems (2.1) and (2.4) consisting of $N = 5$ identical time-delayed Chua's systems with mixed delays coupling to verify the correctness of Theorem 3.3. Choose the following coupling matrices:

$$\bar{C}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 5 & -14 & 4 & 2 & 3 \\ 3 & 5 & -15 & 4 & 3 \\ 6 & 2 & 3 & -16 & 5 \\ 4 & 3 & 3 & 7 & -17 \end{pmatrix}, \qquad \bar{C}^{(2)} = \begin{pmatrix} -0.2 & 0.5 & 0 & 0.1 & -0.3 \\ 0.3 & -0.2 & 0.5 & -0.4 & -0.2 \\ -0.6 & 0.1 & 0.5 & 0 & 0 \\ 0 & -0.4 & -0.1 & 0.3 & 0.2 \\ 0.3 & 0 & -0.1 & 0.1 & -0.3 \end{pmatrix},$$

$$\bar{C}^{(3)} = \begin{pmatrix} 0.1 & -0.2 & 0.6 & -0.5 & 0 \\ 0.3 & -0.6 & 0.2 & 0.3 & -0.2 \\ 0 & 0.5 & 0.1 & -0.7 & 0.1 \\ 0.2 & 0.1 & -0.1 & -0.7 & 0.5 \\ -0.3 & 0.2 & 0 & 0.4 & -0.1 \end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} 4.5 & 0 & 0 \\ 0 & 4.5 & -0.5 \\ 0 & 0.6 & 4.5 \end{pmatrix},$$

$$\Gamma_2 = \begin{pmatrix} -0.2 & -0.1 & 0 \\ 0.1 & 0.3 & 0 \\ 0 & 0 & 0.4 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} 0.2 & -0.4 & 0.1 \\ -0.1 & 0.1 & -0.2 \\ 0 & 0.1 & 0.3 \end{pmatrix}.$$

For $\tau = 0.5$ and $d(t) = 0.5 - 8\sin(0.5t)$, choose the initial conditions $x_i(0) = (-1.8 + 0.5i, -0.9 + 0.5i, -4.7 + 0.5i)^T$ and $y_i(0) = (1.8 + 0.5i, 0.9 + 0.5i, 4.7 + 0.5i)^T$, $1 \le i \le 5$. Fig. 6 shows the curves of error dynamics between the drive-response networks without controllers. It is clear that the complex dynamical networks cannot achieve synchronization.

Then, by applying the adaptive feedback pinning control, we assume $m = 1$. By simple computation, we have $-\frac{\delta}{\gamma_1} = -3.2393$, $\lambda_{max}(\frac{\bar{C}^{(1)}+(\bar{C}^{(1)})^T}{2})_m = -4.4912$. We can see that condition (3.21) of Corollary 3.6 is tenable. Therefore, from Corollary 3.6, the lag synchronization between the drive system (2.1) and the response system (2.4) can be realized by the adaptive controllers (3.10).

In the numerical simulations, we apply the adaptive controllers (3.10) to pin the first node of the response system (2.4) and let $\delta_1 = 1$. The corresponding simulation can be seen in Fig. 7, which shows the drive system (2.1) and response system (2.4) can achieve synchronization by using the adaptive controllers, and the state trajectories of drive system and response system with $\tau = 0.5$ are described in Figs. 8–10.

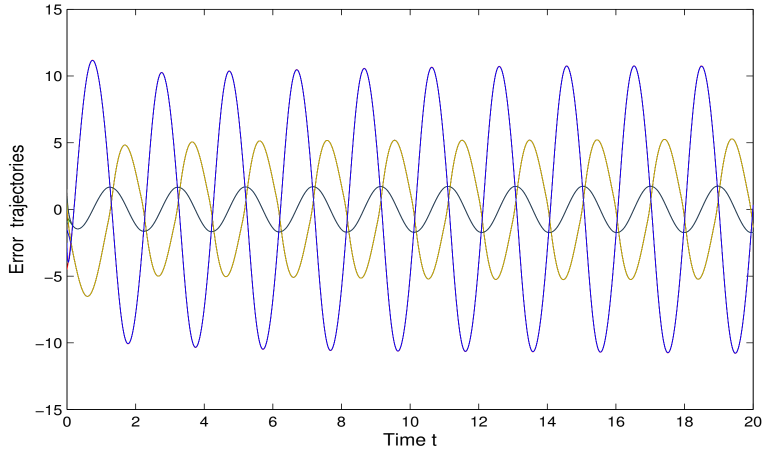

Figure 6: The error-state trajectory without controllers.

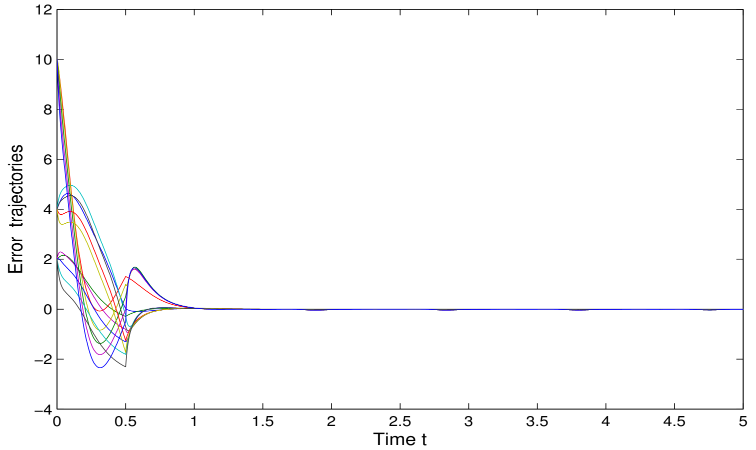

Figure 7: The error-state trajectory by the adaptive pinning control.

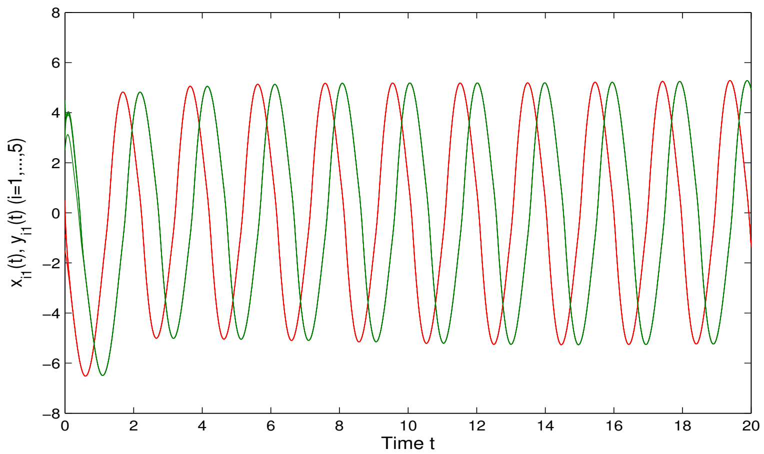

Figure 8: The state trajectory of $x_{i1}(t)$ and $y_{i2}(t)$ $(i = 1, 2, \ldots, 5)$ under the adaptive pinning control.

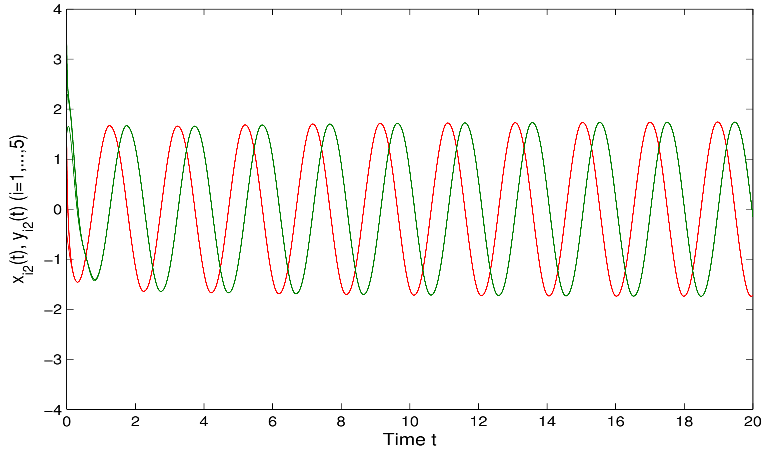

Figure 9: The state trajectory of $x_{i2}(t)$ and $y_{i2}(t)$ $(i = 1, 2, \ldots, 5)$ under the adaptive pinning control.

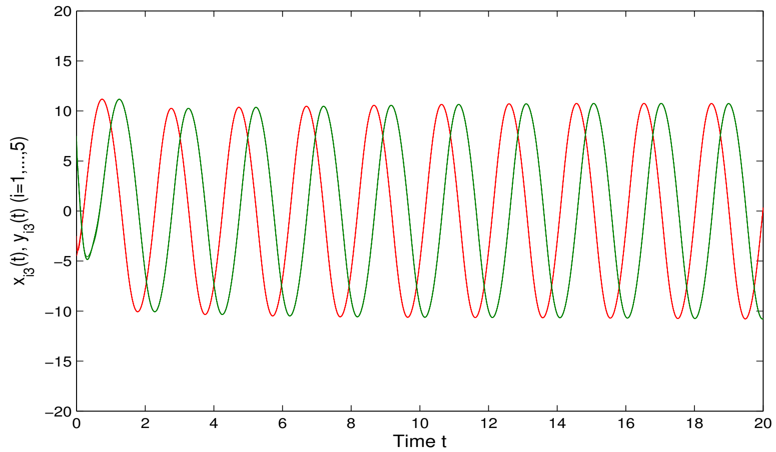

Figure 10: The state trajectory of $x_{i3}(t)$ and $y_{i3}(t)$ $(i = 1, 2, \ldots, 5)$ under the adaptive pinning control.

5. Conclusion

In this paper, the issue of the lag synchronization between drive and response systems with mixed coupling has been investigated. By applying the Lyaponov functional theory and mathematical analysis method, less conservative conditions of lag synchronization are obtained by adding controllers to a part of nodes. Moreover, the coupling configuration matrices are not required to be symmetric or irreducible. It is shown that the lag synchronization of the drive and response systems can be realized via the linear feedback pinning control and adaptive feedback pinning control. These results remove some restrictions on the node dynamics and the number of the pinned nodes. Finally, numerical examples are presented to illustrate the effectiveness of the theoretical results.

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