Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4707–4712
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Some identities of degenerate q-Euler polynomials under the symmetry group of degree n

Taekyun Kim[a,b], D. V. Dolgy[c,d], Lee-Chae Jang[e,*], Hyuck-In Kwon[b]

[a] *Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China.*
[b] *Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea.*
[c] *Hanrimwon, Kwangwoon University, Seoul 139-701, Republic of Korea.*
[d] *Institute of Natural Sciences, Far eastern Federal University, Vladivostok 690950, Russia.*
[e] *Graduate School of Education, Konkuk University, Seoul 143-701, Republic of Korea.*

Communicated by S. H. Rim

Abstract

In this paper, we derive some interesting identities of symmetry for the degenerate q-Euler polynomials under the symmetry group of degree n arising from the fermionic p-adic q-integral on $\mathbb{Z}_p$. ©2016 all rights reserved.

Keywords: Identities of symmetry, degenerate q-Euler polynomial, symmetry group of degree n, fermionic p-adic q-integral.
2010 MSC: 11B68, 11S80, 05A19, 05A30.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let q be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$, where $|\cdot|_p$ is the p-adic norm. As is known, the q-analogue of the number x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Let $f(x)$ be continuous function on $\mathbb{Z}_p$. The fermionic

*Corresponding author
Email addresses: tkkim@kw.ac.kr (Taekyun Kim), d_dol@mail.ru (D. V. Dolgy), lcjang@konkuk.ac.kr (Lee-Chae Jang), sura@kw.ac.kr (Hyuck-In Kwon)

Received 2016-07-30

p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows (see [3–20])

$$\begin{aligned}
I_{-q}(f) &= \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-q}(x+p^N\mathbb{Z}_p) \\
&= \lim_{N\to\infty} \frac{[2]_q}{2} \sum_{x=0}^{p^N-1} f(x)(-q)^x \\
&= \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x.
\end{aligned}$$

For $\lambda, t \in \mathbb{C}_p$ with $|\lambda|_p \le 1, |t|_p < p^{-\frac{1}{p-1}}$, the degenerate Euler polynomials are defined by Carlitz to be

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [2, 12])}. \tag{1.1}$$

Note that $\lim_{\lambda\to 0} \mathcal{E}_{n,\lambda} = E_n(x)$, where $E_n(x)$ are ordinary Euler polynomials which are given by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad \text{(see [1–20])}. \tag{1.2}$$

Recently, Kim proved the following equation:

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}} d\mu_{-1}(y) = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.3}$$

Thus, by (1.3), we get

$$\lambda^n \int_{\mathbb{Z}_p} \left(\frac{x+y}{\lambda}\right)_n d\mu_{-1}(y) = \mathcal{E}_{n,\lambda(x)}, \quad \text{(see [12])}, \tag{1.4}$$

where $\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{q\to 1}\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x)$ and $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \ge 1)$, $(x)_0 = 1$. In [13], the degenerate q-Euler polynomials are defined by Kim as follows

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+y]_q}{\lambda}} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda,q}(x)\frac{t^n}{n!}. \tag{1.5}$$

When $x = 0$, $\mathcal{E}_{n,\lambda,q} = \mathcal{E}_{n,\lambda,q}(0)$ are called the degenerate q-Euler numbers. Note that $\lim_{\lambda\to 0} \mathcal{E}_{n,\lambda,q}(x) = \mathcal{E}_{n,q}(x)$, where $\mathcal{E}_{n,q}(x)$ are called the Carlitz's q-Euler polynomials (see [10, 13]).

In this paper, we study some identities of symmetry for the degenerate q-Euler polynomials arising from the fermionic p-adic q-integral on $\mathbb{Z}_p$ under symmetry group of degree n.

2. Symmetric identities for the degenerate q-Euler polynomials under S_n

Let S_n be the symmetry group of degree n and let $w_1, w_2 \cdots, w_n$ be odd positive integers. Then, we study the following integral equation for the fermionic p-adic q-integral on $\mathbb{Z}_p$:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda}\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right]_q} d\mu_{-q^{w_1w_2\cdots w_{n-1}}}(y) \\
&= \frac{[2]_{q^{w_1\cdots w_{n-1}}}}{2} \lim_{N\to\infty} \sum_{m=0}^{w_n-1}\sum_{y=0}^{p^N-1} (1+\lambda t)^{\frac{1}{\lambda}\left[\left(\prod_{j=1}^{n-1} w_j\right)\left(m+w_ny\right)+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n}\left(\sum_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right]_q} \\
&\quad \times (-1)^{m+w_ny} q^{w_1w_2\cdots w_{n-1}(m+w_ny)}.
\end{aligned} \tag{2.1}$$

From (2.1), we have

$$\begin{aligned}
&\frac{2}{[2]_{q^{w_1\cdots w_{n-1}}}}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\sum_{i=1}^{n-1}k_i}q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\\
&\times\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}\left[\left(\prod_{j=1}^{n-1}w_j\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_q}d\mu_{-q^{w_1w_2\cdots w_{n-1}}}(y)\\
&=\lim_{N\to\infty}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}\sum_{m=0}^{w_n-1}\sum_{y=0}^{p^N-1}(1+\lambda t)^{\frac{1}{\lambda}\left[\left(\prod_{=1}^{n-1}w_j\right)\left(m+w_ny\right)+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_q}\\
&\times(-1)^{\sum_{i=1}^{n-1}k_i+m+y}q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j+\left(\prod_{j=1}^{n-1}w_j\right)m+\left(\prod_{j=1}^{n}w_j\right)y}.
\end{aligned}\tag{2.2}$$

Thus, by (2.2), we note that

$$\begin{aligned}
I(w_1,w_2,\cdots,w_n)=&\frac{2}{[2]_{q^{w_1\cdots w_{n-1}}}}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\sum_{i=1}^{n-1}k_i}q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\\
&\times\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}\left[\left(\prod_{j=1}^{n-1}w_j\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_q}d\mu_{-q^{w_1w_2\cdots w_{n-1}}}(y)
\end{aligned}\tag{2.3}$$

is invariant for any permutation σ in the symmetry group of degree n. Therefore, by (2.2), we obtain the following theorem.

Theorem 2.1. *For $w_1,\cdots,w_n\in\mathbb{N}$ with $w_i\equiv 1\pmod 2$, $i=1,2,\cdots,n$, we note that $I(w_{\sigma(1)},w_{\sigma(2)},\cdots,w_{\sigma(n)})$ are the same for any $\sigma\in S_n,(n\geq 1)$.*

From the definition of $[x]_q$, we note that

$$\Big[(\prod_{j=1}^{n-1}w_j)y+(\prod_{j=1}^{n}w_j)x+w_n\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_j)k_j\Big]_q=\Big[\prod_{j=1}^{n-1}w_j\Big]_q\Big[y+w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\Big]_{q^{w_1w_2\cdots w_{n-1}}}.\tag{2.4}$$

By (2.4), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}\left[\left(\prod_{j=1}^{n-1}w_j\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_j\right)k_j\right]_q}d\mu_{-q^{w_1\cdots w_{n-1}}}(y)\\
&=\int_{\mathbb{Z}_p}\left(1+\frac{\lambda}{[\prod_{j=1}^{n-1}w_j]_q}\Big[\prod_{j=1}^{n-1}w_j\Big]_qt\right)^{\frac{1}{\lambda}\left[\left(\prod_{j=1}^{n-1}w_j\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_j\right)k_j\right]_q}d\mu_{-q^{w_1\cdots w_{n-1}}}(y)\\
&=\sum_{m=0}^{\infty}\Big[\prod_{j=1}^{n-1}w_j\Big]_q^m\mathcal{E}_{m,\frac{\lambda}{[\prod_{j=1}^{n-1}w_j]_q},q^{w_1\cdots w_{n-1}}}\left(w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right)\frac{t^n}{n!},\quad(n\in\mathbb{N}).
\end{aligned}\tag{2.5}$$

Therefore, by Theorem 2.1 and (2.5), we obtain the following theorem.

Theorem 2.2. *Let $w_1,w_2,\cdots,w_n$ be odd integers and let m be a non-negative integer. Then, the following expressions*

$$\begin{aligned}
&\frac{2}{[2]_{q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}}\Big[\prod_{j=1}^{n-1}w_{\sigma(j)}\Big]_q^m\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1}(-1)^{\sum_{i=1}^{n-1}k_i}q^{w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_{\sigma(i)}\right)k_j}\\
&\times\mathcal{E}_{m,\frac{\lambda}{\left[\prod_{j=1}^{n-1}w_{\sigma(j)}\right]_q},q^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}}}\left(w_{\sigma(n)}x+w_{\sigma(n)}\sum_{j=1}^{n-1}\frac{k_j}{w_{\sigma(j)}}\right)
\end{aligned}$$

are the same for any permutation σ in the symmetry group of degree n.

Now, we observe that

$$\begin{aligned}&\Big[y+w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\Big]_{q^{w_1w_2\cdots w_{n-1}}}\\&=\frac{[w_n]_q}{\big[\prod_{j=1}^{n-1}w_j\big]_q}\Big[\sum_{j=1}^{n-1}\big(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\big)k_j\Big]_{q^{w_n}}+q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}[y+w_nx]_{q^{w_1\cdots w_{n-1}}}.\end{aligned}\tag{2.6}$$

From (1.5), we have

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda,q}(x)\frac{t^n}{n!}=\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[x+y]_q}{\lambda}}d\mu_{-q}(y)=\sum_{n=0}^{\infty}\lambda^n\int_{\mathbb{Z}_p}\left(\frac{[x+y]_q}{\lambda}\right)_n d\mu_{-q}(y)\frac{t^n}{n!}.\tag{2.7}$$

By comparing the coefficients on the both sides of (2.7), we get

$$\mathcal{E}_{n,\lambda,q}(x)=\lambda^n\int_{\mathbb{Z}_p}\left(\frac{[x+y]_q}{\lambda}\right)_n d\mu_{-q}(y),\quad (n\geq 0),\tag{2.8}$$

where

$$\left(\frac{[x+y]_q}{\lambda}\right)_n=\sum_{l=0}^{n}S_1(n,l)\left(\frac{[x+y]_q}{\lambda}\right)^l$$

and $S_1(n,l)$ is the Stirling number of the first kind. From (2.8), we have

$$\begin{aligned}&\mathcal{E}_{m,\frac{\lambda}{\left[\prod_{j=1}^{n-1}w_j\right]_q},q^{w_1\cdots w_{n-1}}}\left(w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right)\\&=\left(\frac{\lambda}{\big[\prod_{j=1}^{n-1}w_j\big]_q}\right)^m\int_{\mathbb{Z}_p}\left(\frac{\big[y+w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\big]_{q^{w_1\cdots w_{n-1}}}}{\lambda}\right)_m d\mu_{-q^{w_1\cdots w_{n-1}}}(y).\end{aligned}\tag{2.9}$$

Now, by (2.6), we observe that

$$\begin{aligned}&\left(\frac{\lambda}{\big[\prod_{j=1}^{n-1}w_j\big]_q}\right)^m\left(\frac{\big[y+w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\big]_{q^{w_1\cdots w_{n-1}}}}{\lambda}\right)_m\\&=\left(\frac{\lambda}{\big[\prod_{j=1}^{n-1}w_j\big]_q}\right)^m\left(\frac{1}{\lambda}\left(\frac{[w_n]_q}{\big[\prod_{j=1}^{n-1}w_j\big]_q}\Big[\sum_{j=1}^{n-1}\big(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\big)k_j\Big]_{q^{w_n}}+q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}[y+w_nx]_{q^{w_1\cdots w_{n-1}}}\right)\right)_m\\&=\left(\frac{\lambda}{\big[\prod_{j=1}^{n-1}w_j\big]_q}\right)^m\sum_{l=0}^{m}\lambda^{-l}S_1(m,l)\\&\quad\times\left(\frac{[w_n]_q}{\big[\prod_{j=1}^{n-1}w_j\big]_q}\Big[\sum_{j=1}^{n-1}\big(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\big)k_j\Big]_{q^{w_n}}+q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}[y+w_nx]_{q^{w_1\cdots w_{n-1}}}\right)^l\\&=\left(\frac{\lambda}{\big[\prod_{j=1}^{n-1}w_j\big]_q}\right)^m\sum_{l=0}^{m}\lambda^{-l}S_1(m,l)\sum_{i=0}^{l}\left(\frac{[w_n]_q}{\big[\prod_{j=1}^{n-1}w_j\big]_q}\right)^{l-i}\Big[\sum_{j=1}^{n-1}\big(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\big)k_j\Big]_{q^{w_n}}^{l-i}\\&\quad\times q^{w_ni\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}[y+w_nx]_{q^{w_1\cdots w_{n-1}}}^i\binom{l}{i}.\end{aligned}\tag{2.10}$$

By (2.9) and (2.10), we get

$$\begin{aligned}
&\mathcal{E}_{m,\frac{\lambda}{\left[\prod_{j=1}^{n-1} w_j\right]_q},q^{w_1\cdots w_{n-1}}}\left(w_n x + w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right)\\
&= \frac{1}{\left[\prod_{j=1}^{n-1} w_j\right]_q^m}\sum_{l=0}^{m}\sum_{i=0}^{l}\binom{l}{i}\lambda^{m-l}S_1(m,l)\Big(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j\right]_q}\Big)^{l-i}\Big[\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i)k_j\Big]_{q^{w_n}}^{l-i}\\
&\times q^{w_n i\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j}\int_{\mathbb{Z}_p}[y+w_n x]_{q^{w_1\cdots w_{n-1}}}^{i}\,d\mu_{-q^{w_1\cdots w_{n-1}}}(y)\\
&= \frac{1}{\left[\prod_{j=1}^{n-1} w_j\right]_q^m}\sum_{l=0}^{m}\sum_{i=0}^{l}\binom{l}{i}\lambda^{m-l}S_1(m,l)\Big(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j\right]_q}\Big)^{l-i}\Big[\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i)k_j\Big]_{q^{w_n}}^{l-i}\\
&\times q^{w_n i\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j}\,\mathcal{E}_{i,q^{w_1\cdots w_{n-1}}}(w_n x).
\end{aligned} \tag{2.11}$$

By Theorem 2.2 and (2.11), we get

$$\begin{aligned}
&\frac{2}{[2]_{q^{w_1\cdots w_{n-1}}}}\Big[\prod_{j=1}^{n-1} w_j\Big]_q^m\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\sum_{i=1}^{n-1}k_i}q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j}\\
&\quad\times\mathcal{E}_{m,\frac{\lambda}{\left[\prod_{j=1}^{n-1} w_j\right]_q},q^{w_1\cdots w_{n-1}}}\left(w_n x + w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right)\\
&= \frac{2}{[2]_{q^{w_1\cdots w_{n-1}}}}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\sum_{i=1}^{n-1}k_i}q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j}\\
&\quad\times\sum_{p=0}^{m}\sum_{i=0}^{p}\binom{p}{i}\lambda^{m-p}S_1(m,p)\Big(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j\right]_q}\Big)^{p-i}\Big[\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i)k_j\Big]_{q^{w_n}}^{p-i}\\
&\quad\times q^{w_n i\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j}\,\mathcal{E}_{i,q^{w_1 w_2\cdots w_{n-1}}}(w_n x)\\
&= \sum_{p=0}^{m}\sum_{i=0}^{p}\binom{p}{i}\lambda^{m-p}S_1(m,p)\Big(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j\right]_q}\Big)^{p-i}\mathcal{E}_{i,q^{w_1 w_2\cdots w_{n-1}}}(w_n x)\\
&\quad\times\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\sum_{i=1}^{n-1}k_i}q^{(i+1)w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j}\Big[\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i)k_j\Big]_{q^{w_n}}^{p-i}\\
&= \sum_{p=0}^{m}\sum_{i=0}^{p}\binom{p}{i}\lambda^{m-p}S_1(m,p)\Big(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j\right]_q}\Big)^{p-i}\mathcal{E}_{i,q^{w_1 w_2\cdots w_{n-1}}}(w_n x)T_{n,q^{w_n}}^{(p)}(w_1,\cdots,w_{n-1}|i+1),
\end{aligned} \tag{2.12}$$

where

$$T_{n,q}^{(p)}(w_1,\cdots,w_{n-1}|i) = \prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\sum_{t=1}^{n-1}k_t}q^{i\sum_{j=1}^{n-1}\left(\prod_{\substack{t=1\\ t\neq j}}^{n-1} w_t\right)k_j}\Big[\sum_{j=1}^{n-1}(\prod_{\substack{t=1\\ t\neq j}}^{n-1} w_t)k_j\Big]_q^{p-i}, \tag{2.13}$$

and $\mathcal{E}_{n,q}(x)$ is the Carlitz's q-Euler polynomials which are given by $\int_{\mathbb{Z}_p}[x+y]_q^n d\mu_{-q}(y) = \mathcal{E}_{n,q}(x)$, $(n \geq 0)$, (see [1, 6, 8]). Therefore, by (2.12), we obtain the following theorem.

Theorem 2.3. *For $w_1, w_2, \cdots, w_n \in \mathbb{N}$ with $w_i \equiv 1$ (mod 2), $(i = 1, 2, \cdots, n)$, and $m \geq 0$, the following expressions*

$$\sum_{p=0}^{m}\sum_{i=0}^{p}\binom{p}{i}\lambda^{m-p}S_1(m,p)\left(\frac{[w_{\sigma(n)}]_q}{\left[\prod_{j=1}^{n-1}w_{\sigma(j)}\right]_q}\right)^{p-i}\mathcal{E}_{i,q^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}}}(w_{\sigma(n)}x)T^{(p)}_{n,q^{w_{\sigma(n)}}}(w_{\sigma(1)},\cdots,w_{\sigma(n-1)}|i+1)$$

are the same for any permutation σ in the symmetry group of degree n.

Acknowledgment

This paper is supported by grant NO 14-11-00022 of Russian Scientific Fund.

References

[1] L. Carlitz, *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc., **76** (1954), 332–350. 1.2, 2

[2] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math., **15** (1979), 51–88. 1.1

[3] D. V. Dolgy, T. Kim, H. I. Kwon, J. J. Seo, *Symmetric identities of degenerate q-Bernoulli polynomials under symmetry group S_3*, Proc. Jangjeon Math. Soc., **19** (2016), 1–9. 1

[4] Y. He, *Symmetric identities for Carlitz's q-Bernoulli numbers and polynomials*, Adv. Difference Equ., **2013** (2013), 10 pages.

[5] T. Kim, *Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials*, J. Difference Equ. Appl., **14** (2008), 1267–1277.

[6] T. Kim, *Some identities on the q-Euler polynomials of higher-order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys., **16** (2009), 484–491. 2

[7] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys., **16** (2009), 93–96.

[8] T. Kim, *New approach to q-Euler polynomials of higher-order*, Russ. J. Math. phys., **17** (2010), 218–225. 2

[9] T. Kim, *Symmetric identities of degenerate Bernoulli polynomials*, Proc. Jangjeon Math. Soc., **18** (2015), 593–599.

[10] T. Kim, D. V. Dolgy, J. J. Seo, *Identities of symmetry for degenerate q-Euler polynomials*, Adv. Stud. Contemp. Math., **25** (2015), 577–582. 1

[11] Y. H. Kim, K. W. Hwang, *Symmetry of power sum and twisted Bernoulli polynomials*, Adv. Stud. Contemp. Math., **18** (2009), 127–133.

[12] D. S. Kim, T. Kim, *Some identities of degenerate Euler polynomials arising from p-adic fermionic integral on $\mathbb{Z}_p$*, Integral Transforms Spec. Funct., **26** (2015), 295–302. 1.1, 1.4

[13] D. S. Kim, T. Kim, *Symmetric identities of higher-order degenerate q-Euler polynomials*, J. Nonlinear Sci. Appl., **9** (2016), 443–451. 1, 1

[14] D. S. Kim, T. Kim, *Some identities of symmetry for q-Bernoulli polynomials under symmetric group of degree n*, Ars Combin., **126** (2016), 435–441.

[15] D. S. Kim, T. Kim, *Symmetry identities of higher-order q-Euler polynomials under the symmetric group of degree four*, J. Comput. Anal. Appl., **21** (2016), 521–527.

[16] T. Kim, H. I. Kwon, J. J. Seo, *Identities of symmetry for degenerate q-Bernoulli polynomials*, Proc. Jangjeon Math. Soc., **18** (2015), 495–499.

[17] D. S. Kim, N. Lee, J. Na, K. H. Park, *Identities of symmetry for higher-order Euler polynomials in three variables (I)*, Adv. Stud Contemp. Math., **22** (2012), 51–74.

[18] D. S. Kim, N. Lee, J. Na, K. H. Park, *Abundant symmetry for higher-order Bernoulli polynomials (II)*, Proc. Jangjeon Math. Soc., **16** (2013), 359–378.

[19] H. I. Kwon, T. Kim, J. J. Seo, *Some identities of symmetry for modified degenerate Frobenius-Euler polynomials*, Adv. Stud. Contemp. Math., **26** (2016), 299–305.

[20] E. J. Moon, S. H. Rim, J. H. Jin, S. J. Lee, *On the symmetric properties of higher-order twisted q-Euler numbers and polynomials*, Adv. Difference Equ., **2010** (2010), 8 pages. 1, 1.2