Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4814–4828
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Common best proximity results for multivalued proximal contractions in metric space with applications

Nawab Hussain[a,*], Abdul Rahim Khan[b], Iram Iqbal[c]

[a] *Department of Mathematics, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia.*
[b] *Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals Dhahran, Saudi Arabia.*
[c] *Department of Mathematics, University of Sargodha, Sargodha, Pakistan.*

Communicated by Z. Kadelburg

Abstract

The study of the best proximity points is an interesting topic of optimization theory. We introduce the notion of α_*-proximal contractions for multivalued mappings on a complete metric space and establish the existence of common best proximity point for these mappings in the context of multivalued and single-valued mappings. As an application, we derive some best proximity point and fixed point results for multivalued and single-valued mappings on partially ordered metric spaces. Our results generalize and extend many known results in the literature. Some examples are provided to illustrate the results obtained herein.

Keywords: α_*-proximal admissible mapping, common best proximity point, multivalued mapping.
2010 MSC: 46N40, 47H10, 54H25, 46T99.

1. Introduction and preliminaries

Fixed point theory concerns with some techniques to find a solution of the pattern $\mathcal{T}x = x$, where $\mathcal{T}$ is a self-mapping defined on a subset $\mathcal{A}$ of a metric space (X, d). A well-known principle that guarantees a unique fixed point solution is the Banach contraction principle [9]. Over the years, this principle has been generalized

*Corresponding author
Email addresses: `nhusain@kau.edu.sa` (Nawab Hussain), `arahim@kfupm.edu.sa` (Abdul Rahim Khan), `irami@uos.edu.pk` (Iram Iqbal)

Received 2016-06-03

in many ways (see [5, 7–15, 28, 29]). An interesting generalization of the Banach contraction principle is for multivalued mappings and is known as Nadler's fixed point theorem [24]. In 1982, Sessa [31] defined the concept of weakly commuting mappings to obtain common fixed point for pair of such mappings. Jungck generalized this idea, first to compatible mappings [18] and then to weakly compatible mappings [19]. A mapping $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ does not necessarily have a fixed point, where $\mathcal{A}$ and $\mathcal{B}$ are nonempty subsets of a metric space $\mathcal{X}$. One can proceed to find an element $x \in \mathcal{A}$ in the sense that the distance $d(x, \mathcal{T}x)$ is minimum. Fan's best approximation theorem [13] asserts that if K is a nonempty, compact, and convex subset of a normed space X and $\mathcal{T} : K \to X$ is a continuous mapping, then there exists an element x satisfying the condition $d(x, \mathcal{T}x) = \inf ||y - \mathcal{T}x||, y \in K$. A best approximation theorem guarantees the existence of an approximate solution, while a best proximity point theorem provides an approximate solution which is optimal in the sense that there exists an element x such that $d(x, \mathcal{T}x) = dist(\mathcal{A}, \mathcal{B}) = \inf\{d(x, y) : x \in \mathcal{A} \text{ and } y \in \mathcal{B}\}$; the element x is called a best proximity point of $\mathcal{T}$. Moreover, if the mapping under consideration is a self-mapping, then a best proximity point is reduced to a fixed point. The existence of best proximity points is an interesting aspect of optimization theory and it has attracted the attention of many authors (see [1, 6–8, 12, 15, 16, 20–22] and references therein). Moreover, the best proximity point theorems for several classes of multivalued mappings have been probed in [4, 14, 30].

For non-empty subsets $\mathcal{A}$ and $\mathcal{B}$ of the metric space $\mathcal{X}$, the following notions will be used:

$$\begin{aligned}
&\operatorname{dist}(\mathcal{A}, \mathcal{B}) = \inf\{d(a, b) : a \in \mathcal{A}, b \in \mathcal{B}\},\ D(x, \mathcal{B}) = \inf\{d(x, b) : b \in \mathcal{B}\},\\
&\mathcal{A}_0 = \{a \in \mathcal{A} : d(a, b) = \operatorname{dist}(\mathcal{A}, \mathcal{B}) \text{ for some } b \in \mathcal{B}\},\\
&\mathcal{B}_0 = \{b \in \mathcal{B} : d(a, b) = \operatorname{dist}(\mathcal{A}, \mathcal{B}) \text{ for some } a \in \mathcal{A}\},
\end{aligned}$$

$2^{\mathcal{X}}$ is the set of all nonempty subsets of $\mathcal{X}$, $CL(\mathcal{X})$ is the set of all nonempty closed subsets of $\mathcal{X}$, $K(\mathcal{X})$ is the set of all compact subsets of $\mathcal{X}$ for every $\mathcal{A}, \mathcal{B} \in CL(\mathcal{X})$, $H(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{x\in\mathcal{A}} D(x, \mathcal{B}), \sup_{y\in\mathcal{B}} D(y, \mathcal{A})\right\}$ if the maximum exists and $H(\mathcal{A}, \mathcal{B}) = 0$ otherwise, and let Ψ be the collection of all non-decreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

We present now the necessary definitions and results which will be useful in the sequel.

Definition 1.1 ([23])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of a metric space (X, d). A point x is called a common best proximity point of mappings $\mathcal{T}_i : \mathcal{A} \to \mathcal{B}, (i = 1, 2, ..., n)$ if

$$D(x, \mathcal{T}_i x) = \operatorname{dist}(A, B).$$

Lemma 1.2 ([5])**.** *Let (X, d) be a metric space and $B \in CL(X)$. Then for each $x \in X$ with $d(x, B) > 0$ and $q > 1$, there exists an element $b \in B$ such that*

$$d(x, b) < qd(x, B).$$

Definition 1.3 ([6])**.** Let $(\mathcal{A}, \mathcal{B})$ be a pair of nonempty subsets of a metric space (X, d) with $\mathcal{A}_0 \neq \emptyset$. Then the pair $(\mathcal{A}, \mathcal{B})$ is said to have the weak P-property if and only if for any $x_1, x_2 \in \mathcal{A}$ and $y_1, y_2 \in \mathcal{B}$,

$$\left.\begin{array}{l} d(x_1, y_1) = \operatorname{dist}(A, B)\\ d(x_2, y_2) = \operatorname{dist}(A, B) \end{array}\right\} \quad \Rightarrow \quad d(x_1, x_2) \leq d(y_1, y_2).$$

Definition 1.4 ([6])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of a metric space $(\mathcal{X}, d)$. A mapping $\mathcal{T} : \mathcal{A} \to 2^{\mathcal{B}} \setminus \emptyset$ is called α-proximal admissible if there exists a mapping $\alpha : A \times A \to [0, \infty)$ such that

$$\left.\begin{array}{c} \alpha(x_1, x_2) \geq 1\\ d(u_1, y_1) = \operatorname{dist}(A, B)\\ d(u_2, y_2) = \operatorname{dist}(A, B) \end{array}\right\} \quad \Rightarrow \quad \alpha(u_1, u_2) \geq 1,$$

where $x_1, x_2, u_1, u_2 \in A, y_1 \in Tx_1$ and $y_2 \in Tx_2$.

Definition 1.5 ([6])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of a metric space $(\mathcal{X}, d)$. A mapping $\mathcal{T} : \mathcal{A} \to CL(\mathcal{B})$ is said to be an α-ψ-proximal contraction, if there exist $\psi \in \Psi$ and $\alpha : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that

$$\alpha(x, y)H(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in A. \tag{1.1}$$

In this paper, we generalize the above mentioned notions for a pair of multivalued and single-valued mappings and define α_*-proximal admissible with respect to $\eta : \mathcal{A} \times \mathcal{A} \to [0, \infty)$, α-proximal admissible with respect to $\eta : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ and prove common best proximity point theorems as well as fixed point theorems for these mappings. Our results generalize and improve the results of Ali et al. [6], Jungck ([18], [19]), Samet et al. [29], and Hussain et al. [17].

2. Common best proximity points for multivalued mappings

We begin this section with a definition.

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of a metric space (X, d) and $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{A} \to 2^{\mathcal{B}} \setminus \emptyset$ be multivalued mappings. The pair $(\mathcal{T}_1, \mathcal{T}_2)$ is α_*-proximal admissible with respect to η if there exist $\alpha, \eta : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that for $z_1, z_2, u_1, u_2 \in \mathcal{A}$,

$$\left.\begin{array}{c} \alpha(z_1, z_2) \geq \eta(z_1, z_2) \\ d(u_1, y_1) = \operatorname{dist}(\mathcal{A}, \mathcal{B}) \\ d(u_2, y_2) = \operatorname{dist}(\mathcal{A}, \mathcal{B}) \end{array}\right\} \quad \Rightarrow \quad \alpha(u_1, u_2) \geq \eta(u_1, u_2)$$

for all $y_1 \in \mathcal{T}_i z_1$ and $y_2 \in \mathcal{T}_j z_2$, $i, j \in \{1, 2\}$. When $\alpha(z_1, z_2) = 1$ for all $z_1, z_2 \in \mathcal{A}$, the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is called η_*-proximal sub-admissible, and when $\eta(z_1, z_2) = 1$ for all $z_1, z_2 \in \mathcal{A}$, the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is called α_*-proximal admissible.

Example 2.2. Consider $X = \mathbb{R}^2$ with the usual metric. Suppose $\mathcal{A} = \{(1, x) : 0 \leq x \leq 1\}$ and $\mathcal{B} = \{(0, x) : 0 \leq x \leq 1\}$. Define $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{A} \to 2^{\mathcal{B}} \setminus \emptyset$ by

$$\mathcal{T}_1(1, x) = \begin{cases} \{(0, 1)\} & x = 1, \\ \left\{\left(0, \frac{a}{2}\right) : 0 \leq a \leq x\right\} & \text{otherwise}, \end{cases}$$

$$\mathcal{T}_2(1, x) = \begin{cases} \left\{\left(0, \frac{a}{2}\right) : 0 \leq a \leq x\right\} & x \in \left[0, \frac{1}{2}\right], \\ \left\{\left(0, a^2\right) : 0 \leq a \leq x\right\} & x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

and $\alpha, \eta : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ by

$$\alpha((1, x), (1, y)) = \begin{cases} 4/5 & x, y \in \left[0, \frac{1}{2}\right], \\ 1/2 & \text{otherwise}, \end{cases}$$

$$\eta((1, x), (1, y)) = \frac{3}{4}$$

for all $(1, x), (1, y) \in \mathcal{A} \times \mathcal{A}$. If $z_1 = (1, x_1)$ and $z_2 = (1, x_2)$ in $\mathcal{A}$, then $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$ if $x_1, x_2 \in \left[0, \frac{1}{2}\right]$. So, $\mathcal{T}_1 z_1 = \left\{\left(0, \frac{a}{2}\right) : 0 \leq a \leq x_1\right\}$ and $\mathcal{T}_2 z_2 = \left\{\left(0, \frac{a}{2}\right) : 0 \leq a \leq x_2\right\}$. This shows that $d(u_1, y_1) = 1 = \operatorname{dist}(\mathcal{A}, \mathcal{B})$ and $d(u_2, y_2) = 1 = \operatorname{dist}(\mathcal{A}, \mathcal{B})$ for all $y_1 \in \mathcal{T}_i x_1$ and $y_2 \in \mathcal{T}_j x_2$, $i, j \in \{1, 2\}$ if and only if $u_1, u_2 \in \left\{\left(1, \frac{x}{2}\right) : 0 \leq x \leq \frac{1}{2}\right\}$. Hence $\alpha(u_1, u_2) = \frac{4}{5} > \frac{3}{4} = \eta(u_1, u_2)$. Thus the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is α_*-proximal admissible with respect to η.

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space (X, d) such that $\mathcal{A}_0$ is non-empty and $\mathcal{T}$, $\mathcal{S} : \mathcal{A} \to CL(\mathcal{B})$ be continuous multivalued mappings satisfying the following assertions:*

1. $\alpha(z_1, z_2) \geq \eta(z_1, z_2) \Rightarrow H(\mathcal{T}z_1, \mathcal{S}z_2) \leq \psi(d(z_1, z_2))$;
2. $\mathcal{T}z, \mathcal{S}z \subseteq \mathcal{B}_0$ *for each* $z \in \mathcal{A}_0$ *and* $(\mathcal{A}, \mathcal{B})$ *satisfies the weak P-property;*

3. $(\mathcal{T}, \mathcal{S})$ *is* α_**-proximal admissible with respect to* η*;*
4. *there exists* $z_0, z_1, z_2 \in \mathcal{A}_0$*,* $y_1 \in \mathcal{T}z_0$ *and* $y_2 \in \mathcal{S}z_0$ *such that*

$$d(z_1, y_1) = dist(\mathcal{A}, \mathcal{B}), \qquad \alpha(z_0, z_1) \geq \eta(z_0, z_1)$$

and

$$d(z_2, y_2) = dist(\mathcal{A}, \mathcal{B}), \qquad \alpha(z_0, z_2) \geq \eta(z_0, z_2).$$

Then the mappings $\mathcal{T}$ *and* $\mathcal{S}$ *have a common best proximity point.*

Proof. By the hypothesis, there exists $z_0, z_1 \in \mathcal{A}_0$ and $y_1 \in \mathcal{T}z_0$ such that

$$d(z_1, y_1) = \mathrm{dist}(\mathcal{A}, \mathcal{B}), \alpha(z_0, z_1) \geq \eta(z_0, z_1). \tag{2.1}$$

If $y_1 \in \mathcal{T}z_1 \cap \mathcal{S}z_1$, then z_1 is the common best proximity point of $\mathcal{T}$ and $\mathcal{S}$. If $y_1 \notin \mathcal{S}z_1$, then from condition 1, we have

$$0 < d(y_1, \mathcal{S}z_1) \leq H(\mathcal{T}z_0, \mathcal{S}z_1) \leq \psi(d(z_0, z_1)).$$

For $q > 1$, it follows from Lemma 1.2 that there exists $y_2 \in \mathcal{S}z_1$ such that

$$\begin{aligned} 0 < d(y_1, y_2) &< qd(y_1, \mathcal{S}z_1) \\ &\leq qH(\mathcal{T}z_0, \mathcal{S}z_1) \\ &\leq q\psi((d(z_0, z_1))). \end{aligned} \tag{2.2}$$

As $y_2 \in \mathcal{S}z_1 \subseteq \mathcal{B}_0$, there exists $z_2 \neq z_1 \in \mathcal{A}_0$ such that

$$d(z_2, y_2) = \mathrm{dist}(\mathcal{A}, \mathcal{B}), \tag{2.3}$$

otherwise, z_1 is the common best proximity point of $\mathcal{T}$ and $\mathcal{S}$. As $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property, (2.1) and (2.3) imply that

$$0 < d(z_1, z_2) \leq d(y_1, y_2). \tag{2.4}$$

From (2.2) and (2.4), we have

$$0 < d(z_1, z_2) \leq q\psi(d(z_0, z_1)).$$

Since ψ is non-decreasing, from the above inequality, we have

$$\psi(d(z_1, z_2)) \leq \psi(q\psi(d(z_0, z_1))).$$

Put $q_1 = \frac{\psi(q\psi(d(z_0,z_1)))}{\psi(d(z_1,z_2))}$. As the pair $(\mathcal{T}, \mathcal{S})$ is α_*-proximal admissible with respect to η, so, $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$. Thus, we have

$$d(z_2, y_2) = \mathrm{dist}(\mathcal{A}, \mathcal{B}), \qquad \alpha(z_1, z_2) \geq \eta(z_1, z_2). \tag{2.5}$$

Now, if $y_2 \in \mathcal{T}z_2 \cap \mathcal{S}z_2$, then z_2 is the common best proximity point of $\mathcal{T}$ and $\mathcal{S}$. If $y_2 \notin \mathcal{T}z_2$, then from condition 1, we have

$$0 < d(\mathcal{T}z_2, y_2) \leq H(\mathcal{T}z_2, \mathcal{S}z_1) \leq \psi(d(z_1, z_2)).$$

For $q_1 > 1$, it follows from Lemma 1.2 that there exists $y_3 \in \mathcal{T}z_2$ such that

$$\begin{aligned} 0 < d(y_2, y_3) &< q_1 d(y_2, \mathcal{T}z_2) \\ &\le q_1 H(\mathcal{S}z_1, \mathcal{T}z_2) \\ &\le q_1 \psi((d(z_1, z_2))) \\ &= \psi(q\psi((d(z_0, z_1))). \end{aligned} \tag{2.6}$$

As $y_3 \in \mathcal{T}z_2 \subseteq \mathcal{B}_0$, so there exists $z_3 \neq z_2 \in \mathcal{A}_0$ such that

$$d(z_3, y_3) = \operatorname{dist}(\mathcal{A}, \mathcal{B}), \tag{2.7}$$

otherwise, z_2 is the common best proximity point of $\mathcal{T}$ and $\mathcal{S}$. As $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property, (2.5) and (2.7) imply that

$$0 < d(z_2, z_3) \le d(y_2, y_3). \tag{2.8}$$

From (2.6) and (2.8), we have

$$0 < d(z_2, z_3) \le \psi(q\psi(d(z_0, z_1))).$$

Since ψ is strictly increasing, from the above inequality, we have

$$\psi(d(z_2, z_3)) < \psi^2(q\psi(d(z_0, z_1))).$$

Put $q_2 = \frac{\psi^2(q\psi(d(z_0,z_1)))}{\psi(d(z_2,z_3))}$. As the pair $(\mathcal{T}, \mathcal{S})$ is α_*-proximal admissible with respect to η, so, $\alpha(z_2, z_3) \ge \eta(z_2, z_3)$. Thus, we have

$$d(z_3, y_3) = \operatorname{dist}(\mathcal{A}, \mathcal{B}), \qquad \alpha(z_2, z_3) \ge \eta(z_2, z_3).$$

Now proceeding in the manner described above, we get a sequence $\{z_n\}$ in $\mathcal{A}_0$ and $\{y_n\}$ in $\mathcal{B}_0$ such that for $n \in \mathbb{N}$

$$y_{2n+1} \in \mathcal{T}z_{2n} \qquad \text{and} \qquad y_{2n} \in \mathcal{T}z_{2n-1}, \tag{2.9}$$

where

$$d(z_{n+1}, y_{n+1}) = \operatorname{dist}(\mathcal{A}, \mathcal{B}), \qquad \alpha(z_n, z_{n+1}) \ge \eta(z_n, z_{n+1}), \quad \forall n \in \mathbb{N} \tag{2.10}$$

and

$$d(y_{n+1}, y_{n+2}) < \psi^n(q\psi(d(z_0, z_1))), \quad \forall n \in \mathbb{N}. \tag{2.11}$$

As $y_{n+2} \in \mathcal{T}z_{n+1} \cup \mathcal{S}z_{n+1}$ and $\mathcal{T}z_{n+1}, \mathcal{S}z_{n+1} \subseteq \mathcal{B}_0$ for all $n \in \mathbb{N}$, so there exists $z_{n+2} \neq z_{n+1} \in \mathcal{A}_0$ such that

$$d(z_{n+2}, y_{n+2}) = \operatorname{dist}(\mathcal{A}, \mathcal{B}), \qquad \forall n \in \mathbb{N}. \tag{2.12}$$

Since $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property, from (2.10) and (2.12), we have

$$d(z_{n+1}, z_{n+2}) \le d(y_{n+1}, y_{n+2}), \qquad \forall n \in \mathbb{N}. \tag{2.13}$$

From (2.11) and (2.13), we get

$$d(z_{n+1}, z_{n+2}) < \psi^n(q\psi(d(z_0, z_1))), \quad \forall n \in \mathbb{N}.$$

Now for $n > m$, we have

$$d(z_n, z_m) \le \sum_{i=n}^{m-1} d(z_i, z_{i+1}) < \sum_{i=n}^{m-1} \psi^{i-1}(q\psi(d(z_0, z_1))).$$

Hence $\{z_n\}$ is a Cauchy sequence in $\mathcal{A}$. Similarly, $\{y_n\}$ is a Cauchy sequence in $\mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B}$ are closed subsets of a complete metric space (X, d), there exist $z^* \in \mathcal{A}$ and $y^* \in \mathcal{B}$ such that $z_n \to z^*$ and $y_n \to y^*$ as $n \to \infty$. By taking limit as $n \to \infty$ in equation (2.12), we get that

$$d(z^*, y^*) = \operatorname{dist}(\mathcal{A}, \mathcal{B}).$$

Since $\mathcal{T}$ and $\mathcal{S}$ are continuous, therefore from (2.9), we get that $y^* \in \mathcal{T}z^* \cap \mathcal{S}z^*$. Hence

$$\text{dist}(\mathcal{A},\mathcal{B}) \leq D(z^*,\mathcal{T}z^*) \leq d(z^*,y^*) = \text{dist}(\mathcal{A},\mathcal{B})$$

and

$$\text{dist}(\mathcal{A},\mathcal{B}) \leq D(z^*,\mathcal{S}z^*) \leq d(z^*,y^*) = \text{dist}(\mathcal{A},\mathcal{B}).$$

This implies that $D(z^*,\mathcal{T}z^*) = D(z^*,\mathcal{S}z^*) = \text{dist}(\mathcal{A},\mathcal{B})$, that is, z^* is a common best proximity point of $\mathcal{T}$ and $\mathcal{S}$. □

Example 2.4. Consider X, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{T}_1,\mathcal{T}_2 : \mathcal{A} \to 2^{\mathcal{B}} \setminus \emptyset$ and $\alpha,\eta : \mathcal{A}\times\mathcal{A} \to [0,\infty)$ as in Example 2.2. Then $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{B}_0 = \mathcal{B}$, $\text{dist}(\mathcal{A},\mathcal{B}) = 1$ and $\mathcal{T}_1z,\mathcal{T}_2z \subseteq \mathcal{B}_0$ for each $z \in \mathcal{A}_0$. As $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{B}_0 = \mathcal{B}$, so for $z_1 = (1,x_1), z_2 = (1,x_2) \in \mathcal{A}$, there exist $y_1 = (0,x_1), y_2 = (0,x_2) \in \mathcal{B}$ such that $d(z_1,y_1) = d(z_2,y_2) = \text{dist}(\mathcal{A},\mathcal{B})$ and $d(z_1,z_2) = |x_1 - x_2| = d(y_1,y_2)$. Hence the pair $(\mathcal{A},\mathcal{B})$ satisfies the weak P-property and the pair $(\mathcal{T}_1,\mathcal{T}_2)$ is α_*-proximal admissible map with respect to η (see Example 2.2). Let $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. Note that $\alpha(z_1,z_2) \geq \eta(z_1,z_2)$ if $x_1,x_2 \in \left[0,\frac{1}{2}\right]$. Therefore,

$$\begin{aligned} H(\mathcal{T}_1z_1,\mathcal{T}_2z_2) &= \left|\frac{x_1}{2} - \frac{x_2}{2}\right| \\ &= \frac{1}{2}|x_1 - x_2| \\ &= \psi(d(z_1,z_2)). \end{aligned}$$

Also, for $z_0 = \left(1,\frac{1}{2}\right) \in \mathcal{A}_0$, $y_1 = \left(0,\frac{1}{4}\right) \in \mathcal{T}_1x_0$ and $y_2 = \left(0,\frac{1}{8}\right) \in \mathcal{T}_2x_0$, we have $z_1 = \left(1,\frac{1}{4}\right), z_2 = \left(1,\frac{1}{8}\right) \in \mathcal{A}_0$ such that $d(z_1,y_1) = d(z_2,y_2) = 1 = \text{dist}(\mathcal{A},\mathcal{B})$, $\alpha(z_0,z_1) = \frac{4}{5} \geq \frac{3}{4} = \eta(z_0,z_1)$ and $\alpha(z_0,z_2) = \frac{4}{5} \geq \frac{3}{4} = \eta(z_0,z_2)$. Thus all the conditions of Theorem 2.3 are satisfied and $(1,1)$ is a common best proximity point of $\mathcal{T}_1$ and $\mathcal{T}_2$.

The case $\eta(z_1,z_2) = 1$, reduces Theorem 2.3 to the following:

Corollary 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space (X,d) such that $\mathcal{A}_0$ is non-empty and $\mathcal{T},\mathcal{S} : \mathcal{A} \to CL(\mathcal{B})$ be continuous multivalued mappings satisfying the following assertions:*

1. $\alpha(z_1,z_2) \geq 1 \Rightarrow H(\mathcal{T}z_1,\mathcal{S}z_2) \leq \psi(d(z_1,z_2))$;
2. $\mathcal{T}z,\mathcal{S}z \subseteq \mathcal{B}_0$ *for each* $z \in \mathcal{A}_0$ *and* $(\mathcal{A},\mathcal{B})$ *satisfies the weak P-property;*
3. $(\mathcal{T},\mathcal{S})$ *is* α_**-proximal admissible;*
4. *there exist* $z_0,z_1,z_2 \in \mathcal{A}_0$, $y_1 \in \mathcal{T}z_0$ *and* $y_2 \in \mathcal{S}z_0$ *such that*

$$d(z_1,y_1) = \mathit{dist}(\mathcal{A},\mathcal{B}), \qquad \alpha(z_0,z_1) \geq 1$$

and

$$d(z_2,y_2) = \mathit{dist}(\mathcal{A},\mathcal{B}), \qquad \alpha(z_0,z_2) \geq 1.$$

Then the mappings $\mathcal{T}$ and $\mathcal{S}$ have a common best proximity point.

If we take $\alpha(z_1,z_2) = 1$ in Theorem 2.3, then we have the following:

Corollary 2.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space (X,d) such that $\mathcal{A}_0$ is non-empty and $\mathcal{T},\mathcal{S} : \mathcal{A} \to CL(\mathcal{B})$ be continuous multivalued mappings satisfying the following assertions:*

1. $\eta(z_1,z_2) \leq 1 \Rightarrow H(\mathcal{T}z_1,\mathcal{S}z_2) \leq \psi(d(z_1,z_2))$;
2. $\mathcal{T}z,\mathcal{S}z \subseteq \mathcal{B}_0$ *for each* $z \in \mathcal{A}_0$ *and* $(\mathcal{A},\mathcal{B})$ *satisfies the weak P-property;*
3. $(\mathcal{T},\mathcal{S})$ *is* η_**-proximal subadmissible;*
4. *there exist* $z_0,z_1,z_2 \in \mathcal{A}_0$, $y_1 \in \mathcal{T}z_0$ *and* $y_2 \in \mathcal{S}z_0$ *such that*

$$d(z_1,y_1) = \mathit{dist}(\mathcal{A},\mathcal{B}), \qquad \eta(z_0,z_1) < 1$$

and

$$d(z_2,y_2) = \mathit{dist}(\mathcal{A},\mathcal{B}), \qquad \eta(z_0,z_2) < 1.$$

Then the mappings $\mathcal{T}$ and $\mathcal{S}$ have a common best proximity point.

In case, $\mathcal{T}_1 = \mathcal{T}_2$, Definition 2.1 and Theorem 2.3 is reduced to the following:

Definition 2.7. Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of a metric space (X, d) and $\mathcal{T} : \mathcal{A} \to 2^{\mathcal{B}} \setminus \emptyset$ be a multivalued mapping. We say that $\mathcal{T}$ is α_*-proximal admissible with respect to η if there exist two functions $\alpha, \eta : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that for $z_1, z_2, u_1, u_2 \in \mathcal{A}$,

$$\left.\begin{array}{r}\alpha(z_1, z_2) \geq \eta(z_1, z_2)\\ d(u_1, y_1) = \operatorname{dist}(\mathcal{A}, \mathcal{B})\\ d(u_2, y_2) = \operatorname{dist}(\mathcal{A}, \mathcal{B})\end{array}\right\} \quad \Rightarrow \quad \alpha(u_1, u_2) \geq \eta(u_1, u_2)$$

for all $y_1 \in \mathcal{T} z_1$ and $y_2 \in \mathcal{T} z_2$. When $\alpha(z_1, z_2) = 1$ for all $z_1, z_2 \in \mathcal{A}$, $\mathcal{T}$ is called η-proximal sub-admissible.

Theorem 2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space (X, d) such that $\mathcal{A}_0$ is nonempty and $\mathcal{T} : A \to CL(\mathcal{B})$ be a continuous multivalued mapping satisfying the following assertions:*

1. $\alpha(z_1, z_2) \geq \eta(z_1, z_2) \Rightarrow H(\mathcal{T} z_1, \mathcal{T} z_2) \leq \psi(d(z_1, z_2))$;
2. $\mathcal{T} z \subseteq \mathcal{B}_0$ *for each* $z \in \mathcal{A}_0$ *and* $(\mathcal{A}, \mathcal{B})$ *satisfies the weak P-property;*
3. $\mathcal{T}$ *is* α_**-proximal admissible with respect to* η*;*
4. *there exist* $z_0, z_1 \in \mathcal{A}_0$, $y_1 \in \mathcal{T} z_0$ *such that*

$$d(z_1, y_1) = dist(\mathcal{A}, \mathcal{B}), \qquad \alpha(z_0, z_1) \geq \eta(z_0, z_1).$$

Then the mapping $\mathcal{T}$ has a best proximity point.

If we take $\eta(z_1, z_2) = 1$ in Theorem 2.8, then we have the following:

Corollary 2.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space (X, d) such that $\mathcal{A}_0$ is nonempty and $\mathcal{T} : A \to CL(\mathcal{B})$ be a continuous multivalued mapping satisfying the following assertions:*

1. $\alpha(z_1, z_2) \geq 1 \Rightarrow H(\mathcal{T} z_1, \mathcal{T} z_2) \leq \psi(d(z_1, z_2))$;
2. $\mathcal{T} z \subseteq \mathcal{B}_0$ *for each* $z \in \mathcal{A}_0$ *and* $(\mathcal{A}, \mathcal{B})$ *satisfies the weak P-property;*
3. $\mathcal{T}$ *is* α*-proximal admissible;*
4. *there exist* $z_0, z_1 \in \mathcal{A}_0$, $y_1 \in \mathcal{T} z_0$ *such that*

$$d(z_1, y_1) = dist(\mathcal{A}, \mathcal{B}), \qquad \alpha(z_0, z_1) \geq 1.$$

Then the mapping $\mathcal{T}$ has a best proximity point.

Remark 2.10. The special case of Theorem 2.8 for $\alpha(z_1, z_2) = 1$ can be obtained as in Corollary 2.6.

Remark 2.11. When $\eta(z_1, z_2) = 1$ for all $z_1, z_2 \in \mathcal{A}$, Definition 2.7 reduces to Definition 10 in [6]. As the condition 1 is more general than the inequality (1.1) (see Remark 3.5 in [5]), so Corollary 2.9 extends Theorem 13 in [6].

Remark 2.12. When $\mathcal{A} = \mathcal{B}$, Theorem 2.8 is reduced to the Theorem 3.3 in [5].

Remark 2.13. Note that the uniqueness of the common best proximity points of multivalued mappings $\mathcal{T}$ and $\mathcal{S}$ is not given in Theorem 2.3. Thus, we can present the following problem: Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T}, \mathcal{S} : \mathcal{A} \to CL(\mathcal{B})$ be continuous multivalued mappings satisfying all the assertions of Theorem 2.3. Does $\mathcal{T}$ and $\mathcal{S}$ have a unique common best proximity point? By adding a condition and taking mappings $\mathcal{T}, \mathcal{S} : \mathcal{A} \to K(\mathcal{B})$, we can give a partial answer of this problem as follows:

Theorem 2.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space (X, d) such that $\mathcal{A}_0$ is non-empty and $\mathcal{T}, \mathcal{S} : \mathcal{A} \to K(\mathcal{B})$ be continuous multivalued mappings satisfying all the assertions of Theorem 2.3 and also satisfy*

H. $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$ *for all common best proximity points of $\mathcal{T}$ and $\mathcal{S}$.*

Then the mappings $\mathcal{T}$ and $\mathcal{S}$ have a unique common best proximity point.

Proof. We will only prove the part of uniqueness. Let z_1, z_2 be two common best proximity points of $\mathcal{T}$ and $\mathcal{S}$ such that $z_1 \neq z_2$, then by hypothesis H we have $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$ and $D(z_1, \mathcal{T}z_1) = \text{dist}(\mathcal{A}, \mathcal{B}) = D(z_1, \mathcal{S}z_1) = D(z_2, \mathcal{T}z_2) = D(z_2, \mathcal{S}z_2)$. Since $\mathcal{T}z_1$ and $\mathcal{S}z_2$ are compact, so there exist an element $u_1 \in \mathcal{T}z_1$ and $u_2 \in \mathcal{S}z_2$ such that

$$d(z_1, u_1) = D(z_1, \mathcal{T}z_1)$$

and

$$d(z_2, u_2) = D(z_2, \mathcal{S}z_2).$$

Since the pair $(\mathcal{T}, \mathcal{S})$ satisfies the weak P-property, so we have

$$d(z_1, z_2) = d(u_1, u_2).$$

So by using condition 1 and Lemma 1.2 there exists $q > 1$ such that

$$\begin{aligned} d(z_1, z_2) = d(u_1, u_2) &< qD(u_1, \mathcal{S}z_2) \\ &< qH(\mathcal{T}z_1, \mathcal{S}z_2) \\ &< q\psi(d(z_1, z_2)) \\ &< qd(z_1, z_2), \end{aligned}$$

which is a contradiction. This implies that $d(z_1, z_2) = 0$, consequently, $\mathcal{T}$ and $\mathcal{S}$ have a unique common best proximity point. □

By similar arguments as in Theorem 2.14, we state the following:

Theorem 2.15. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space (X, d) such that $\mathcal{A}_0$ is nonempty and $\mathcal{T} : A \to K(\mathcal{B})$ be a continuous multivalued mapping satisfying all the assertions of Theorem 2.8 with condition H, then $\mathcal{T}$ has a unique common best proximity point.*

3. Common best proximity points for single-valued mappings

We start with the following definition:

Definition 3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of a metric space (X, d) and $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{A} \to \mathcal{B}$ be mappings. The pair $(\mathcal{T}_1, \mathcal{T}_2)$ is α-proximal admissible with respect to η if there exist two functions $\alpha, \eta : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that for $z_1, z_2, u_1, u_2 \in \mathcal{A}$,

$$\left.\begin{array}{r} \alpha(z_1, z_2) \geq \eta(z_1, z_2) \\ d(u_1, \mathcal{T}_1 z_1) = \text{dist}(\mathcal{A}, \mathcal{B}) \\ d(u_2, \mathcal{T}_2 z_2) = \text{dist}(\mathcal{A}, \mathcal{B}) \end{array}\right\} \quad \Rightarrow \quad \alpha(u_1, u_2) \geq \eta(u_1, u_2).$$

When $\alpha(z_1, z_2) = 1$ for all $z_1, z_2 \in \mathcal{A}$, the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is called η-proximal subadmissible and when $\eta(z_1, z_2) = 1$ for all $z_1, z_2 \in \mathcal{A}$, the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is called α-proximal admissible.

Example 3.2. Consider $\mathcal{X} = \mathbb{R}^2$ with the usual metric. Let $\mathcal{A} = \{(-6, 0), (0, -6), (0, 5)\}$ and $\mathcal{B} = \{(-1, 0), (0, -1), (0, 0), (-1, 1), (1, 1)\}$ be closed subsets of (X, d). Then $d(\mathcal{A}, \mathcal{B}) = 5$, $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{B}_0 = \mathcal{B}$. Define $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{A} \to \mathcal{B}$ by

$$\begin{array}{rcl} \mathcal{T}_1(-6, 0) &=& (-1, 0), \\ \mathcal{T}_1(0, -6) &=& (0, -1), \\ \mathcal{T}_1(0, 5) &=& (1, 1), \end{array} \qquad \begin{array}{rcl} \mathcal{T}_2(-6, 0) &=& (0, 0), \\ \mathcal{T}_2(0, -6) &=& (-1, 1), \\ \mathcal{T}_1(0, 5) &=& (1, 1), \end{array}$$

and $\alpha, \eta : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ by

$$\alpha(z_1, z_2) = \begin{cases} 1 & \text{if} \quad y_1, y_2 \neq 0, \\ 0 & \text{otherwise}, \end{cases} \qquad \eta(z_1, z_2) = \frac{1}{2},$$

for all $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \mathcal{A}$.

Note that $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$ if $z_1, z_2 \in \{(0, -6), (0, 5)\}$. For $z_1 = (0, -6)$, $d(u_1, \mathcal{T}_1 z_1) = \text{dist}(\mathcal{A}, \mathcal{B})$ if $u_1 \in \{(0, -6)\}$ and $d(u_2, \mathcal{T}_2 z_1) = \text{dist}(\mathcal{A}, \mathcal{B})$ if $u_2 \in \{(0, 5)\}$. This implies that $\alpha(u_1, u_2) = 1 > \frac{1}{2} = \eta(u_1, u_2)$. For $z_2 = (0, 5)$, $d(u_1, \mathcal{T}_1 z_1) = \text{dist}(\mathcal{A}, \mathcal{B}) = d(u_2, \mathcal{T}_2 z_1)$ if $u_1, u_2 \in \{(0, 5)\}$. This shows that $\alpha(u_1, u_2) = 1 > \frac{1}{2} = \eta(u_1, u_2)$. Thus the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is α-proximal admissible with respect to η.

By Theorem 2.3, we immediately obtain the following result.

Theorem 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space (X, d) such that $\mathcal{A}_0$ is nonempty and let $\mathcal{T}, \mathcal{S} : \mathcal{A} \to \mathcal{B}$ be continuous mappings satisfying the following assertions for all $z_1, z_2 \in \mathcal{A}$:*

1. *$\alpha(z_1, z_2) \geq \eta(z_1, z_2) \Rightarrow d(\mathcal{T} z_1, \mathcal{S} z_2) \leq \psi(d(z_1, z_2))$;*
2. *$\mathcal{T}(\mathcal{A}_0), \mathcal{S}(\mathcal{A}_0) \subseteq \mathcal{B}_0$ and $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property;*
3. *$(\mathcal{T}, \mathcal{S})$ is α-proximal admissible with respect to η;*
4. *there exist $z_0, z_1, z_2 \in \mathcal{A}_0$ such that*

$$d(z_1, \mathcal{T} z_0) = dist(\mathcal{A}, \mathcal{B}), \qquad \alpha(z_0, z_1) \geq \eta(z_0, z_1)$$

   and

$$d(z_2, \mathcal{S} z_0) = dist(\mathcal{A}, \mathcal{B}), \qquad \alpha(z_0, z_2) \geq \eta(z_0, z_2).$$

Then the mappings $\mathcal{T}$ and $\mathcal{S}$ have a common best proximity point.

The case $\mathcal{A} = \mathcal{B} = X$ reduces Definition 3.1 and Theorem 3.3 into the following:

Definition 3.4. Let (X, d) be a metric space and $\mathcal{T}_1, \mathcal{T}_2 : X \to X$ be mappings. The pair $(\mathcal{T}_1, \mathcal{T}_2)$ is α-admissible with respect to η if there exist functions $\alpha, \eta : X \times X \to [0, \infty)$ such that for $z_1, z_2 \in X$,

$$\alpha(z_1, z_2) \geq \eta(z_1, z_2) \Rightarrow \alpha(\mathcal{T}_1 z_1, T_2 z_2) \geq \eta(\mathcal{T}_1 z_1, T_2 z_2).$$

When $\alpha(z_1, z_2) = 1$ for all $z_1, z_2 \in X$, the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is called η-subadmissible and when $\eta(z_1, z_2) = 1$ for all $z_1, z_2 \in X$, the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is called α-admissible.

Remark 3.5. Definition 3.4 generalizes the concepts of compatibility and weak compatibility by Jungck ([18] and [19]). Every weakly compatible pair is α- admissible with respect to η. Indeed, let $(\mathcal{T}_1, \mathcal{T}_2)$ be weakly compatible pair. Then $\mathcal{T}_1(\mathcal{T}_2 z) = \mathcal{T}_2(\mathcal{T}_1 z)$ for all z belonging to $C(\mathcal{T}_1, \mathcal{T}_2)$ as the set of all coincidence points of mappings $\mathcal{T}_1$ and $\mathcal{T}_2$. Define

$$\alpha(z_1, z_2) = \begin{cases} 1 & \text{if} \quad z_1, z_2 \in C(\mathcal{T}_1, \mathcal{T}_2), \\ 0 & \qquad \text{otherwise,} \end{cases} \qquad \text{and} \quad \eta(z_1, z_2) = \frac{1}{2} \quad \text{for all} \quad z_1, z_2 \in X.$$

Then $\alpha(z_1, z_2) > \eta(z_1, z_2)$ if $z_1, z_2 \in C(\mathcal{T}_1, \mathcal{T}_2)$. Since $(\mathcal{T}_1, \mathcal{T}_2)$ is weakly compatible pair, so for all $z_1, z_2 \in C(\mathcal{T}_1, \mathcal{T}_2)$, we have $\mathcal{T}_1(\mathcal{T}_1 z_1) = \mathcal{T}_1(\mathcal{T}_2 z_1) = \mathcal{T}_2(\mathcal{T}_1 z_1)$ and $\mathcal{T}_1(\mathcal{T}_2 z_2) = \mathcal{T}_2(\mathcal{T}_1 z_2) = \mathcal{T}_2(\mathcal{T}_2 z_2)$. This implies that $\mathcal{T}_1 z_1, \mathcal{T}_2 z_2 \in C(\mathcal{T}_1, \mathcal{T}_2)$. Hence $\alpha(\mathcal{T}_1 z_1, \mathcal{T}_2 z_2) = 1 > \frac{1}{2} = \eta(\mathcal{T}_1 z_1, \mathcal{T}_2 z_2)$, that is, the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is α- admissible with respect to η. But the converse is not true which is clear from the following:

Example 3.6. Consider $X = \mathbb{R}$ with the usual metric. Define $\mathcal{T}_1, \mathcal{T}_2 : X \to X$ by

$$\mathcal{T}_1(z) = z^3, \qquad \mathcal{T}_2(z) = \frac{z^2}{4}$$

and $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(z_1, z_2) = \begin{cases} 2 & \text{if} \quad z_1, z_2 \geq 0, \\ 0 & \text{if} \quad z_1, z_2 < 0, \end{cases} \qquad \eta(z_1, z_1) = \frac{1}{4}$$

for all $z_1, z_2 \in X$. Note that $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$ when $z_1, z_2 \geq 0$. This implies that $\alpha(\mathcal{T}_1 z_1, T_2 z_2) = 2 > \frac{1}{4} = \eta(\mathcal{T}_1 z_1, T_2 z_2)$. Hence the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is α- admissible with respect to η. On the other hand, the coincidence points of $\mathcal{T}_1$ and $\mathcal{T}_2$ are 0 and $\frac{1}{4}$ such that $\mathcal{T}_1\left(\mathcal{T}_2\left(\frac{1}{4}\right)\right) = \frac{1}{(64)^3} \neq \mathcal{T}_2\left(\mathcal{T}_1\left(\frac{1}{4}\right)\right) = \frac{1}{4}(\frac{1}{64})^2$. Thus, the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is not weakly compatible.

Theorem 3.7. *Let (X, d) be a complete metric space and $\mathcal{T}, \mathcal{S} : X \to X$ be continuous mappings satisfying the following assertions for all $z_1, z_2 \in X$:*

1. $\alpha(z_1, z_2) \geq \eta(z_1, z_2) \Rightarrow d(\mathcal{T}z_1, \mathcal{S}z_2) \leq \psi(d(z_1, z_2))$;
2. $(\mathcal{T}, \mathcal{S})$ *is* α*-admissible with respect to* η;
3. *there exist* $z_0, z_1 \in X$ *such that* $\alpha(z_0, \mathcal{T}z_0) \geq \eta(z_0, \mathcal{T}z_0)$ *and* $\alpha(z_1, \mathcal{S}z_1) \geq \eta(z_1, \mathcal{S}z_1)$.

Then the mappings $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point.

Taking $\eta(z_1, z_2) = 1$ in Theorem 3.7, we get the following:

Corollary 3.8. *Let (X, d) be a complete metric space and $\mathcal{T}, \mathcal{S} : X \to X$ be continuous mappings satisfying the following assertions for all $z_1, z_2 \in X$:*

1. $\alpha(z_1, z_2) \geq 1 \Rightarrow d(\mathcal{T}z_1, \mathcal{S}z_2) \leq \psi(d(z_1, z_2))$;
2. $(\mathcal{T}, \mathcal{S})$ *is* α*-admissible;*
3. *there exist* $z_0, z_1 \in X$ *such that* $\alpha(z_0, \mathcal{T}z_0) \geq 1$ *and* $\alpha(z_1, \mathcal{S}z_1) \geq 1$.

Then the mappings $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point.

Remark 3.9. When $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}$ in Definition 3.4, we get Definition 2.1 in [28] and in case $\mathcal{T} = \mathcal{S}$, (with the help of Remark 3.5 in [5]), Corollary 3.8 generalizes Theorem 2.1 in [29].

When $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}$, Definition 3.1 and Theorem 3.3 are reduced to Definition 8 in [15] and the following result, respectively.

Theorem 3.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space (X, d) such that $\mathcal{A}_0$ is nonempty and $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ be a continuous mapping satisfying the following assertions for all $z_1, z_2 \in \mathcal{A}$:*

1. $\alpha(z_1, z_2) \geq \eta(z_1, z_2) \Rightarrow d(\mathcal{T}z_1, \mathcal{T}z_2) \leq \psi(d(z_1, z_2))$;
2. $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$ *and* $(\mathcal{A}, \mathcal{B})$ *satisfies the weak P-property;*
3. $\mathcal{T}$ *is* α*-proximal admissible with respect to* η;
4. *there exist* $z_0, z_1 \in \mathcal{A}_0$ *such that*

$$d(z_1, \mathcal{T}z_0) = dist(\mathcal{A}, \mathcal{B}), \qquad \alpha(z_0, z_1) \geq \eta(z_0, z_1).$$

Then $\mathcal{T}$ has a best proximity point.

Remark 3.11. The special cases of Theorems 3.3 and 3.10 for $\eta(z_1, z_2) = 1$ and $\alpha(z_1, z_2) = 1$ can be obtained as in Corollaries 2.5 and 2.6.

4. Generalization

In this section we generalize the results of Sections 2 and 3 for a sequence of mappings.

Definition 4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of a metric space (X, d) and $\{\mathcal{T}_i : \mathcal{A} \to 2^{\mathcal{B}} \setminus \emptyset\}_{i=1}^{\infty}$ be a sequence of multivalued mappings. The sequence $\{\mathcal{T}_i\}$ is α_*-proximal admissible with respect to η if there exist functions $\alpha, \eta : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that for $z_1, z_2, u_1, u_2 \in \mathcal{A}$,

$$\left.\begin{array}{r} \alpha(z_1, z_2) \geq \eta(z_1, z_2) \\ d(u_1, y_1) = \mathrm{dist}(\mathcal{A}, \mathcal{B}) \\ d(u_2, y_2) = \mathrm{dist}(\mathcal{A}, \mathcal{B}) \end{array}\right\} \quad \Rightarrow \quad \alpha(u_1, u_2) \geq \eta(u_1, u_2)$$

for all $y_1 \in \mathcal{T}_i z_1$ and $y_2 \mathcal{T}_j z_2$, and for all $i, j \in \mathbb{N}$. When $\alpha(z_1, z_2) = 1$ for all $z_1, z_2 \in \mathcal{A}$, the sequence $\{\mathcal{T}_i\}$ is called η_*-proximal sub-admissible and when $\eta(z_1, z_2) = 1$ for all $z_1, z_2 \in \mathcal{A}$, the sequence $\{\mathcal{T}_i\}$ is called α_*-proximal admissible.

Theorem 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space (X, d) such that $\mathcal{A}_0$ is nonempty and $\{\mathcal{T}_i : \mathcal{A} \to CL(\mathcal{B})\}_{i=1}^{\infty}$ be a sequence of continuous multivalued mappings satisfying the following assertions:*

1. *$\alpha(z_1, z_2) \geq \eta(z_1, z_2) \Rightarrow H(\mathcal{T}_i z_1, \mathcal{T}_j z_2) \leq \psi(d(z_1, z_2))$ for each $i, j \in \mathbb{N}$;*
2. *$\mathcal{T}_i z \subseteq \mathcal{B}_0$ for each $z \in \mathcal{A}_0$, $i \in \mathbb{N}$ and $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property;*
3. *$\{\mathcal{T}_i\}$ is α_*-proximal admissible with respect to η;*
4. *there exist $z_0, z_i \in \mathcal{A}$, and $y_i \in \mathcal{T}_i z_0$ for each $i \in \mathbb{N}$ such that*

$$d(z_i, y_i) = dist(\mathcal{A}, \mathcal{B}), \qquad \alpha(z_0, z_i) \geq \eta(z_0, z_i).$$

Then the mappings $\mathcal{T}_i$ have a common best proximity point.

Proof. It is similar to the proof of Theorem 2.3 and is omitted. □

Taking $\eta(z_1, z_2) = 1$ in Theorem 4.2, we get the following:

Corollary 4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space (X, d) such that $\mathcal{A}_0$ is nonempty and $\{\mathcal{T}_i : \mathcal{A} \to CL(\mathcal{B})\}_{i=1}^{\infty}$ be a sequence of continuous multivalued mappings satisfying the following assertions:*

1. *$\alpha(z_1, z_2) \geq 1 \Rightarrow H(\mathcal{T}_i z_1, \mathcal{T}_j z_2) \leq \psi(d(z_1, z_2))$ for each $i, j \in \mathbb{N}$;*
2. *$\mathcal{T}_i z \subseteq \mathcal{B}_0$, for each $z \in \mathcal{A}_0$, $i \in \mathbb{N}$ and $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property;*
3. *$\{\mathcal{T}_i\}$ is α_*-proximal admissible;*
4. *there exists $z_0, z_i \in \mathcal{A}$, and $y_i \in \mathcal{T}_i z_0$ for each $i \in \mathbb{N}$ such that*

$$d(z_i, y_i) = dist(\mathcal{A}, \mathcal{B}), \qquad \alpha(z_0, z_i) \geq 1.$$

Then the mappings $\mathcal{T}_i$ have a common best proximity point.

Taking $\alpha(z_1, z_2) = 1$ in Theorem 4.2, we get the following:

Corollary 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space (X, d) such that $\mathcal{A}_0$ is nonempty and $\{\mathcal{T}_i : \mathcal{A} \to CL(\mathcal{B})\}_{i=1}^{\infty}$ be a sequence of continuous multivalued mappings satisfying the following assertions:*

1. *$\eta(z_1, z_2) \leq 1 \Rightarrow H(\mathcal{T}_i z_1, \mathcal{T}_j z_2) \leq \psi(d(z_1, z_2))$ for each $i, j \in \mathbb{N}$;*
2. *$\mathcal{T}_i z \subseteq \mathcal{B}_0$ for each $z \in \mathcal{A}_0$, $i \in \mathbb{N}$ and $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property;*
3. *$\{\mathcal{T}_i\}$ is η_*-proximal subadmissible;*
4. *there exist $z_0, z_i \in \mathcal{A}$, and $y_i \in \mathcal{T}_i z_0$ for each $i \in \mathbb{N}$ such that*

$$d(z_i, y_i) = dist(\mathcal{A}, \mathcal{B}), \qquad \eta(z_0, z_i) \leq 1.$$

Then the mappings $\mathcal{T}_i$ have a common best proximity point.

Remark 4.5. The choice $\mathcal{A} = \mathcal{B} = X$ reduces Definition 4.1 and Theorem 4.2 into the Definition 3.1 and Theorem 3.2 in [5], respectively, and generalizes Theorem 4.1 in [17]. When $\mathcal{A} = \mathcal{B} = X$, Corollaries 4.3 and 4.4 generalize Corollaries 4.1 and 4.2 in [17], respectively.

Theorem 4.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space (X, d) such that $\mathcal{A}_0$ is nonempty and $\{\mathcal{T}_i : \mathcal{A} \to K(\mathcal{B})\}_{i=1}^{\infty}$ be a sequence of continuous multivalued mappings satisfying all assertions of Theorem 4.2 with condition H. Then the mappings $\mathcal{T}_i$ have a unique common best proximity.*

Definition 4.7. Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of a metric space (X, d) and $\{\mathcal{T}_i : \mathcal{A} \to \mathcal{B}\}_{i=1}^{\infty}$ be a sequence of mappings. The sequence $\{\mathcal{T}_i\}$ is α_*-proximal admissible with respect to η if there exists two functions $\alpha, \eta : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that for $z_1, z_2, u_1, u_2 \in \mathcal{A}$,

$$\left.\begin{array}{r}\alpha(z_1, z_2) \geq \eta(z_1, z_2) \\ d(u_1, \mathcal{T}_i z_1) = \operatorname{dist}(\mathcal{A}, \mathcal{B}) \\ d(u_2, \mathcal{T}_j z_2) = \operatorname{dist}(\mathcal{A}, \mathcal{B})\end{array}\right\} \quad \Rightarrow \quad \alpha(u_1, u_2) \geq \eta(u_1, u_2)$$

for each $i, j \in \mathbb{N}$. When $\alpha(z_1, z_2) = 1$ for all $z_1, z_2 \in \mathcal{A}$, the sequence $\{\mathcal{T}_i\}$ is called η_*-proximal subadmissible and when $\eta(z_1, z_2) = 1$ for all $z_1, z_2 \in \mathcal{A}$, the sequence $\{\mathcal{T}_i\}$ is called α_*-proximal admissible.

From Definition 4.1 and Theorem 4.2, we obtain the following result for a sequence of single-valued mappings.

Theorem 4.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a complete metric space (X, d) such that $\mathcal{A}_0$ is nonempty and $\{\mathcal{T}_i : \mathcal{A} \to \mathcal{B}\}_{i=1}^{\infty}$ be a sequence of continuous mappings satisfying the following assertions:*

1. $\alpha(z_1, z_2) \geq \eta(z_1, z_2) \Rightarrow d(\mathcal{T}_i z_1, \mathcal{T}_j z_2) \leq \psi(d(z_1, z_2))$ *for each* $i, j \in \mathbb{N}$;
2. $\mathcal{T}_i z \subseteq \mathcal{B}_0$ *for each* $z \in \mathcal{A}_0$, $i \in \mathbb{N}$ *and* $(\mathcal{A}, \mathcal{B})$ *satisfies the weak P-property;*
3. $\{\mathcal{T}_i\}$ *is* α_**-proximal admissible with respect to* η;
4. *there exist* $z_0, z_i \in \mathcal{A}_0$ *such that for each* $i \in \mathbb{N}$

$$d(z_i, \mathcal{T}_i z_0) = dist(\mathcal{A}, \mathcal{B}), \qquad \alpha(z_0, z_i) \geq \eta(z_0, z_i).$$

Then the mappings $\mathcal{T}_i$ have a common best proximity point.

5. Common best proximity point results in partially ordered metric space

Let $(\mathcal{X}, d, \preceq)$ be a partially ordered metric space and $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of $\mathcal{X}$. The existence of best proximity point in the setting of a partially order metric space has been established in [2, 3, 10, 11, 25–27]. In this section, we derive new results in partially order metric spaces as an application of our results in Sections 2, and 3. Recall that a mapping $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ is said to be proximally increasing if it satisfies the condition

$$\left.\begin{array}{r}z_1 \preceq z_2 \\ d(u_1, \mathcal{T} z_1) = \operatorname{dist}(\mathcal{A}, \mathcal{B}) \\ d(u_2, \mathcal{T} z_2) = \operatorname{dist}(\mathcal{A}, \mathcal{B})\end{array}\right\} \quad \Rightarrow \quad u_1 \preceq u_2,$$

where $z_1, z_2, u_1, u_2 \in \mathcal{A}$ (see [10]). Very recently, Pragadeeswarar et al. [27] defined the notion of proximal relation between two subsets of $\mathcal{X}$ as follows:

Definition 5.1 ([27])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of a partially ordered metric space $(\mathcal{X}, d, \preceq)$ such that $\mathcal{A}_0 \neq \emptyset$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two nonempty subsets of $\mathcal{B}_0$. The proximal relation between $\mathcal{B}_1$ and $\mathcal{B}_2$ is denoted and defined by $\mathcal{B}_1 \preceq_{(1)} \mathcal{B}_2$, if for every $b_1 \in \mathcal{B}_1$ with $d(a_1, b_1) = d(\mathcal{A}, \mathcal{B})$, there exists $b_2 \in \mathcal{B}_2$ with $d(a_2, b_2) = d(\mathcal{A}, \mathcal{B})$ such that $a_1 \preceq a_2$.

Now we present our main results of this section.

Theorem 5.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a partially ordered complete metric space $(\mathcal{X}, d, \preceq)$ such that $\mathcal{A}_0$ is nonempty and $\mathcal{T}, \mathcal{S} : \mathcal{A} \to CL(\mathcal{B})$ be continuous mappings satisfying the following assertions for all $z_1, z_2 \in \mathcal{A}$ with $z_1 \preceq z_2$:*

1. $H(\mathcal{T} z_1, \mathcal{S} z_2) \leq \psi(d(z_1, z_2))$;
2. $\mathcal{T} z, \mathcal{S} z \subseteq \mathcal{B}_0$ *for each* $z \in \mathcal{A}_0$ *and* $(\mathcal{A}, \mathcal{B})$ *satisfies the weak P-property;*
3. $z_1, z_2 \in \mathcal{A}_0$, $z_1 \preceq z_2$ *implies* $\mathcal{T} z_1 \preceq_{(1)} \mathcal{S} z_2$;

4. *there exist* $z_0, z_1, z_2 \in \mathcal{A}_0$, $y_1 \in \mathcal{T} z_0$ *and* $y_2 \in \mathcal{S} z_0$ *such that*

$$d(z_1, y_1) = dist(\mathcal{A}, \mathcal{B}), \qquad z_0 \preceq z_1$$

and

$$d(z_2, y_2) = dist(\mathcal{A}, \mathcal{B}), \qquad z_0 \preceq z_2.$$

Then $\mathcal{T}$ *and* $\mathcal{S}$ *have a common best proximity point.*

Proof. Define $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\alpha(z_1, z_2) = \begin{cases} 1 & z_1 \preceq z_2, \\ 0 & \text{otherwise}, \end{cases} \qquad \eta(z_1, z_2) = \begin{cases} \frac{1}{2} & z_1 \preceq z_2, \\ 0 & \text{otherwise}. \end{cases}$$

Since $\mathcal{T} z_1 \preceq_{(1)} \mathcal{S} z_2$, therefore for $z_1, z_2, u_1, u_2 \in \mathcal{X}$, $y_1 \in \mathcal{T} z_1$, $y_2 \in \mathcal{S} z_2$ with

$$\left.\begin{array}{r} \alpha(z_1, z_2) \geq \eta(z_1, z_2) \\ d(u_1, y_1) = \mathrm{dist}(\mathcal{A}, \mathcal{B}) \\ d(u_2, y_2) = \mathrm{dist}(\mathcal{A}, \mathcal{B}) \end{array}\right\},$$

we have $u_1 \preceq u_2$. This implies that $\alpha(u_1, u_2) = 1 > \frac{1}{2} = \eta(u_1, u_2)$ for $z_1 \preceq z_2$ and $\alpha(u_1, u_2) = 0 = \eta(u_1, u_2)$ otherwise. Thus, all the conditions of Theorem 2.3 are satisfied and hence mappings $\mathcal{T}$ and $\mathcal{S}$ have a common best proximity point. □

By considering $\mathcal{T} = \mathcal{S}$, Theorem 5.2 is reduced to the following:

Theorem 5.3. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two nonempty closed subsets of a partially ordered complete metric space* $(\mathcal{X}, d, \preceq)$ *such that* $\mathcal{A}_0$ *is non-empty and* $\mathcal{T} : \mathcal{A} \to CL(\mathcal{B})$ *be a continuous mapping satisfying the following assertions for all* $z_1, z_2 \in \mathcal{A}$ *with* $z_1 \preceq z_2$*:*

1. $H(\mathcal{T} z_1, \mathcal{T} z_2) \leq \psi(d(z_1, z_2))$;
2. $\mathcal{T} z \subseteq \mathcal{B}_0$ *for each* $z \in \mathcal{A}_0$ *and* $(\mathcal{A}, \mathcal{B})$ *satisfies the weak P-property;*
3. $z_1, z_2 \in \mathcal{A}_0$, $z_1 \preceq z_2$ *implies* $\mathcal{T} z_1 \preceq_{(1)} \mathcal{T} z_2$;
4. *there exist* $z_0, z_1 \in \mathcal{A}_0$, $y_1 \in \mathcal{T} z_0$ *such that*

$$d(z_1, y_1) = dist(\mathcal{A}, \mathcal{B}), \qquad z_0 \preceq z_1.$$

Then the mapping $\mathcal{T}$ *has a best proximity point.*

Following the arguments in the proof of Theorem 5.2, we obtain the following result.

Theorem 5.4. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two nonempty closed subsets of a partially ordered complete metric space* $(\mathcal{X}, d, \preceq)$ *such that* $\mathcal{A}_0$ *is nonempty and* $\{\mathcal{T}_i : \mathcal{A} \to CL(\mathcal{B})\}_1^\infty$ *be sequence of continuous mappings satisfying the following assertions for all* $z_1, z_2 \in \mathcal{A}$ *with* $z_1 \preceq z_2$*:*

1. $H(\mathcal{T}_i z_1, \mathcal{T}_j z_2) \leq \psi(d(z_1, z_2))$ *for each* $i, j \in \mathbb{N}$;
2. $\mathcal{T}_i z \subseteq \mathcal{B}_0$ *for each* $z \in \mathcal{A}_0$, $i \in \mathbb{N}$ *and* $(\mathcal{A}, \mathcal{B})$ *satisfies the weak P-property;*
3. $z_1, z_2 \in \mathcal{A}_0$, $z_1 \preceq z_2$ *implies* $\mathcal{T}_i z_1 \preceq_{(1)} \mathcal{T}_j z_2$ *for each* $i, j \in \mathbb{N}$;
4. *there exist* $z_0, z_i \in \mathcal{A}_0$ *and* $y_i \in \mathcal{T}_i z_0$ *for each* $i \in \mathbb{N}$ *such that*

$$d(z_i, y_i) = dist(\mathcal{A}, \mathcal{B}), \qquad z_0 \preceq z_i.$$

Then the mappings $\mathcal{T}_i$ *have a common best proximity point.*

For single valued mappings, from Theorems 5.2-5.4 we obtain the following results.

Theorem 5.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a partially ordered complete metric space $(\mathcal{X}, d, \preceq)$ such that $\mathcal{A}_0$ is nonempty and $\mathcal{T}, \mathcal{S} : \mathcal{A} \to \mathcal{B}$ be continuous mappings satisfying the following assertions for all $z_1, z_2 \in \mathcal{A}$ with $z_1 \preceq z_2$:*

1. *$d(\mathcal{T}z_1, \mathcal{S}z_2) \leq \psi(d(z_1, z_2))$;*
2. *$\mathcal{T}z, \mathcal{S}z \subseteq \mathcal{B}_0$ for each $z \in \mathcal{A}_0$ and $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property;*
3. *$z_1, z_2 \in \mathcal{A}_0$ $z_1 \preceq z_2$ implies $\mathcal{T}z_1 \preceq \mathcal{S}z_2$;*
4. *there exist $z_0, z_1, z_2 \in \mathcal{A}_0$ such that*

$$d(z_1, \mathcal{T}z_0) = dist(\mathcal{A}, \mathcal{B}), \qquad z_0 \preceq z_1$$

and

$$d(z_2, \mathcal{T}z_0) = dist(\mathcal{A}, \mathcal{B}), \qquad z_0 \preceq z_2.$$

Then $\mathcal{T}$ and $\mathcal{S}$ have a common best proximity point.

Theorem 5.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a partially ordered complete metric space $(\mathcal{X}, d, \preceq)$ such that $\mathcal{A}_0$ is non-empty and $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ be a continuous mapping satisfying the following assertions for all $z_1, z_2 \in \mathcal{A}$ with $z_1 \preceq z_2$:*

1. *$d(\mathcal{T}z_1, \mathcal{T}z_2) \leq \psi(d(z_1, z_2))$;*
2. *$\mathcal{T}z \subseteq \mathcal{B}_0$ for each $z \in \mathcal{A}_0$ and $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property;*
3. *$z_1, z_2 \in \mathcal{A}_0$, $z_1 \preceq z_2$ implies $\mathcal{T}z_1 \preceq \mathcal{T}z_2$;*
4. *there exist $z_0, z_1 \in \mathcal{A}_0$ such that*

$$d(z_1, \mathcal{T}z_0) = dist(\mathcal{A}, \mathcal{B}), \qquad z_0 \preceq z_1.$$

Then $\mathcal{T}$ has a best proximity point.

Theorem 5.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty closed subsets of a partially ordered complete metric space $(\mathcal{X}, d, \preceq)$ such that $\mathcal{A}_0$ is nonempty and $\{\mathcal{T}_i : \mathcal{A} \to \mathcal{B}\}_1^\infty$ be sequence of continuous mappings satisfying the following assertions for all $z_1, z_2 \in \mathcal{A}$ with $z_1 \preceq z_2$:*

1. *$d(\mathcal{T}_i z_1, \mathcal{T}_j z_2) \leq \psi(d(z_1, z_2))$ for each $i, j \in \mathbb{N}$;*
2. *$\mathcal{T}_i z \subseteq \mathcal{B}_0$ for each $z \in \mathcal{A}_0$, $i \in \mathbb{N}$ and $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property;*
3. *$z_1, z_2 \in \mathcal{A}_0$, $z_1 \preceq z_2$ implies $\mathcal{T}_i z_1 \preceq \mathcal{T}_j z_2$ for each $i, j \in \mathbb{N}$;*
4. *there exist $z_0, z_i \in \mathcal{A}_0$ for each $i \in \mathbb{N}$ such that*

$$d(z_i, \mathcal{T}_i z_0) = dist(\mathcal{A}, \mathcal{B}), \qquad z_0 \preceq z_i.$$

Then the mappings $\mathcal{T}_i$ have a common best proximity point.

Acknowledgment

The author A. R. Khan is grateful to King Fahd University of Petroleum & Minerals for supporting research project IN141047.

References

[1] M. Abbas, A. Hussain, P. Kumam, *A coincidence best proximity point problem in G-metric spaces*, Abstr. Appl. Anal., **2015** (2015), 12 pages. 1

[2] A. Abkar, M. Gabeleh, *Best proximity points for cyclic mappings in ordered metric spaces*, J. Optim. Theory Appl., **150** (2011), 188–193. 5

[3] A. Abkar, M. Gabeleh, *Generalized cyclic contractions in partially ordered metric spaces*, Optim. Lett., **6** (2012), 1819–1830. 5

[4] A. Abkar, M. Gabeleh, *The existence of best proximity points for multivalued non-self-mappings*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, **107** (2013), 319–325. 1

[5] M. U. Ali, T. Kamram, E. Karapinar, *Further discussion on modified multivalued α_*-ψ-contractive type mapping*, Filomat, **29** (2015), 1893–1900. 1, 1.2, 2.11, 2.12, 3.9, 4.5

[6] M. U. Ali, T. Kamram, N. Shahzad, *Best proximity point for α-ψ-proximal contractive multimaps*, Abstr. Appl. Anal., **2014** (2014), 6 pages. 1, 1.3, 1.4, 1.5, 1, 2.11

[7] A. Amini-Harandi, *Best proximity points for proximal generalized contractions in metric spaces*, Optim. Lett., **7** (2013), 913–921. 1

[8] A. Amini-Harandi, M. Fakhar, H. R. Hajisharifi, N. Hussain, *Some new results on fixed and best proximity points in preordered metric spaces*, Fixed Point Theory Appl., **2013** (2013), 15 pages. 1

[9] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181. 1

[10] S. S. Basha, *Discrete optimization in partially ordered sets*, J. Global Optim., **54** (2012), 511–517. 5

[11] S. S. Basha, *Best proximity point theorems on partially ordered sets*, Optim. Lett., **7** (2013), 1035–1043. 5

[12] A. A. Eldred, P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl., **323** (2006), 1001–1006. 1

[13] K. Fan, *Extensions of two fixed point theorems of F. E. Browder*, Math. Z., **122** (1969), 234–240. 1

[14] M. Gabeleh, *Best proximity points: global minimization of multivalued non-self mappings*, Optim. Lett., **8** (2014), 1101–1112. 1

[15] N. Hussain, M. A. Kutbi, P. Salimi, *Best proximity point results for modified α-ψ-proximal rational contractions*, Abstr. Appl. Anal., **2013** (2013), 14 pages. 1, 3

[16] N. Hussain, A. Latif, P. Salimi, *Best proximity point results for modified Suzuki α-ψ-proximal contractions*, Fixed Point Theory Appl., **2014** (2014), 16 pages. 1

[17] N. Hussain, P. Salimi, A. Latif, *Fixed point results for single and set-valued α-η-ψ-contractive mappings*, Fixed Point Theory Appl., **2013** (2013), 23 pages. 1, 4.5

[18] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci., **9** (1986), 771–779. 1, 1, 3.5

[19] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci., **4** (1996), 199–215. 1, 1, 3.5

[20] E. Karapınar, *Best proximity points of cyclic mappings*, Appl. Math. Lett., **25** (2007), 79–92. 1

[21] A. R. Khan, S. A. Shukri, *Best proximity points in the Hilbert ball*, J. Nonlinear Convex Anal., (Accepted).

[22] A. Latif, M. Hezarjaribi, P. Salimi, N. Hussain, *Best proximity point theorems for α-ψ-proximal contractions in intuitionistic fuzzy metric spaces*, J. Inequal. Appl., **2014** (2014), 19 pages. 1

[23] P. Lo'lo', S. M. Vaezpour, J. Esmaily, *Common best proximity points theorem for four mappings in metric-type spaces*, Fixed Point Theory Appl., **2015** (2015), 7 pages. 1.1

[24] S. B. Nadler Jr, *Multivalued contraction mappings*, Pacific J. Math., **30** (1969), 475–488. 1

[25] V. Pragadeeswarar, M. Marudai, *Best proximity points: approximation and optimization in partially ordered metric spaces*, Optim. Lett., **7** (2013), 1883–1892. 5

[26] V. Pragadeeswarar, M. Marudai, *Best proximity points for generalized proximal weak contractions in partially ordered metric spaces*, Optim. Lett., **9** (2015), 105–118.

[27] V. Pragadeeswarar, M. Marudai, P. Kumam, *Best proximity point theorems for multivalued mappings on partially ordered metric spaces*, J. Nonlinear Sci. Appl., **9** (2016), 1911–1921. 5, 5.1

[28] P. Salimi, A. Latif, N. Hussain, *Modified α-ψ-contractive mappings with applications*, Fixed Point Theory Appl., **2013** (2013), 19 pages. 1, 3.9

[29] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for $\alpha\psi$-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165. 1, 1, 3.9

[30] V. M. Sehgal, S. P. Singh, *A generalization to multifunctions of Fan's best approximation theorem*, Proc. Amer. Math. Soc., **102** (1988), 534–537. 1

[31] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. (Beograd), **32** (1982), 149–153. 1