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Approximate analytical solutions of Goursat problem within local fractional operators

Dumitru Baleanu[a,*], Hassan Kamil Jassim[b], Maysaa Al Qurashi[c]

[a]*Department of Mathematics, Faculty of Arts and Sciences, Çankaya University, Ankara, Turkey.*
[b]*Department of Mathematics, Faculty of Education for Pure Sciences, University of Thi-Qar, Nasiriyah, Iraq.*
[c]*Department of Mathematics, College of Science, King Saud University, Ryad, Saudia Arabia.*

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Abstract

The local fractional differential transform method (LFDTM) and local fractional decomposition method (LFDM) are applied to implement the homogeneous and nonhomogeneous Goursat problem involving local fractional derivative operators. The approximate analytical solution of this problem is calculated in form of a series with easily computable components. Examples are studied in order to show the accuracy and reliability of presented methods. We demonstrate that the two approaches are very effective and convenient for finding the analytical solutions of partial differential equations with local fractional derivative operators.

Keywords: Goursat problem, local fractional differential transform method, local fractional decomposition method, analytical solutions, local fractional derivative operators.
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1. Introduction

The local fractional differential transform and Adomian decomposition methods are accurate approximation techniques for solving the partial differential equations with local fractional derivative operators. The concept of local fractional differential transform method (LFDTM) was introduced first by Yang et al. in 2016 [7]. This scheme is based on the local fractional Taylor's theorem to construct analytical solutions

*Corresponding author
Email addresses: dumitru@cankaya.edu.tr (Dumitru Baleanu), hassan.kamil28@yahoo.com (Hassan Kamil Jassim), maysaa@ksu.edu.sa (Maysaa Al Qurashi)

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in the form of a polynomial by means of an iterative procedure. Recently, these methods (LFDTM) and (LFDM) are introduced by Yang, Jafari and Baleanu et al. [1, 2, 4, 5, 7] and used for solving the diffusion equation [1, 7], Laplace equation [5], Fokker Planck equations [4], and telegraph equation [2].

In the present work we shall find the analytical solutions of the Goursat problem that arises in the partial differential equations with mixed derivatives in its standard form which is given by:

$$\frac{\partial^{2\vartheta}}{\partial\eta^{\vartheta}\partial\kappa^{\vartheta}}\psi(\eta,\kappa)=G\left(\eta,\kappa,\psi,\psi_{\eta}^{(\vartheta)},\psi_{\kappa}^{(\vartheta)}\right), \tag{1.1}$$

subject to the conditions

$$\begin{aligned}\psi(\eta,0)&=f_1(\eta),\\ \psi(0,\kappa)&=f_2(\kappa),\\ \psi(0,0)&=f_1(0)=f_2(0),\end{aligned}$$

by using the local fractional differential transform and Adomian decomposition methods.

2. Basic definitions of local fractional calculus

Below we present some basic definitions and properties of local fractional calculus theory utilized in our manuscript ([3, 6]).

Definition 2.1. The local fractional derivative of $\varphi(\eta)$ of order ϑ at $\eta=\eta_0$ is given by

$$\varphi^{(\vartheta)}(\eta_0)=\lim_{\eta\to\eta_0}\frac{\triangle^{\vartheta}(\varphi(\eta)-\varphi(\eta_0))}{(\eta-\eta_0)^{\vartheta}},$$

where $\triangle^{\vartheta}(\varphi(\eta)-\varphi(\eta_0)\cong\Gamma(\vartheta+1)(\varphi(\eta)-\varphi(\eta_0))$.

The formulas of local fractional derivatives of special functions used in the paper are as follows:

$$\begin{aligned}D_{\eta}^{(\vartheta)}a\varphi(\eta)&=aD_{\eta}^{(\vartheta)}\varphi(\eta),\\ L_{\eta}^{(\vartheta)}\left[\frac{\eta^{n\vartheta}}{\Gamma(1+n\vartheta)}\right]&=\frac{\eta^{(n-1)\vartheta}}{\Gamma(1+(n-1)\vartheta)},\ n\in N.\end{aligned}$$

Definition 2.2. The local fractional integral of $\varphi(\eta)$ of order ϑ in the interval $[a,b]$ is given by

$${}_aI_b^{(\vartheta)}\varphi(\eta)=\frac{1}{\Gamma(1+\vartheta)}\int_a^b\varphi(t)(dt)^{\vartheta}=\frac{1}{\Gamma(1+\vartheta)}\lim_{\triangle t\longrightarrow 0}\sum_{j=0}^{N-1}\varphi(t_j)(\triangle t_j)^{\vartheta},$$

where the partitions of the interval $[a,b]$ are denoted as $(t_j,t_{j+1}), j=0,...,N-1, t_0=a$ and $t_N=b$ with $\triangle t_j=t_{j+1}-t_j$ and $\triangle t=\max\{\triangle t_0,\triangle t_1,\ldots\}$.

The formulas of local fractional integrals of special functions used in the paper are as follows:

$$\begin{aligned}{}_aI_b^{(\vartheta)}a\varphi(\eta)&=a\ {}_aI_b^{(\vartheta)}\varphi(\eta),\\ {}_aI_b^{(\vartheta)}\left[\frac{\eta^{n\vartheta}}{\Gamma(1+n\vartheta)}\right]&=\frac{b^{(n+1)\vartheta}}{\Gamma(1+(n+1)\vartheta)}-\frac{a^{(n+1)\vartheta}}{\Gamma(1+(n+1)\vartheta)}.\end{aligned}$$

Definition 2.3. In fractal space, the Mittage Leffler function, sine function, and cosine function are defined as:

$$E_{\vartheta}(\eta^{\vartheta})=\sum_{n=0}^{\infty}\frac{\eta^{n\vartheta}}{\Gamma\left(1+n\vartheta\right)},$$

$$\sin_\vartheta(\eta^\vartheta) = \sum_{n=0}^{\infty}(-1)^n \frac{\eta^{(2n+1)\vartheta}}{\Gamma(1+(2n+1)\vartheta)},$$
$$\cos_\vartheta(\eta^\vartheta) = \sum_{n=0}^{\infty}(-1)^n \frac{\eta^{2n\vartheta}}{\Gamma(1+2n\vartheta)}.$$

3. Local fractional differential transform method (LFDTM)

In the following, the basic definitions and fundamental operations of the local fractional differential transform method are shown.

Definition 3.1. The two dimensional differential transform of the local fractional analytic function $\psi(\eta,\kappa)$ via the local fractional operator is defined by the following formula:

$$\Psi(\beta,\varepsilon) = \frac{1}{\Gamma(1+\beta\vartheta)}\frac{1}{\Gamma(1+\varepsilon\vartheta)}\left[\frac{\partial^{(\beta+\varepsilon)\vartheta}\psi(\eta,\kappa)}{\partial\eta^{\beta\vartheta}\partial\kappa^{\varepsilon\vartheta}}\right]_{\eta=\eta_0,\kappa=\kappa_0}, \tag{3.1}$$

where $\beta,\varepsilon = 0,1,\ldots,n$ and $0<\vartheta\le 1$.

Definition 3.2. The two dimensional differential inverse transform of $\Psi(\beta,\varepsilon)$ via the local fractional operator is defined as:

$$\psi(\eta,\kappa) = \sum_{\beta=0}^{\infty}\sum_{\varepsilon=0}^{\infty}\Psi(\beta,\varepsilon)(\eta-\eta_0)^{\beta\vartheta}(\kappa-\kappa_0)^{\varepsilon\vartheta}. \tag{3.2}$$

Combining (3.1) and (3.2), it can be obtained that

$$\psi(\eta,\kappa) = \sum_{\beta=0}^{\infty}\sum_{\varepsilon=0}^{\infty}\frac{1}{\Gamma(1+\beta\vartheta)}\frac{1}{\Gamma(1+\varepsilon\vartheta)}\left[\frac{\partial^{(\beta+\varepsilon)\vartheta}\psi(\eta,\kappa)}{\partial\eta^{\beta\vartheta}\partial\kappa^{\varepsilon\vartheta}}\right]_{\eta=\eta_0,\kappa=\kappa_0}(\eta-\eta_0)^{\beta\vartheta}(\kappa-\kappa_0)^{\varepsilon\vartheta}.$$

If $\eta_0 = 0$ and $\kappa_0 = 0$, then (3.1) is shown as follows:

$$\Psi(\beta,\varepsilon) = \frac{1}{\Gamma(1+\beta\vartheta)}\frac{1}{\Gamma(1+\varepsilon\vartheta)}\left[\frac{\partial^{(\beta+\varepsilon)\vartheta}\psi(\eta,\kappa)}{\partial\eta^{\beta\vartheta}\partial\kappa^{\varepsilon\vartheta}}\right]_{\eta=0,\kappa=0}$$

and (3.2) is expressed as follows:

$$\psi(\eta,\kappa) = \sum_{\beta=0}^{\infty}\sum_{\varepsilon=0}^{\infty}\Psi(\beta,\varepsilon)\eta^{\beta\vartheta}\kappa^{\varepsilon\vartheta}. \tag{3.3}$$

Theorem 3.3. *Suppose that $\psi(\eta,\kappa), \varphi(\eta,\kappa)$ and $\theta(\eta,\kappa)$ are local fractional analytic functions and $\Psi(\beta,\varepsilon)$, $\Phi(\beta,\varepsilon)$and $\Theta(\beta,\varepsilon)$ are their corresponding local fractional differential transforms with order of fraction ϑ, respectively, then we have*

1. *If $\psi(\eta,\kappa) = \varphi(\eta,\kappa)+\theta(\eta,\kappa)$ then $\Psi(\beta,\varepsilon) = \Phi(\beta,\varepsilon)+\Theta(\beta,\varepsilon)$.*
2. *If $\psi(\eta,\kappa) = \varphi(\eta,\kappa)+\theta(\eta,\kappa)$ then $\Psi(\beta,\varepsilon) = \sum_{r=0}^{\beta}\sum_{s=0}^{\varepsilon}\Phi(\beta,\varepsilon-s)\Theta(\beta-r,\varepsilon)$.*
3. *If $\psi(\eta,\kappa) = a\varphi(\eta,\kappa)$, where a is a constant, then $\Psi(\beta,\varepsilon) = \Phi(\beta,\varepsilon)$.*
4. *If $\psi(\eta,\kappa) = \frac{\partial^\vartheta}{\partial\eta^\vartheta}\varphi(\eta,\kappa)$ then $\Psi(\beta,\varepsilon) = \frac{\Gamma(1+(\beta+1)\vartheta)}{\Gamma(1+\beta\vartheta)}\Phi(\beta+1,\varepsilon)$.*
5. *If $\psi(\eta,\kappa) = \frac{\partial^\vartheta}{\partial\kappa^\vartheta}\varphi(\eta,\kappa)$ then $\Psi(\beta,\varepsilon) = \frac{\Gamma(1+(\varepsilon+s)\vartheta)}{\Gamma(1+\varepsilon\vartheta)}\Phi(\beta,\varepsilon+1)$.*

6. *If* $\psi(\eta,\kappa)=\dfrac{\partial^{(r+s)\vartheta}}{\partial\eta^{r\vartheta}\partial\kappa^{s\vartheta}}\varphi(\eta,\kappa)$ *then* $\Psi(\beta,\varepsilon)=\dfrac{\Gamma(1+(\beta+r)\vartheta)}{\Gamma(1+\beta\vartheta)}\dfrac{\Gamma(1+(\varepsilon+s)\vartheta)}{\Gamma(1+\varepsilon\vartheta)}\Phi(\beta+r,\varepsilon+s)$.

7. *If* $\psi(\eta,\kappa)=\dfrac{(\eta-\eta_0)^{r\vartheta}}{\Gamma(1+r\vartheta)}\dfrac{(\kappa-\kappa_0)^{s\vartheta}}{\Gamma(1+s\vartheta)}$ *then* $\Psi(\beta,\varepsilon)=\dfrac{\delta_\vartheta(\beta-r)}{\Gamma(1+r\vartheta)}\dfrac{\delta_\vartheta(\varepsilon-s)}{\Gamma(1+s\vartheta)}$,

   where the local fractional Dirac-delta function is given by

$$\delta_\vartheta(\beta-r)=\begin{cases}1, & \beta=r,\\ 0, & \beta\neq r,\end{cases}\quad \textit{and}\quad \delta_\vartheta(\varepsilon-s)=\begin{cases}1, & \varepsilon=s,\\ 0, & \varepsilon\neq s.\end{cases}$$

4. Local fractional decomposition method (LFDM).

It is obvious that (1.1) can be rewritten in the following form:

$$L_\eta^{(\vartheta)}L_\kappa^{(\vartheta)}\psi(\eta,\kappa)=G\left(\eta,\kappa,\psi,\psi_\eta^{(\vartheta)},\psi_\kappa^{(\vartheta)}\right), \tag{4.1}$$

where

$$L_\eta^{(\vartheta)}=\frac{\partial^\vartheta}{\partial\eta^\vartheta},\quad L_\kappa^{(\vartheta)}=\frac{\partial^\vartheta}{\partial\kappa^\vartheta}$$

are the local fractional differential operators of order ϑ with respect to η and κ, respectively.

We now define the inverse operators $L_\eta^{(-\vartheta)}$ and $L_\kappa^{(-\vartheta)}$ in the form:

$$L_\eta^{(-\vartheta)}(\cdot)=\frac{1}{\Gamma(1+\vartheta)}\int_0^\eta(\cdot)\,(d\eta)^\vartheta,\quad L_\kappa^{(-\vartheta)}(\cdot)=\frac{1}{\Gamma(1+\vartheta)}\int_0^\kappa(\cdot)\,(d\kappa)^\vartheta.$$

Applying $L_\kappa^{(-\vartheta)}$ on (4.1), we construct

$$L_\eta^{(\vartheta)}\left[L_\kappa^{(-\vartheta)}L_\kappa^{(\vartheta)}\psi(\eta,\kappa)\right]=L_\kappa^{(-\vartheta)}G\left(\eta,\kappa,\psi,\psi_\eta^{(\vartheta)},\psi_\kappa^{(\vartheta)}\right). \tag{4.2}$$

Making use of (4.2), we get

$$L_\eta^{(\vartheta)}\psi(\eta,\kappa)=L_\eta^{(\vartheta)}\psi(\eta,0)+L_\kappa^{(-\vartheta)}G\left(\eta,\kappa,\psi,\psi_\eta^{(\vartheta)},\psi_\kappa^{(\vartheta)}\right). \tag{4.3}$$

Taking $L_\eta^{(-\vartheta)}$ to the both sides of (4.3), we have

$$L_\eta^{(-\vartheta)}L_\eta^{(\vartheta)}\psi(\eta,\kappa)=L_\eta^{(-\vartheta)}L_\eta^{(\vartheta)}\psi(\eta,0)+L_\eta^{(-\vartheta)}L_\kappa^{(-\vartheta)}G\left(\eta,\kappa,\psi,\psi_\eta^{(\vartheta)},\psi_\kappa^{(\vartheta)}\right).$$

This gives

$$\psi(\eta,\kappa)=\psi(\eta,0)+\psi(0,\kappa)-\psi(0,0)+L_\eta^{(-\vartheta)}L_\kappa^{(-\vartheta)}G\left(\eta,\kappa,\psi,\psi_\eta^{(\vartheta)},\psi_\kappa^{(\vartheta)}\right),$$

or equivalently

$$\psi(\eta,\kappa)=f_1(\eta)+f_2(\kappa)-f_1(0)+L_\eta^{(-\vartheta)}L_\kappa^{(-\vartheta)}G\left(\eta,\kappa,\psi,\psi_\eta^{(\vartheta)},\psi_\kappa^{(\vartheta)}\right).$$

In the LFDM we express the solution $\psi(\eta,\kappa)$ of local fractional differential equation (4.1) in a series form defined by

$$\psi(\eta,\kappa)=\sum_{m=0}^{\infty}\psi_m(\eta,\kappa).$$

The components $\psi_m(\eta,\kappa)$ are obtained by the recurrence relation:

$$\psi_0(\eta,\kappa)=g(\eta,\kappa), \tag{4.4}$$

$$\psi_{m+1}(\eta,\kappa) = L_{\eta}^{(-\vartheta)} L_{\kappa}^{(-\vartheta)} H\left(\psi_m, \psi_{m\eta}^{(\vartheta)}, \psi_{m\kappa}^{(\vartheta)}\right), \tag{4.5}$$

where

$$g(\eta,\kappa) = \begin{cases} f_1(\eta) + f_2(\kappa) - f_1(0), & G = H\left(\psi, \psi_{\eta}^{(\vartheta)}, \psi_{\kappa}^{(\vartheta)}\right), \\ f_1(\eta) + f_2(\kappa) - f_1(0) + L_{\eta}^{(-\vartheta)} L_{\kappa}^{(-\vartheta)} w(\eta,\kappa), & G = w(\eta,\kappa) + H\left(\psi, \psi_{\eta}^{(\vartheta)}, \psi_{\kappa}^{(\vartheta)}\right). \end{cases}$$

5. Illustrative examples.

In this section, two examples for Goursat models involving the local fractional differential operators are presented in order to demonstrate the simplicity and the efficiency of the above methods.

Example 5.1. Let us consider the homogeneous Goursat problem with local fractional differential operators

$$\frac{\partial^{2\vartheta}}{\partial\eta^{\vartheta}\partial\kappa^{\vartheta}}\psi(\eta,\kappa) = \psi(\eta,\kappa), \tag{5.1}$$

subject to the initial conditions

$$\begin{aligned} \psi(\eta,0) &= E_{\vartheta}(\eta^{\vartheta}), \\ \psi(0,\kappa) &= E_{\vartheta}(\kappa^{\vartheta}), \\ \psi(0,0) &= 1. \end{aligned} \tag{5.2}$$

I. Below we present the LFDTM.

Taking the LFDTM of (5.1), by using the basic operation in Theorem 3.3, we have

$$\frac{\Gamma(1+(\beta+1)\vartheta)}{\Gamma(1+\beta\vartheta)}\frac{\Gamma(1+(\varepsilon+1)\vartheta)}{\Gamma(1+\varepsilon\vartheta)}\Psi(\beta+1,\varepsilon+1) = \Psi(\beta,\varepsilon),$$

or equivalently

$$\Psi(\beta+1,\varepsilon+1) = \frac{\Gamma(1+\beta\vartheta)}{\Gamma(1+(\beta+1)\vartheta)}\frac{\Gamma(1+\varepsilon\vartheta)}{\Gamma(1+(\varepsilon+1)\vartheta)}\Psi(\beta,\varepsilon), \tag{5.3}$$

where

$$\Psi(\beta,0) = \frac{1}{\Gamma(1+\beta\vartheta)}, \quad \Psi(0,\varepsilon) = \frac{1}{\Gamma(1+\varepsilon\vartheta)}, \quad \Psi(0,0) = 1. \tag{5.4}$$

In view of (5.3) and (5.4), the results are listed as follows:

$$\begin{aligned}
\Psi(1,1) &= \frac{1}{\Gamma(1+\vartheta)}\frac{1}{\Gamma(1+\vartheta)}, & \Psi(1,2) &= \frac{1}{\Gamma(1+\vartheta)}\frac{1}{\Gamma(1+2\vartheta)}, \\
\Psi(1,3) &= \frac{1}{\Gamma(1+\vartheta)}\frac{1}{\Gamma(1+3\vartheta)}, & \Psi(1,4) &= \frac{1}{\Gamma(1+\vartheta)}\frac{1}{\Gamma(1+4\vartheta)}, \\
\Psi(2,1) &= \frac{1}{\Gamma(1+2\vartheta)}\frac{1}{\Gamma(1+\vartheta)}, & \Psi(2,2) &= \frac{1}{\Gamma(1+2\vartheta)}\frac{1}{\Gamma(1+2\vartheta)}, \\
\Psi(2,3) &= \frac{1}{\Gamma(1+2\vartheta)}\frac{1}{\Gamma(1+3\vartheta)}, & \Psi(2,4) &= \frac{1}{\Gamma(1+2\vartheta)}\frac{1}{\Gamma(1+4\vartheta)}, \\
\Psi(3,1) &= \frac{1}{\Gamma(1+3\vartheta)}\frac{1}{\Gamma(1+\vartheta)}, & \Psi(3,2) &= \frac{1}{\Gamma(1+3\vartheta)}\frac{1}{\Gamma(1+2\vartheta)}, \\
\Psi(3,3) &= \frac{1}{\Gamma(1+3\vartheta)}\frac{1}{\Gamma(1+3\vartheta)}, & \Psi(3,4) &= \frac{1}{\Gamma(1+3\vartheta)}\frac{1}{\Gamma(1+4\vartheta)}, \\
\Psi(4,4) &= \frac{1}{\Gamma(1+4\vartheta)}\frac{1}{\Gamma(1+4\vartheta)}
\end{aligned}$$

and so on. In general, we obtain

$$\Psi(\beta,\varepsilon)=\frac{1}{\Gamma(1+\beta\vartheta)}\frac{1}{\Gamma(1+\varepsilon\vartheta)}. \tag{5.5}$$

Utilizing (3.3) and (5.5), we have

$$\begin{aligned}\psi(\eta,\kappa)&=\sum_{\beta=0}^{\infty}\sum_{\varepsilon=0}^{\infty}\Psi(\beta,\varepsilon)\eta^{\beta\vartheta}\kappa^{\varepsilon\vartheta}\\&=\sum_{\beta=0}^{\infty}\sum_{\varepsilon=0}^{\infty}\frac{\eta^{\beta\vartheta}}{\Gamma(1+\beta\vartheta)}\frac{\kappa^{\varepsilon\vartheta}}{\Gamma(1+\varepsilon\vartheta)}\\&=E_\vartheta(\eta^\vartheta)E_\vartheta(\kappa^\vartheta).\end{aligned}$$

II. As a next step, we apply the LFDM.

From(5.1), (5.2), (4.4), and (4.5) we get the following iterative formula:

$$\psi_0(\eta,\kappa)=E_\vartheta(\eta^\vartheta)+E_\vartheta(\kappa^\vartheta)-1, \tag{5.6}$$

$$\psi_{m+1}(\eta,\kappa)=L_\eta^{(-\vartheta)}L_\kappa^{(-\vartheta)}\psi_m(\eta,\kappa),\quad m\geq 0. \tag{5.7}$$

Utilizing (5.6) and (5.7), we obtain the following approximations:

$$\begin{aligned}\psi_1(\eta,\kappa)&=L_\eta^{(-\vartheta)}L_\kappa^{(-\vartheta)}\psi_0(\eta,\kappa)\\&=L_\eta^{(-\vartheta)}L_\kappa^{(-\vartheta)}\left[E_\vartheta(\eta^\vartheta)+E_\vartheta(\kappa^\vartheta)-1\right]\\&=\frac{\kappa^\vartheta}{\Gamma(1+\vartheta)}E_\vartheta(\eta^\vartheta)+\frac{\eta^\vartheta}{\Gamma(1+\vartheta)}E_\vartheta(\kappa^\vartheta)-\frac{\eta^\vartheta}{\Gamma(1+\vartheta)}\frac{\kappa^\vartheta}{\Gamma(1+\vartheta)}-\frac{\eta^\vartheta}{\Gamma(1+\vartheta)}-\frac{\kappa^\vartheta}{\Gamma(1+\vartheta)},\end{aligned}$$

$$\begin{aligned}\psi_2(\eta,\kappa)&=L_\eta^{(-\vartheta)}L_\kappa^{(-\vartheta)}\psi_1(\eta,\kappa)\\&=\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)}E_\vartheta(\eta^\vartheta)+\frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}E_\vartheta(\kappa^\vartheta)-\frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)}-\frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)}\\&\quad-\frac{\eta^{\vartheta}}{\Gamma(1+\vartheta)}\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)}-\frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}-\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)},\end{aligned}$$

$$\begin{aligned}\psi_3(\eta,\kappa)&=L_\eta^{(-\vartheta)}L_\kappa^{(-\vartheta)}\psi_2(\eta,\kappa)\\&=\frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)}E_\vartheta(\eta^\vartheta)+\frac{\eta^{3\vartheta}}{\Gamma(1+3\vartheta)}E_\vartheta(\kappa^\vartheta)-\frac{\eta^{3\vartheta}}{\Gamma(1+3\vartheta)}\frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)}-\frac{\eta^{3\vartheta}}{\Gamma(1+3\vartheta)}\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)}\\&\quad-\frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}\frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)}-\frac{\eta^{3\vartheta}}{\Gamma(1+3\vartheta)}\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)}-\frac{\eta^{\vartheta}}{\Gamma(1+\vartheta)}\frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)}-\frac{\eta^{3\vartheta}}{\Gamma(1+3\vartheta)},\end{aligned}$$

and so on. Therefore, we get the solution of (5.1) as follows:

$$\begin{aligned}\psi(\eta,\kappa)&=E_\vartheta(\eta^\vartheta)\left[1+\frac{\kappa^\vartheta}{\Gamma(1+\vartheta)}+\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)}+\frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)}+\cdots\right]\\&\quad+E_\vartheta(\kappa^\vartheta)\left[1+\frac{\eta^\vartheta}{\Gamma(1+\vartheta)}+\frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}+\frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)}+\cdots\right]\\&\quad-\left[1+\frac{\eta^\vartheta}{\Gamma(1+\vartheta)}+\frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}+\cdots\right]\left[1+\frac{\kappa^\vartheta}{\Gamma(1+\vartheta)}+\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)}+\cdots\right]\\&=E_\vartheta(\eta^\vartheta)E_\vartheta(\kappa^\vartheta).\end{aligned}$$

Example 5.2. Consider the following nonhomogeneous Goursat problem with local fractional differential operators:

$$\frac{\partial^{2\vartheta}}{\partial\eta^{\vartheta}\partial\kappa^{\vartheta}}\psi(\eta,\kappa)=\psi(\eta,\kappa)-\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)}, \tag{5.8}$$

subject to the initial conditions

$$\begin{aligned}&\psi(\eta,0)=E_{\vartheta}(\eta^{\vartheta}),\\&\psi(0,\kappa)=\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)}+E_{\vartheta}(\kappa^{\vartheta}),\\&\psi(0,0)=1.\end{aligned} \tag{5.9}$$

I. By using LFDTM.

Taking the LFDTM of (5.8) and using the basic operation in Theorem 3.3 one can observe that

$$\frac{\Gamma(1+(\beta+1)\vartheta)}{\Gamma(1+\beta\vartheta)}\frac{\Gamma(1+(\varepsilon+1)\vartheta)}{\Gamma(1+\varepsilon\vartheta)}\Psi(\beta+1,\varepsilon+1)=\Psi(\beta,\varepsilon)-\frac{\delta_{\vartheta}(\beta,\varepsilon-1)}{\Gamma(1+\vartheta)},$$

or equivalently

$$\Psi(\beta+1,\varepsilon+1)=\frac{\Gamma(1+\beta\vartheta)}{\Gamma(1+(\beta+1)\vartheta)}\frac{\Gamma(1+\varepsilon\vartheta)}{\Gamma(1+(\varepsilon+1)\vartheta)}\left[\Psi(\beta,\varepsilon)-\frac{\delta_{\vartheta}(\beta,\varepsilon-1)}{\Gamma(1+\vartheta)}\right], \tag{5.10}$$

where

$$\Psi(\beta,0)=\frac{1}{\Gamma(1+\beta\vartheta)},\quad \Psi(0,\varepsilon)=\frac{1}{\Gamma(1+\varepsilon\vartheta)}+\frac{\delta_{\vartheta}(\beta,\varepsilon-1)}{\Gamma(1+\vartheta)},\quad \Psi(0,0)=1. \tag{5.11}$$

In view of (5.10) and (5.11), the results are listed as follows:

$$\begin{aligned}
&\Psi(0,1)=\frac{1}{\Gamma(1+\vartheta)}\frac{1}{\Gamma(1+\vartheta)}, && \Psi(0,2)=\frac{1}{\Gamma(1+2\vartheta)}, && \Psi(0,3)=\frac{1}{\Gamma(1+3\vartheta)},\\
&\Psi(0,4)=\frac{1}{\Gamma(1+4\vartheta)}, && \Psi(1,1)=\frac{1}{\Gamma(1+\vartheta)}\frac{1}{\Gamma(1+\vartheta)}, && \Psi(1,2)=\frac{1}{\Gamma(1+\vartheta)}\frac{1}{\Gamma(1+2\vartheta)},\\
&\Psi(1,3)=\frac{1}{\Gamma(1+\vartheta)}\frac{1}{\Gamma(1+3\vartheta)}, && \Psi(1,4)=\frac{1}{\Gamma(1+\vartheta)}\frac{1}{\Gamma(1+4\vartheta)}, && \Psi(2,1)=\frac{1}{\Gamma(1+2\vartheta)}\frac{1}{\Gamma(1+\vartheta)},\\
&\Psi(2,2)=\frac{1}{\Gamma(1+2\vartheta)}\frac{1}{\Gamma(1+2\vartheta)}, && \Psi(2,3)=\frac{1}{\Gamma(1+2\vartheta)}\frac{1}{\Gamma(1+3\vartheta)}, && \Psi(2,4)=\frac{1}{\Gamma(1+2\vartheta)}\frac{1}{\Gamma(1+4\vartheta)},\\
&\Psi(3,1)=\frac{1}{\Gamma(1+3\vartheta)}\frac{1}{\Gamma(1+\vartheta)}, && \Psi(3,2)=\frac{1}{\Gamma(1+3\vartheta)}\frac{1}{\Gamma(1+2\vartheta)}, && \Psi(3,3)=\frac{1}{\Gamma(1+3\vartheta)}\frac{1}{\Gamma(1+3\vartheta)},\\
&\Psi(3,4)=\frac{1}{\Gamma(1+3\vartheta)}\frac{1}{\Gamma(1+4\vartheta)}, && \Psi(4,4)=\frac{1}{\Gamma(1+4\vartheta)}\frac{1}{\Gamma(1+4\vartheta)}
\end{aligned}$$

and so on. In general, we obtain

$$\Psi(\beta,\varepsilon)=\begin{cases}\dfrac{1}{\Gamma(1+\vartheta)}\dfrac{1}{\Gamma(1+\vartheta)}, & \beta=0,\varepsilon=1,\\ \dfrac{1}{\Gamma(1+\beta\vartheta)}\dfrac{1}{\Gamma(1+\varepsilon\vartheta)}, & \text{e.w.}\end{cases}$$

Hence, we get the solution of (5.8) as follows:

$$\begin{aligned}\psi(\eta,\kappa)&=\sum_{\beta=0}^{\infty}\sum_{\varepsilon=0}^{\infty}\Psi(\beta,\varepsilon)\eta^{\beta\vartheta}\kappa^{\varepsilon\vartheta}\\&=\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)}\sum_{\beta=0}^{\infty}\sum_{\varepsilon=0}^{\infty}\frac{\eta^{\beta\vartheta}}{\Gamma(1+\beta\vartheta)}\frac{\kappa^{\varepsilon\vartheta}}{\Gamma(1+\varepsilon\vartheta)}\\&=\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)}E_{\vartheta}(\eta^{\vartheta})E_{\vartheta}(\kappa^{\vartheta}).\end{aligned}$$

II. By using LFDM.

From (5.8), (5.9), (4.4) and (4.5), we get the following iterative formula:

$$\psi_0(\eta,\kappa) = E_\vartheta(\eta^\vartheta) + E_\vartheta(\kappa^\vartheta) + \frac{\kappa^\vartheta}{\Gamma(1+\vartheta)} - 1 - \frac{\eta^\vartheta}{\Gamma(1+\vartheta)}\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)}, \tag{5.12}$$

$$\psi_{m+1}(\eta,\kappa) = L_\eta^{(-\vartheta)} L_\kappa^{(-\vartheta)} \psi_m(\eta,\kappa), \quad m \geq 0. \tag{5.13}$$

Utilizing (5.12) and (5.13), we obtain the following approximations:

$$\begin{aligned}
\psi_1(\eta,\kappa) &= L_\eta^{(-\vartheta)} L_\kappa^{(-\vartheta)} \psi_0(\eta,\kappa)\\
&= L_\eta^{(-\vartheta)} L_\kappa^{(-\vartheta)} \left[E_\vartheta(\eta^\vartheta) + E_\vartheta(\kappa^\vartheta) + \frac{\kappa^\vartheta}{\Gamma(1+\vartheta)} - 1 - \frac{\eta^\vartheta}{\Gamma(1+\vartheta)}\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)}\right]\\
&= \frac{\kappa^\vartheta}{\Gamma(1+\vartheta)} E_\vartheta(\eta^\vartheta) + \frac{\eta^\vartheta}{\Gamma(1+\vartheta)} E_\vartheta(\kappa^\vartheta) + \frac{\eta^\vartheta}{\Gamma(1+\vartheta)}\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)} - \frac{\eta^\vartheta}{\Gamma(1+\vartheta)}\frac{\kappa^\vartheta}{\Gamma(1+\vartheta)}\\
&\quad - \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}\frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)} - \frac{\eta^\vartheta}{\Gamma(1+\vartheta)} - \frac{\kappa^\vartheta}{\Gamma(1+\vartheta)},
\end{aligned}$$

$$\begin{aligned}
\psi_2(\eta,\kappa) &= L_\eta^{(-\vartheta)} L_\kappa^{(-\vartheta)} \psi_1(\eta,\kappa)\\
&= \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)} E_\vartheta(\eta^\vartheta) + \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)} E_\vartheta(\kappa^\vartheta) + \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}\frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)} - \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)}\\
&\quad - \frac{\eta^{3\vartheta}}{\Gamma(1+3\vartheta)}\frac{\kappa^{4\vartheta}}{\Gamma(1+4\vartheta)} - \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}\frac{\kappa^\vartheta}{\Gamma(1+\vartheta)} - \frac{\eta^\vartheta}{\Gamma(1+\vartheta)}\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)} - \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}\\
&\quad - \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)},
\end{aligned}$$

and so on. Therefore, we get the solution of (5.8) as follows:

$$\begin{aligned}
\psi(\eta,\kappa) &= \frac{\kappa^\vartheta}{\Gamma(1+\vartheta)} + E_\vartheta(\eta^\vartheta)\left[1 + \frac{\kappa^\vartheta}{\Gamma(1+\vartheta)} + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)} + \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)} + \cdots\right]\\
&\quad + E_\vartheta(\kappa^\vartheta)\left[1 + \frac{\eta^\vartheta}{\Gamma(1+\vartheta)} + \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)} + \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)} + \cdots\right]\\
&\quad - \left[1 + \frac{\eta^\vartheta}{\Gamma(1+\vartheta)} + \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)} + \cdots\right]\left[1 + \frac{\kappa^\vartheta}{\Gamma(1+\vartheta)} + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)} + \cdots\right]\\
&= \frac{\kappa^\vartheta}{\Gamma(1+\vartheta)} + E_\vartheta(\eta^\vartheta) E_\vartheta(\kappa^\vartheta).
\end{aligned}$$

6. Conclusions

The LFDTM and LFDM have been successfully applied to obtain the analytical solutions of problems that arise in partial differential equations with mixed derivatives. The examples show that the results of local fractional differential transform method are in excellent agreement with the results given by the local fractional decomposition method. The LFDTM reduces the computational difficulties existing in the LFDM and all the calculations can be done by simple manipulations.

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