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Some results about Krasnosel'skiĭ-Mann iteration process

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Abstract

We introduce a Mann type iteration method and give a result about strongly convergence of this iteration method to a fixed point of nonexpansive mappings on Banach spaces. Also, by using idea of Ishikawa iteration method, we introduce a new iteration method via two mappings on uniformly convex Banach spaces and we provide a result about strongly convergence of the iteration method to a common fixed points of the mappings.

Keywords: Fixed point, Krasnosel'skiĭ-Mann iteration, nonexpansive mapping.
2010 MSC: 47H10, 54H25.

1. Introduction

In 1953, Mann [11] defined an iterative method. Let C be a nonempty convex subset of a linear space X. Let $\{\alpha_n\}_{n\geq 1}$ be a sequence in $[0,1)$ satisfying $\sum_{n=1}^{\infty}\alpha_n = \infty$. Consider T a selfmap on C and $x_1 \in C$. Define a sequence $\{x_n\}_{n\geq 1}$ in C by

$$x_{n+1} = M(x_n, \alpha_n, T) = (1-\alpha_n)x_n + \alpha_n T x_n$$

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for all $n \geq 1$. Then the sequence $\{x_n\}_{n\geq 1}$ is called the Mann iteration. In 1955, Krasnosel'skiĭ [10] defined a modified Mann iteration method. Let $\{\alpha_n\}_{n\geq 1}$ be a sequence in $[0,1)$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < a \leq \alpha_n \leq b < 1$ for some constants a and b. Then the sequence $\{x_n\}_{n\geq 1}$ defined by

$$x_{n+1} = M(x_n, \alpha_n, T) = (1-\alpha_n)x_n + \alpha_n T^n x_n$$

is called a modified Mann iteration. Later, Ishikawa [7] defined an iteration method in 1974. Let $x_1 \in C$. Take $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ two sequences in $[0,1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$ and $0 \leq \alpha_n \leq \beta_n \leq 1$ for all $n \geq 1$. Then the sequence $\{x_n\}_{n\geq 1}$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned}$$

is called the Ishikawa iteration. The following definition is needed in the sequel.

Definition 1.1. Let X be a Banach space and $T : X \to X$ a selfmap. We have the following:

(i) T is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in X$.

(ii) Let X^* be the dual of a Banach space X. Then a multivalued mapping $J : X \to 2^{X^*}$ is said to be a (normalized) duality mapping, where for each $x \in X$,

$$Jx = \{j \in X^* : j(x) =< x, j >= \|x\|^2 = \|j\|^2\}.$$

(iii) T is said pseudocontractive if $\|x - y\| \leq \|(1+t)(x-y) - t(Tx - Ty)\|$ for all $x, y \in X$ and $t > 0$.

(iv) T is called Lipschitzian if for all $x, y \in X$, there exists $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|.$$

(v) T is called Lipschitzian pseudocontractive whenever T is Lipschitzian and pseudocontractive.

Remark 1.2. A nonexpansive mapping is continuous.

In 1976, Ishikawa [8] proved some results about fixed points of nonexpansive mappings by using his iteration method. Later, by using this idea and variant modified Mann iteration methods, some authors established many results about fixed points of strictly pseudocontractive mappings [3], common fixed points of an infinite family of nonexpansive mappings [5, 12] and asymptotically nonexpansive mappings [4, 6, 9, 13, 14].

In this paper, we introduce a Mann type iteration method. Using it, we give a result about strongly convergence of the iteration method to a fixed point of nonexpansive mappings. Also, by using the idea of Ishikawa iteration method, we introduce a new iteration method via two mappings on uniformly convex Banach spaces and we prove a result about strongly convergence of the iteration method to a common fixed points of the mappings.

The following results will be needed in the sequel.

Lemma 1.3 ([1]). *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a uniformly convex Banach space X such that $x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n$ and $\| y_n \| \leq \| x_n \|$ for all $n \geq 1$, where $\{\alpha_n\}$ is a sequence of nonnegative numbers in $[0,1]$ such that $\sum_{n=1}^{\infty} \min\{\alpha_n, 1-\alpha_n\} = \infty$. Then $0 \in \overline{\{x_n - y_n,\ n \geq 1\}}$.*

Lemma 1.4 ([2]). *Let $\delta \in [0,1)$ and $\{\varepsilon_n\}_{n\geq 1}$ be a positive sequence satisfying $\lim_{n\to\infty} \varepsilon_n = 0$. Then, for any positive sequence $\{u_n\}_{n\geq 1}$ satisfying $u_{n+1} \leq \delta u_n + \varepsilon_n$, it follows that $\lim_{n\to\infty} u_n = 0$.*

2. Main results

Now, we are ready to state our main results via Mann type and Krasnosel'skiĭ type iterations.

2.1. *Strong convergence to a fixed point of nonexpansive selfmaps on nonempty convex closed subset of a Banach space X*

First, we introduce a Mann type iteration method.

Definition 2.1. Let C be a nonempty convex subset of a linear space X. Let $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$ be three sequences in $[0, 1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$. Consider T as a selfmap on C and $x_1 \in C$. Define a sequence $\{x_n\}_{n\geq 1}$ in C by

$$x_{n+1} = GM(x_n, \alpha_n, \beta_n, \gamma_n, T) = \alpha_n x_n + \beta_n T x_n + \gamma_n T^2 x_n$$

for all $n \geq 1$. Then the sequence $\{x_n\}_{n\geq 1}$ is called a Mann type iteration.

Note that, $GM(x_n, \alpha_n, \beta_n, \gamma_n, T) \in conv(C) = C$ for all $n \geq 1$. If $\gamma_n = 0$ for all $n \geq 1$, then the Mann type iteration is reduced to the Mann iteration. Let C be a nonempty convex closed subset of a Banach space X and T be a continuous selfmap on C. It has been proved that if the Mann iteration $\{x_n\}_{n\geq 1}$ converges strongly to a point $p \in C$, then p is a fixed point of T. Now, we prove it for our Mann type iteration on nonexpansive mappings.

Theorem 2.2. *Let C be a nonempty convex closed subset of a Banach space X and T be a nonexpansive selfmap on C. If the Mann type iteration $\{x_n\}_{n\geq 1}$ converges strongly to a point $p \in C$, then p is a fixed point of T.*

Proof. Assume that $\{x_n\}_{n\geq 1}$ converges to p. To prove that p is a fixed point of T, we argue by contradiction. Suppose that $p \neq Tp$. Define $\varepsilon_n := x_n - Tx_n - (p - Tp)$ and $\delta_n := T^2x_n - Tx_n - (T^2p - Tp)$ for all $n \geq 1$. The map T is nonexpansive and for all n we have

$$\|\varepsilon_n\| \leq \|x_n - Tx_n - (p - Tp)\| \leq \|x_n - p\| + \|Tx_n - Tp\| \leq 2\|x_n - p\|,$$

and

$$\|\delta_n\| \leq \|T^2x_n - T^2p - (Tx_n - Tp)\| \leq \|T^2x_n - T^2p\| + \|Tx_n - Tp\| \leq 2\|x_n - p\|.$$

Since $\lim_{n\to\infty} x_n = p$, we find that $\lim_{n\to\infty} \varepsilon_n = 0$ and $\lim_{n\to\infty} \delta_n = 0$. Due to the fact that $\| p - Tp \| > 0$, there exists a natural number n_0 such that $\| \varepsilon_n \| < \frac{\|p-Tp\|}{3}$, $\| \delta_n \| < \frac{\|p-Tp\|}{2}$ and $\| x_n - x_m \| < \frac{\|p-Tp\|}{7}$ for all $n, m \geq n_0$. Choose a natural number N such that $\sum_{i=n_0}^{n_0+N}(1-\alpha_i) \geq 1$ and $\sum_{i=n_0}^{n_0+N}\gamma_i \leq \frac{1}{3}$. Note that

$$\begin{aligned}
x_{i+1} - x_i &= \alpha_i x_i + \beta_i Tx_i + \gamma_i T^2x_i - \alpha_{i-1}x_{i-1} - \beta_{i-1}Tx_{i-1} - \gamma_{i-1}T^2x_{i-1}\\
&= \alpha_i x_i - \alpha_i Tx_i + \alpha_i Tx_i - \alpha_{i-1}x_{i-1} + \beta_i Tx_i - \beta_{i-1}Tx_{i-1}\\
&\quad + \gamma_i T^2x_i - \gamma_i Tx_i + \gamma_i Tx_i - \gamma_{i-1}T^2x_{i-1}\\
&= \alpha_i x_i - \alpha_i Tx_i + (\alpha_i Tx_i + \beta_i Tx_i + \gamma_i Tx_i) - (\alpha_{i-1}x_{i-1} + \beta_{i-1}Tx_{i-1} + \gamma_{i-1}T^2x_{i-1})\\
&\quad + \gamma_i T^2x_i - \gamma_i Tx_i\\
&= \alpha_i x_i - \alpha_i Tx_i + Tx_i(\alpha_i + \beta_i + \gamma_i) - x_i + \gamma_i T^2x_i - \gamma_i Tx_i\\
&= \alpha_i(x_i - Tx_i) + Tx_i - x_i + \gamma_i(T^2x_i - Tx_i)\\
&= (1-\alpha_i)(Tx_i - x_i) + \gamma_i(T^2x_i - Tx_i).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\| x_{n_0} - x_{n_0+N+1} \| &= \| \sum_{i=n_0}^{n_0+N} (x_i - x_{i+1}) \| \\
&= \| \sum_{i=n_0}^{n_0+N} ((1-\alpha_i)(p - Tp + \varepsilon_i) + \gamma_i(T^2p - Tp + \delta_i)) \| \\
&\geq \| \sum_{i=n_0}^{n_0+N} (1-\alpha_i)(p - Tp) \| - \| \sum_{i=n_0}^{n_0+N} (1-\alpha_i)\varepsilon_i \| \\
&\quad - \| \sum_{i=n_0}^{n_0+N} \gamma_i(T^2p - Tp) \| - \| \sum_{i=n_0}^{n_0+N} \gamma_i\delta_i \| \\
&\geq \sum_{i=n_0}^{n_0+N} (1-\alpha_i)(\| p - Tp \| - \frac{\| p - Tp \|}{3}) \\
&\quad - \sum_{i=n_0}^{n_0+N} \gamma_i(\| p - Tp \| + \| \delta_i \|) \\
&\geq 2\frac{\| p - Tp \|}{3} - \frac{\| p - Tp \|}{2} = \frac{\| p - Tp \|}{6}.
\end{aligned}$$

This contradiction completes the proof. □

Question. Does above result hold for the continuous mappings?

2.2. Strong convergence to a common fixed point of nonexpansive maps from nonempty closed convex bounded subsets C of a uniformly convex Banach space X into a compact subset of C

Now, let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and T be a nonexpansive mapping from C into a compact subset of C. Taking $x_1 \in C$, Krasnosel'skiĭ [10] proved that the sequence $\{x_n\}_{n\geq 1}$ defined by

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n) = M(x_n, \frac{1}{2}, T)$$

converges strongly to a fixed point of T [1, Theorem 6.4.1]. Now, we extend it by using two nonexpansive mappings.

Theorem 2.3. *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $x_1 \in C$. Let T_1 and T_2 be two nonexpansive mappings from C into a compact subset of C with $F(T_1) \cap F(T_2) \neq \phi$. If $\| x_n - T_1x_n \| \to 0$ or $\| x_n - T_2x_n \| \to 0$, then the sequence $\{x_n\}$ defined by $x_{n+1} = \frac{1}{4}(2x_n + T_1x_n + T_2x_n)$ converges strongly to a common fixed point of T_1 and T_2.*

Proof. Let $p \in F(T_1) \cap F(T_2)$. Then, for each $n \geq 1$ we have

$$\begin{aligned}
\| 2x_{n+1} - T_1x_{n+1} - T_2x_{n+1} \| &\leq \frac{1}{2}[\| 2x_n - T_1x_n - T_2x_n \| \\
&\quad + 2 \| T_1x_n + T_2x_n - T_1x_{n+1} - T_2x_{n+1} \|] \\
&\leq \frac{1}{2} \| 2x_n - T_1x_n - T_2x_n \| + 2 \| x_n - x_{n+1} \| \\
&= \frac{1}{2} \| 2x_n - T_1x_n - T_2x_n \| + \frac{1}{2} \| 2x_n - T_1x_n - T_2x_n \| \\
&= \| 2x_n - T_1x_n - T_2x_n \| .
\end{aligned}$$

Hence, $\{\| 2x_n - T_1x_n - T_2x_n \|\}_{n\geq 1}$ is a nonincreasing sequence and so it converges. Now, note that

$$\begin{aligned} x_{n+1} - p &= \frac{1}{4}(2x_n + T_1x_n + T_2x_n - 4p) = \frac{1}{4}[2(x_n - p) + (T_1x_n + T_2x_n - 2p)] \\ &= \frac{1}{2}(x_n - p) + \frac{1}{2}(\frac{1}{2}T_1x_n + \frac{1}{2}T_2x_n - p). \end{aligned}$$

Since $\|\frac{1}{2}T_1x_n+\frac{1}{2}T_2x_n-p\| \leq \|x_n-p\|$, by using Lemma 1.3, we deduce that $\liminf_{n\to\infty} \| 2x_n-T_1x_n-T_2x_n \|= 0$.

Suppose that $\lim_{n\to\infty} \| x_n - T_1x_n \|= 0$. Since $\{T_1x_n\}$ is a sequence in a compact set, there exist a subsequence $\{T_1x_{n_k}\}$ of $\{T_1x_n\}$ and $v \in C$ such that $T_1x_{n_k} \to v$. Again, $\lim_{k\to\infty} \| x_{n_k} - T_1x_{n_k} \|= 0$, so $x_{n_k} \to v$ as $k \to \infty$. The map T_1 is continuous, hence v is a fixed point of T_1. But,

$$|\| x_n - T_2x_n \| - \| x_n - T_1x_n \||\leq\| 2x_n - T_1x_n - T_2x_n \|$$

for all $n \geq 1$. Hence, we have $\lim_{n\to\infty} \| x_n - T_2x_n \|= 0$. This implies that v is a fixed point of T_2. On the other hand, X is a uniformly convex Banach space, then $\lim_{n\to\infty} \| x_n - v \|= \lim_{k\to\infty} \| x_{nk} - v \|= 0$. Therefore, the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2. □

Remark 2.4. If $T_1 = T_2$, then our iteration method is reduced to $M(x_n, \frac{1}{2}, T_1)$, that is, the main result of Krasnosel'skiĭ [10].

Theorem 2.5. *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $x_1, y_1 \in C$. Let T_1 and T_2 be two nonexpansive mappings from C into a compact subset of C with $F(T_1) \cap F(T_2) \neq \emptyset$. Define the sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ by $x_{n+1} = \frac{1}{2}(x_n + T_1(z_n))$ and $y_{n+1} = \frac{1}{2}(y_n+T_2(z_n))$, where $z_n = \frac{1}{2}(x_n+y_n)$. If $\| z_n-T_1z_n \|\to 0$ or $\| z_n-T_2z_n \|\to 0$, then the sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ converge strongly to a common fixed point of T_1 and T_2.*

Proof. First, we write $z_{n+1} = \frac{1}{4}(2z_n + T_1(z_n) + T_2(z_n))$ for all $n \geq 1$. By using Theorem 2.3, $\{z_n\}$ converges strongly to a common fixed point of T_1 and T_2. On the other hand, we have

$$\begin{aligned} \| x_{n+1} - y_{n+1} \| &= \frac{1}{2} \| x_n - y_n + T_1(\frac{x_n + y_n}{2}) - T_2(\frac{x_n + y_n}{2}) \| \\ &\leq \frac{1}{2} \| x_n - y_n \| +\frac{1}{2} \| T_1(\frac{x_n + y_n}{2}) - T_2(\frac{x_n + y_n}{2}) \| . \end{aligned}$$

But,

$$\begin{aligned} \| T_1(\frac{x_n + y_n}{2}) - T_2(\frac{x_n + y_n}{2}) \| &=\| T_1(z_n) - T_2(z_n) \| \\ &\leq\| T_1(z_n) - z_n \| + \| z_n - T_2(z_n) \| . \end{aligned}$$

Since $\{z_n\}$ converges to a common fixed point of T_1 and T_2, by continuity of T_1 and T_2, the right hand-side converges to 0. Hence, by Lemma 1.4, we obtain $\lim_{n\to\infty} \| x_n - y_n \|= 0$. Therefore, the sequences $\{x_n\}$ and $\{y_n\}$ converge to a common fixed point of T_1 and T_2. □

Again, by using similar proofs, we can prove the following results. First, as Theorem 2.3, we state:

Theorem 2.6. *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $x_1 \in C$. Let T_1, T_2 and T_3 be nonexpansive mappings from C into a compact subset of C with $F(T_1) \cap F(T_2) \cap F(T_3) \neq \phi$. Define the sequence $\{x_n\}_{n\geq 1}$ by $x_{n+1} = \frac{1}{6}(3x_n + T_1x_n + T_2(x_n) + T_3(x_n))$ for all $n \geq 1$. Suppose also that one of the following statements holds:*

(i) $\| x_n - T_1x_n \|\to 0$ *and* $\| x_n - T_2x_n \|\to 0$*;*

(ii) $\| x_n - T_1 x_n \| \to 0$ *and* $\| x_n - T_3 x_n \| \to 0$;

(iii) $\| x_n - T_2 x_n \| \to 0$ *and* $\| x_n - T_3 x_n \| \to 0$.

Then the sequence $\{x_n\}$ *converges strongly to a common fixed point of* T_1, T_2 *and* T_3.

Similar to Theorem 2.5, we state the following result:

Theorem 2.7. *Let* C *be a nonempty closed convex bounded subset of a uniformly convex Banach space* X *and* $x_1, y_1, z_1 \in C$. *Let* T_1, T_2 *and* T_3 *be nonexpansive mappings from* C *into a compact subset of* C *with* $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. *Define the sequences* $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$ *and* $\{z_n\}_{n\geq 1}$ *by*

$$\begin{aligned} x_{n+1} &= \tfrac{1}{2}(x_n + T_1(\tfrac{x_n+y_n+z_n}{3})), \\ y_{n+1} &= \tfrac{1}{2}(y_n + T_2(\tfrac{x_n+y_n+z_n}{3})), \\ z_{n+1} &= \tfrac{1}{2}(z_n + T_3(\tfrac{x_n+y_n+z_n}{3})), \end{aligned}$$

for all $n \geq 1$. *Suppose also that one of the following statements holds:*

(i) $\| \frac{x_n+y_n+z_n}{3} - T_1(\frac{x_n+y_n+z_n}{3}) \| \to 0$ *and* $\| \frac{x_n+y_n+z_n}{3} - T_2(\frac{x_n+y_n+z_n}{3}) \| \to 0$;

(ii) $\| \frac{x_n+y_n+z_n}{3} - T_1(\frac{x_n+y_n+z_n}{3}) \| \to 0$ *and* $\| \frac{x_n+y_n+z_n}{3} - T_3(\frac{x_n+y_n+z_n}{3}) \| \to 0$;

(iii) $\| \frac{x_n+y_n+z_n}{3} - T_2(\frac{x_n+y_n+z_n}{3}) \| \to 0$ *and* $\| \frac{x_n+y_n+z_n}{3} - T_3(\frac{x_n+y_n+z_n}{3}) \| \to 0$.

Then the sequences $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$ *and* $\{z_n\}_{n\geq 1}$ *converge strongly to a common fixed point of* T_1, T_2 *and* T_3.

2.3. Strong convergence to a common fixed point of two Lipschitzian pseudocontractive mappings on Hilbert spaces

Here, by using above idea, we provide another new iteration method in order to obtain a strongly convergent sequence to a common fixed point of two Lipschitzian pseudocontractive mappings on Hilbert spaces.

Definition 2.8. Let C be a nonempty convex subset of a linear space X and $x_1 \in C$. Let T_1 and T_2 be two selfmaps on C and $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 1}$, $\{\alpha'_n\}$ and $\{\beta'_n\}$ be four sequences in $[0,1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \beta'_n = 0$, $0 \leq \alpha_n \leq \beta_n \leq 1$ and $0 \leq \alpha'_n \leq \beta'_n \leq 1$ for all $n \geq 1$. Now, define the sequence $\{x_n\}_{n\geq 1}$ by

$$\begin{aligned} x_{n+1} &= \frac{1}{2}((1-\alpha_n)x_n + \alpha_n T_1 y_n) + \frac{1}{2}((1-\alpha'_n)x_n + \alpha'_n T_2 z_n), \\ y_n &= (1-\beta_n)x_n + \beta_n T_1 x_n, \\ z_n &= (1-\beta'_n)x_n + \beta'_n T_2 x_n. \end{aligned}$$

We denote this iteration method by $x_{n+1} = R(x_n, \alpha_n, \beta_n, \alpha'_n, \beta'_n, T_1, T_2)$.

Theorem 2.9. *Let* C *be a nonempty compact convex subset of a Hilbert space* H. *Let* T_1 *and* T_2 *be two Lipschitzian pseudocontractive selfmaps on* C *such that* $F(T_1) \cap F(T_2) \neq \emptyset$. *Then the sequence*

$$\{R(x_n, \alpha_n, \beta_n, \alpha'_n, \beta'_n, T_1, T_2)\}$$

converges strongly to a common fixed point of T_1 *and* T_2.

Proof. Let $p \in F(T_1) \cap F(T_2)$. Then we have

$$\begin{aligned} x_{n+1} &= \frac{1}{2}((1-\alpha_n)x_n + \alpha_n T_1((1-\beta_n)x_n + \beta_n T_1 x_n)) \\ &\quad + \frac{1}{2}((1-\alpha'_n)x_n + \alpha'_n T_2((1-\beta'_n)x_n + \beta'_n T_2 x_n)). \end{aligned}$$

Therefore,

$$\begin{aligned} \| x_{n+1} - p \|^2 \leq \frac{1}{4}\| &(1-\alpha_n)x_n + \alpha_n T_1((1-\beta_n)x_n + \beta_n T_1 x_n) - p \|^2 + \frac{1}{2} \\ &\times \| (1-\alpha_n)x_n + \alpha_n T_1((1-\beta_n)x_n + \beta_n T_1 x_n) - p \| \\ &\times \| (1-\alpha'_n)x_n + \alpha'_n T_2((1-\beta'_n)x_n + \beta'_n T_2 x_n) - p \| \\ &+ \frac{1}{4}\| (1-\alpha'_n)x_n + \alpha'_n T_2((1-\beta'_n)x_n + \beta'_n T_2 x_n) - p \|^2. \end{aligned}$$

Since $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$, we obtain

$$\begin{aligned} \| x_{n+1} - p \|^2 &\leq \frac{1}{2}\| (1-\alpha_n)x_n + \alpha_n T_1((1-\beta_n)x_n + \beta_n T_1 x_n) - p \|^2 \\ &\quad + \frac{1}{2}\| (1-\alpha'_n)x_n + \alpha'_n T_2((1-\beta'_n)x_n + \beta'_n T_2 x_n) - p \|^2 \\ &\leq \frac{1}{2}(\| x_n - p \|^2 \\ &\quad - \alpha_n\beta_n(1-2\beta_n)\| x_n - T_1 x_n \|^2 - \alpha_n(\beta_n - \alpha_n)\| x_n - T_1 y_n \|^2 \\ &\quad + \alpha_n\beta_n\| T_1 x_n - T_1 y_n \|^2) + \frac{1}{2}(\| x_n - p \|^2 \\ &\quad - \alpha'_n\beta'_n(1-2\beta'_n)\| x_n - T_2 x_n \|^2 \\ &\quad - \alpha'_n(\beta'_n - \alpha'_n)\| x_n - T_2 z_n \|^2 + \alpha'_n\beta'_n\| T_2 x_n - T_2 z_n \|^2). \end{aligned}$$

In view of the fact that $\alpha_n \leq \beta_n$ and $\alpha'_n \leq \beta'_n$ for all $n \geq 1$, we deduce that

$$\begin{aligned} \| x_{n+1} - p \|^2 \leq \frac{1}{2}(&\| x_n - p \|^2 - \alpha_n\beta_n(1-2\beta_n)\| x_n - T_1 x_n \|^2 \\ &+ \alpha_n\beta_n\| T_1 x_n - T_1 y_n \|^2) \\ &+ \frac{1}{2}(\| x_n - p \|^2 - \alpha'_n\beta'_n(1-2\beta'_n)\| x_n - T_2 x_n \|^2 \\ &+ \alpha'_n\beta'_n\| T_2 x_n - T_2 z_n \|^2). \end{aligned} \tag{2.1}$$

Suppose that T_1 and T_2 are η_1-Lipschitzian and η_2-Lipschitzian mappings, respectively. Then

$$\begin{aligned} &\| T_1 x_n - T_1 y_n \| \leq \eta_1 \| x_n - y_n \| \leq \eta_1\beta_n \| x_n - T_1 x_n \|, \\ &\| T_2 x_n - T_2 z_n \| \leq \eta_2 \| x_n - z_n \| \leq \eta_2\beta'_n \| x_n - T_2 x_n \| . \end{aligned}$$

Hence, by using (2.1), we have

$$\begin{aligned} \| x_{n+1} - p \|^2 \leq \frac{1}{2}(&\| x_n - p \|^2 - \alpha_n\beta_n(1-2\beta_n - {\eta_1}^2{\beta_n}^2) \| x_n - T_1 x_n \|^2) \\ &+ \frac{1}{2}(\| x_n - p \|^2 - \alpha'_n\beta'_n(1-2\beta'_n - {\eta_2}^2{\beta'_n}^2)\| x_n - T_2 x_n \|^2). \end{aligned}$$

Having in mind that $\lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} \beta'_n = 0$, there exists a natural number n_0 such that $2\beta_n + {\eta_1}^2{\beta_n}^2 \leq \frac{1}{2}$ and $2\beta'_n + {\eta_2}^2{\beta'_n}^2 \leq \frac{1}{2}$ for all $n \geq n_0$. Hence,

$$\| x_{n+1} - p \|^2 \leq \| x_n - p \|^2 - \frac{1}{4}\alpha_n\beta_n \| x_n - T_1 x_n \|^2 - \frac{1}{4}\alpha'_n\beta'_n\| x_n - T_2 x_n \|^2).$$

This implies that

$$\frac{1}{4}\sum_{i=n_0}^{n}(\alpha_i\beta_i\| x_i - T_1x_i \|^2 + \alpha'_i\beta'_i\| x_i - T_2x_i \|^2) \leq \| x_{n_0} - p \|^2 - \| x_{n+1} - p \|^2$$

for all $n \geq n_0$. Since C is bounded, $\| x_{n+1} - p \|$ is a bounded sequence. Thus, the series

$$\sum_{i=1}^{\infty}(\alpha_i\beta_i\| x_i - T_1x_i \|^2 + \alpha'_i\beta'_i\| x_i - T_2x_i \|^2)$$

is convergent. This yields that

$$\lim_{n\to\infty} \| x_n - T_1x_n \| = 0 \quad \text{and} \quad \lim_{n\to\infty} \| x_n - T_2x_n \| = 0.$$

Now, by using a similar proof in the last part of Theorem 2.3, one can show that the sequence converges strongly to a common fixed point of T_1 and T_2. □

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