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A novel approach of variable order derivative: Theory and Methods

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Abstract

In order to solve problems posed while using the concept of fractional variable order derivative, we introduce in this work a novel fractional variable order derivative. Our derivative has no singular kernel, this allows it to well-describe the effect of memory. We present the relationship between the new derivative with the well-known integral transforms. We present exact solution of some basic associated differential equations. We presented the numerical approximation of the derivative for first and second order approximation.

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1. Introduction

In many discipline and manufacturing fields, the concept of memory effect or properties have been extensively found in complicated classifications [11, 12]. To mention, the memory effect have been found in uncharacteristic transmission, in viscoelastic deformation medium for instance an aquifer, stock market, bacterial chemo-taxis and many other complex networks [4–6, 8, 14]. Nonetheless, how to portray truthfully the memory property of systems is still remain a thought-provoking subject in objective modeling and phenomenological portrayal. From widespread hypothetical and investigational analysis, the concept

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of fractional operators has been contemplated as one of the superlative mathematical tools to exemplify the memory property of complicated systems and certain materials [3, 10, 16]. But these fractional operators with constant order have failed to accurately characterize more complex system as for instance the movement of pollution within a deformable aquifer [9]. But the variable-order fractional derivative, which are extension of constant-order fractional derivative have been suggested and have been considered as the best mathematical operators to depicting the memory property which changes with time or spatial location [2, 15]. However, there are couples of problems associate to these variable order derivatives. For instance differential equations with these derivatives cannot be solved analytically, we cannot find a clear relation between these derivatives with the well-known integral transform like Mellin transform, Laplace transform, Fourier transform and Sumudu transform. With these derivatives, one cannot calculate the derivative of a simple function for instance sinus or exponential function. Therefore a new fractional variable order is needed which can be handled analytically and also have a relationship with the well-known integral transform. In the work, we will propose a fractional variable order derivative, that can be handled analytically and also have relationship with some other integral transform. The rest of the paper with be structured as follows: In Section 2, we present some definitions of existing fractional variable order operators.

1.1. Preliminaries

Let's recall the relevant definitions for variable order fractional calculus [1, 7, 13, 17].

Definition 1.1. Left and right Riemann-Liouville integrals of variable order: Let $0 < \alpha(x,t) < 1$ for all $(x,t) \in [a,b]$ and $f \in L_1[a,b]$ then,

$$ {}_aI_t^{\alpha(.,.)}(f(t)) = \int_a^t \frac{1}{\Gamma[\alpha(t,x)]}(t-x)^{\alpha(t,x)-1} f(x)dx, \qquad (t > a) $$

is called the left Riemann-Liouville integral of variable fractional order $\alpha(.,.)$. While

$$ {}_bI_t^{\alpha(.,.)}(f(t)) = \int_t^b \frac{1}{\Gamma[\alpha(t,x)]}(t-x)^{\alpha(t,x)-1} f(x)dx, \qquad (t < b) $$

is referred to as the right Riemann-Liouville integral of variable fractional order $\alpha(.,.)$.

Definition 1.2. Left and right Riemann-Liouville derivatives of variable order: Let $0 < \alpha(x,t) < 1$ for all $(x,t) \in [a,b]$. If ${}_aI_t^{1-\alpha(.,.)} f \in AC[a,b]$ then, the left Riemann-Liouville derivative of variable fractional order $\alpha(.,.)$ is given as:

$$ {}_aD_t^{\alpha(.,.)}(f(t)) = \frac{d}{dt}\int_a^t \frac{1}{\Gamma[1-\alpha(t,x)]}(t-x)^{-\alpha(t,x)} f(x)dx, \qquad (t > a). $$

Likewise, we have the following expression referred to as the right Riemann-Liouville derivative of variable fractional order $\alpha(.,.)$,

$$ {}_bD_t^{\alpha(.,.)}(f(t)) = \frac{d}{dt}\int_t^b \frac{1}{\Gamma[1-\alpha(t,x)]}(t-x)^{-\alpha(t,x)} f(x)dx, \qquad (t < b). $$

Definition 1.3. Left and right Caputo derivatives of variable order: Let $0 < \alpha(x,t) < 1$ for all $(x,t) \in [a,b]$. If ${}_aI_t^{1-\alpha(.,.)} f \in AC[a,b]$ then, the left Caputo derivative of variable fractional order $\alpha(.,.)$ is given as:

$$ {}_aD_t^{\alpha(.,.)}(f(t)) = \int_a^t \frac{1}{\Gamma[1-\alpha(t,x)]}(t-x)^{-\alpha(t,x)} \frac{d}{dx} f(x)dx, \qquad (t > a). $$

Likewise, we have the following expression referred to as the right Caputo derivative of variable fractional order $\alpha(.,.)$,

$$
{}_bD_t^{\alpha(.,.)}\left(f(t)\right) = \int_t^b \frac{-1}{\Gamma[1-\alpha(t,x)]}(t-x)^{-\alpha(t,x)}\frac{d}{dx}f(x)dx, \qquad (t < b).
$$

However, we will use throught this work the following definition:

Definition 1.4. Let $f : \mathbb{R} \to \mathbb{R}$, $x \to f(x)$ denote a continuous and necessary differentiable function, let $\alpha(x)$ be a continuous function in $(0, 1]$. Then its variable order differential in $[0, M)$ is defined as:

$$
D_0^{\alpha(x)}\left(f(x)\right) = \frac{1}{\Gamma(1-\alpha(x))}\int_0^x (x-t)^{-\alpha(t)}\frac{df(t)}{dt}dt.
$$

The above derivative is called the Caputo variable order differential operator, and computes the derivative of a constant to be zero.

2. Novel fractional variable order derivative

In this section, we provide a derivative with fractional order that has no singularity, and also easy to use analytically.

Definition 2.1. Let a positive function, $f(x) \in C^1[a, b]$, let $g(t)$ be a differentiable function in an open interval I, then, the derivative of fractional variable order $f(x)$ of $g(t)$ is given as follows:

$$
{}_0^{AK}D_t^{f(x)}[g(t)] = \int_0^t \frac{dg(y)}{dy}\exp[-f(x)(t-y)]dy. \tag{2.1}
$$

The above definition is very easy to handle analytically than the existing derivative with fractional variable order in the literature. With the above operator, if the function $g(t)$ is a constant, we obtain zero at the right hand side of equation (2.1). It is very important to notice that, the equation (2.1) can be seen as the convolution. We shall now present some interesting relationship of the proposed derivative with existing integral operators.

3. Relation with integral transforms

Theorem 3.1. *The Laplace transform of equation* (2.1) *produces*

$$
L\left({}_0^{AK}D_t^{f(x)}[g(t)]\right) = \frac{sL(g(t)) - g(0)}{s + f(x)}.
$$

Proof. By definition, we have

$$
L\left({}_0^{AK}D_t^{f(x)}[g(t)]\right) = \int_0^\infty \left(\int_0^z \frac{dg(y)}{dy}\exp[-f(x)(t-y)]dy\right)\exp[-sz]dz.
$$

Using the convolution theorem for Laplace transform, we obtain

$$
L\left({}_0^{AK}D_t^{f(x)}[g(t)]\right) = L\left(\frac{dg(y)}{dy}\right)L\left(\exp[-f(x)(t)]\right) = \frac{sL(g(t)) - g(0)}{s + f(x)}.
$$

□

Theorem 3.2. *The Sumudu transform of equation* (2.1) *produces*

$$S\left({}_0^{AK}D_t^{f(x)}[g(t)]\right) = \frac{S(g(t)) - g(0)}{s\,(1 + sf(x))}.$$

Proof. By definition of Sumudu transform, we have

$$S\left({}_0^{AK}D_t^{f(x)}[g(t)]\right) = \int_{-\infty}^{\infty}\left(\int_0^z \frac{dg(y)}{dy}\exp[-f(x)(t-y)]dy\right)\frac{\exp[-\frac{z}{s}]}{s}dz.$$

Using the convolution theorem for Sumudu transform, we obtain

$$S\left({}_0^{AK}D_t^{f(x)}[g(t)]\right) = S\left(\frac{dg(y)}{dy}\right)S\left(\exp[-f(x)(t)]\right) = \frac{S(g(t)) - g(0)}{s\,(1 + sf(x))}.$$

□

Theorem 3.3. *Let $g(t)$ be a function for which ${}_0^{AK}D_t^{f(x)}[g(t)]$ is exists, the Fourier transform of the derivative of fractional variable order $f(x)$ of $g(t)$ is given as:*

$$\left[F\left({}_0^{AK}D_t^{f(x)}[g(t)]\right) = 2\pi ikG(k)\sqrt{2\pi}\delta(k + if(x))\right].$$

Proof. By definition of Fourier transform, we have the following

$$F\left({}_0^{AK}D_t^{f(x)}[g(t)]\right) = \int_{-\infty}^{\infty}\left(\int_0^t \frac{dg(y)}{dy}\exp[-f(x)(t-y)]dy\right)\exp(-2\pi ikt)dt.$$

Using the convolution theorem for fourier transform, we obtain

$$F\left({}_0^{AK}D_t^{f(x)}[g(t)]\right) = F\left(\frac{dg(y)}{dy}\right)F\left(\exp[-f(x)(t)]\right).$$

Then we obtain the following

$$\left[F\left({}_0^{AK}D_t^{f(x)}[g(t)]\right) = 2\pi ikG(k)\sqrt{2\pi}\delta(k + if(x))\right].$$

This completes the proof. □

Theorem 3.4. *Let $g(t)$ be a function for which ${}_0^{AK}D_t^{f(x)}[g(t)]$ is exists, the Mellin transform of the derivative of fractional variable order $f(x)$ of $g(t)$ is given as:*

$$M\left({}_0^{AK}D_t^{f(x)}[g(t)]\right)(s) = \frac{\Gamma(s)^2}{\Gamma(s-1)}f(x)^{-s}.$$

Proof. By definition of Mellin transform, we have the following

$$\varphi(s) = M\left({}_0^{AK}D_t^{f(x)}[g(t)]\right)(s) = \int_0^{\infty} t^{s-1}\left(\int_0^t \frac{dg(y)}{dy}\exp[-f(x)(t-y)]dy\right)dt.$$

Using the convolution theorem for Mellin transform, we obtain

$$M\left({}_0^{AK}D_t^{f(x)}[g(t)]\right)(s) = M\left(\frac{dg(y)}{dy}\right)(s)M\left(\exp[-f(x)(t)]\right)(s).$$

Then we obtain the following

$$M\left({}_0^{AK}D_t^{f(x)}[g(t)]\right)(s) = \frac{\Gamma(s)^2}{\Gamma(s-1)}f(x)^{-s}.$$

This completes the proof. □

Theorem 3.5. *Let a positive function, $f(x) \in C^1[a,b]$, let $g(t)$ be a differentiable function in an open interval I, then*

$$\left[{}_0^{AK}D_t^{f(x)}[g(t)] = h(t)\right],$$

and we have the following

$$\left[g(t) = g(0) + h(t) + f(x)\int_0^t h(y)dy\right].$$

Proof. Let taking the Laplace transform of ${}_0^{AK}D_t^{f(x)}[g(t)] = h(t)$. Then we have

$$G(s) = \frac{g(0)}{s} + \left(1 + \frac{f(x)}{s}\right)H(s).$$

Taking the Inverse Laplace operator on both sides and using the linearity we obtain

$$g(t) = g(0) + h(t) + L^{-1}\left(\frac{f(x)}{s}H(s)\right).$$

However

$$L^{-1}\left(\frac{f(x)}{s}H(s)\right) = f(x)\int_0^t h(y)dy.$$

Then

$$g(t) = g(0) + h(t) + f(x)\int_0^t h(y)dy.$$

□

Theorem 3.6. *Let a positive function, $f(x) \in C^1[a,b]$, let $g(t)$ be a differentiable function in an open interval I, then*

$$\left[{}_0^{AK}D_t^{f(x)}[g(t)] = g(t)\right],$$

and we have the following

$$\left[g(t) = -\frac{g(0)}{f(x)}\delta(t)\right].$$

Proof. Let taking the Laplace transform of ${}_0^{AK}D_t^{f(x)}[g(t)] = g(t)$. Then we have

$$G(s) = (1 + \frac{f(x)}{s})G(s) + \frac{g(0)}{s}.$$

Then we have

$$G(s) = -\frac{g(0)}{f(x)}.$$

Now taking the Inverse Laplace operator on both sides we obtain

$$g(t) = -\frac{g(0)}{f(x)}\delta(t),$$

□

which completes the proof.

Theorem 3.7. *Let a positive function, $f(x) \in C^1[a,b]$, let $g(t)$ be a differentiable function in an open interval I, then*

$$\left[{}_0^{AK}D_t^{f(x)}[g(t)] = c\right],$$

and we have the following

$$\left[g(t) = g(0) + c + cf(x)t\right].$$

Proof. Let taking the Laplace transform of ${}_0^{AK}D_t^{f(x)}[g(t)] = c$. Then we have

$$G(s) = \frac{g(0)}{s} + \frac{c}{s} + \frac{cf(x)}{s^2}.$$

Now taking the Inverse Laplace operator on both sides and using the laplace transforms linearity, we obtain

$$g(t) = g(0) + c + cf(x)t.$$

This completes the proof. □

Theorem 3.8. *Let a positive function, $f(x) \in C^1[a,b]$, let $g(t)$ be a differentiable function in an open interval I, then, the derivative of fractional variable order $f(x)$ of $g(t)$, ${}_0^{AK}D_t^{f(x)}[g(t)]$ is satisfy the Lipchitz condition.*

Proof.

$$\left\| {}_0^{AK}D_t^{f(x)}\left(g(t)\right) - {}_0^{AK}D_t^{f(x)}\left(h(t)\right) \right\| = \left\| \int_0^t \frac{d}{dy}\left(g(y)-h(y)\right)\exp[-f(x)(t-y)]dy \right\|.$$

Using the triangular inequality for integral operator, we obtain

$$\left\| {}_0^{AK}D_t^{f(x)}\left(g(t)\right) - {}_0^{AK}D_t^{f(x)}\left(h(t)\right) \right\| < \int_0^t \left\| \frac{d}{dy}\left(g(y)-h(y)\right)\exp[-f(x)(t-y)]dy \right\|. \tag{3.1}$$

But the expression with exponential is positive then

$$\left\| {}_0^{AK}D_t^{f(x)}\left(g(t)\right) - {}_0^{AK}D_t^{f(x)}\left(h(t)\right) \right\| < \int_0^t \left\| \frac{d}{dy}\left(g(y)-h(y)\right) \right\| \exp[-f(x)(t-y)]dy. \tag{3.2}$$

Using the Lipchitz condition of the derivative, we can find a positive constant such that,

$$\left\| \frac{d}{dy}\left(g(y)-h(y)\right) \right\| < K\left\| g-h \right\|.$$

Replacing equation (3.2) in equation (3.1), we obtain

$$\left\| {}_0^{AK}D_t^{f(x)}\left(g(t)\right) - {}_0^{AK}D_t^{f(x)}\left(h(t)\right) \right\| < K\left\| g-h \right\| \int_0^t \exp[-f(x)(t-y)]dy.$$

However,

$$\int_0^t \exp[-f(x)(t-y)]dy = \frac{1}{f(x)}\left[1-\exp[-f(x)t]\right] > 0.$$

Therefore by letting

$$L = K\frac{1}{f(x)}\left[1-\exp[-f(x)t]\right].$$

We obtain

$$\left\| {}_0^{AK}D_t^{f(x)}\left(g(t)\right) - {}_0^{AK}D_t^{f(x)}\left(h(t)\right) \right\| < L\left\| g-h \right\|.$$

□

4. Numerical approximation of derivative with fractional variable orde

It was observed in many situations that models of complex real world problem could only be solved via numerical methods. To use these numerical methods, one needs to have numerical representations of the used derivatives. We therefore devote this section to the discussion underpinning the numerical approximation of the proposed fractional variable order derivative.

$$ {}_{0}^{AK}D_t^{f(x)}[g(t)] = \int_0^t \frac{dg(y)}{dy} \exp[-f(x)(t-y)]dy. $$

Now at a particular $t = t_n$ $(n > 1)$, we have the following

$$ \begin{aligned} {}_{0}^{AK}D_t^{f(x)}[g(t_n)] &= \int_0^{t_n} \frac{dg(y)}{dy} \exp[-f(x)(t_n-y)]dy \\ &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{dg(y)}{dy} \exp[-f(x)(t_n-y)]dy \\ &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{g^{j+1}-g^j}{\Delta t} \exp[-f(x)(t_n-y)]dy \\ &= \sum_{j=1}^{n} \frac{g^{j+1}-g^j}{\Delta t} \int_{t_{j-1}}^{t_j} \exp[-f(x)(t_n-y)]dy \\ &= \sum_{j=1}^{n} \frac{g^{j+1}-g^j}{\Delta t} \frac{1}{f(x)} \left[\exp[-f(x)(t_n-t_j)] - \exp[-f(x)(t_n-t_{j+1})]\right]. \end{aligned} $$

Then for $0 < f(x) < 1$, the approximate numerical representation of the proposed derivative is given as

$$ \left[{}_{0}^{AK}D_t^{f(x)}\left(g(t_n)\right) = \sum_{j=1}^{n} \frac{g^{j+1}-g^j}{\Delta t} \delta_{(n,j)}^{f(x)} \right]. $$

Theorem 4.1. *Let $g(t) \in C^1[a,b]$ and let the fractional variable order $0 < f(x) < 1$, then the numerical approximation of the new variable order derivative is*

$$ {}_{0}^{AK}D_t^{f(x)}\left[g(t_n)\right] = \sum_{j=1}^{n} \frac{g^{j+1}-g^j}{\Delta t} \delta_{(n,j)}^{f(x)} + O(\Delta t). $$

Proof. Let the conditions of the theorem be satisfied then

$$ \begin{aligned} {}_{0}^{AK}D_t^{f(x)}[g(t_n)] &= \int_0^{t_n} \frac{dg(y)}{dy} \exp[-f(x)(t_n-y)]dy \\ &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{dg(y)}{dy} \exp[-f(x)(t_n-y)]dy \\ &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left(\frac{g^{j+1}-g^j}{\Delta t} + O(\Delta t)\right) \exp[-f(x)(t_n-y)]dy \end{aligned} $$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \frac{g^{j+1} - g^{j}}{\Delta t} \int\limits_{t_{j-1}}^{t_j} \exp[-f(x)(t_n - y)]dy \\
&\quad + O(\Delta t) \sum_{j=1}^{n} \int\limits_{t_{j-1}}^{t_j} \exp[-f(x)(t_n - y)]dy \\
&= \sum_{j=1}^{n} \frac{g^{j+1} - g^{j}}{\Delta t} \delta_{(n,j)}^{f(x)} + O(\Delta t) \sum_{j=1}^{n} \delta_{(n,j)}^{f(x)} \\
&= \sum_{j=1}^{n} \frac{g^{j+1} - g^{j}}{\Delta t} \delta_{(n,j)}^{f(x)} + O(\Delta t).
\end{aligned}
$$

This completes the proof. □

4.1. Numerical approximation of second order variable order derivative

In this section, we present the numerical approximation of the space fractional variable order derivative. However, for the second order variable order derivative, we defined, we proposed the following definition.

Definition 4.2. Let a positive function $1 < f(x) < 2 \in C^2[0, 1]$, let $g(t)$ be a two times differentiable function on an open interval $[a, b]$ then, the derivative of fractional variable order $f(x)$ of $g(t)$ is given as follows:

$$
{}_{0}^{AK}D_t^{f(x)}[g(t)] = \int\limits_{0}^{t} \frac{d^2 g(y)}{dy^2} \exp[-[f(x)(t - y)]^2]dy.
$$

At a particular $t = t_n$ $(n \geq 1)$, we have the following expressions

$$
\begin{aligned}
{}_{0}^{AK}D_t^{f(x)}[g(t_n)] &= \int\limits_{0}^{t_n} \frac{d^2 g(y)}{dy^2} \exp[-[f(x)(t_n - y)]^2]dy, \\
{}_{0}^{AK}D_t^{f(x)}[g(t_n)] &= \sum_{j=1}^{n} \int\limits_{t_j}^{t_j+1} \frac{d^2 g(y)}{dy^2} \exp[-[f(x)(t_n - y)]^2]dy, \\
{}_{0}^{AK}D_t^{f(x)}[g(t_n)] &= \sum_{j=1}^{n} \int\limits_{t_j}^{t_j+1} \left(\frac{g^{j+1} - 2g^{j} + g^{j-1}}{(\Delta t)^2} \right) \exp[-[f(x)(t_n - y)]^2]dy, \\
{}_{0}^{AK}D_t^{f(x)}[g(t_n)] &= \sum_{j=1}^{n} \left(\frac{g^{j+1} - 2g^{j} + g^{j-1}}{(\Delta t)^2} \right) \int\limits_{t_j}^{t_j+1} \exp[-[f(x)(t_n - y)]^2]dy, \\
{}_{0}^{AK}D_t^{f(x)}[g(t_n)] &= \sum_{j=1}^{n} \left(\frac{g^{j+1} - 2g^{j} + g^{j-1}}{(\Delta t)^2} \right) \frac{\sqrt{\pi}\left(Erf[(t_{j+1} - t_n) f[x]] + Erf[(-t_j + t_n) f[x]] \right)}{2 f[x]}, \\
{}_{0}^{AK}D_t^{f(x)}[g(t_n)] &= \sum_{j=1}^{n} \left(\frac{g^{j+1} - 2g^{j} + g^{j-1}}{(\Delta t)^2} \right) \omega_{(n,j)}^{f(x)}.
\end{aligned}
$$

Theorem 4.3. *Let $g(t)$ be twice differentiable, then the second derivative approximation of the variable order derivative of order $1 < f(x) < 2$ is given as*

$$
{}_{0}^{AK}D_t^{f(x)}[g(t_n)] = \sum_{j=1}^{n} \left(\frac{g^{j+1} - 2g^{j} + g^{j-1}}{(\Delta t)^2} \right) \omega_{(n,j)}^{f(x)} + O\left((\Delta t)^2 \right).
$$

Proof. By definition, we have the following relation

$$
{}_{0}^{AK}D_t^{f(x)}[g(t_n)] = \int_0^t \frac{d^2 g(y)}{dy^2} \exp[-[f(x)(t_n - y)]^2] dy,
$$

$$
{}_{0}^{AK}D_t^{f(x)}[g(t_n)] = \sum_{j=1}^{n} \int_{t_j}^{t_j+1} \frac{d^2 g(y)}{dy^2} \exp[-[f(x)(t_n - y)]^2] dy,
$$

$$
{}_{0}^{AK}D_t^{f(x)}[g(t_n)] = \sum_{j=1}^{n} \int_{t_j}^{t_j+1} \left(\frac{g^{j+1} - 2g^j + g^{j-1}}{(\Delta t)^2} + O\left((\Delta t)^2\right) \right) \exp[-[f(x)(t_n - y)]^2] dy,
$$

$$
\begin{aligned}
{}_{0}^{AK}D_t^{f(x)}[g(t_n)] =& \sum_{j=1}^{n} \int_{t_j}^{t_j+1} \left(\frac{g^{j+1} - 2g^j + g^{j-1}}{(\Delta t)^2} \right) \exp[-[f(x)(t_n - y)]^2] dy \\
&+ O\left((\Delta t)^2\right) \sum_{j=1}^{n} \int_{t_j}^{t_j+1} \exp[-[f(x)(t_n - y)]^2] dy,
\end{aligned}
$$

$$
\begin{aligned}
{}_{0}^{AK}D_t^{f(x)}[g(t_n)] =& \sum_{j=1}^{n} \int_{t_j}^{t_j+1} \left(\frac{g^{j+1} - 2g^j + g^{j-1}}{(\Delta t)^2} \right) \exp[-[f(x)(t_n - y)]^2] dy \\
&+ O\left((\Delta t)^2\right) \sum_{j=1}^{n} \omega_{(n,j)}^{f(x)},
\end{aligned}
$$

$$
\begin{aligned}
{}_{0}^{AK}D_t^{f(x)}[g(t_n)] =& \sum_{j=1}^{n} \int_{t_j}^{t_j+1} \left(\frac{g^{j+1} - 2g^j + g^{j-1}}{(\Delta t)^2} \right) \exp[-[f(x)(t_n - y)]^2] dy \\
&+ O\left((\Delta t)^2\right).
\end{aligned}
$$

This completes the proof. □

5. Conclusion

In the recent decade, the concept of variable order fractional derivative has been introduced and used to describe the anomalous diffusion problems that could not been investigated using the concept of fractional constant order derivative. However, these derivatives have many problems for instance, one could not establish the relationship between these derivatives with the well-know integral transform as in the case of constant order fractional derivative and local derivative. In addition, one could not find the derivative of simple function like $\sin(x)$ and also the associated differential equations cannot be solved analytically. To solve these problems, we proposed in this work a new derivative that can be used numerically and analytically.

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