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A homotopy algorithm for computing the fixed point of self-mapping with inequality and equality constraints

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Abstract

In this paper, to compute the fixed point of self-mapping on general non-convex sets, a modified constraint shifting homotopy algorithm for perturbing simultaneously both equality constraints and inequality constraints is proposed and the global convergence of the smooth homotopy pathways is proven under some mild conditions. The advantage of the newly constructed homotopy is that the initial point needs to be only in the shifted feasible set, not necessarily, an interior point in the original feasible set, and hence it is more convenient to be implemented than the existing results. Some numerical examples are also given to show its feasibility and effectiveness.

Keywords: Homotopy method, general non-convex sets, self-mapping, fixed point.
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1. Introduction and preliminaries

Fixed point theory has been broadly implemented in nonlinear analysis, integral and differential equations, dynamical system theory, game theory, optimization problems and other fields. Recently, lots of results

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on the fixed point theory and algorithms have appeared and attracted many attentions, see references such as [2, 5, 7, 12–15, 17]. Among all of the fixed point theorems, the famous Brouwer fixed point theorem only requires that the self-mapping $\Psi : X \to X$ is continuous but not contractive and has been extensively applied in game theory, equilibrium problems and other across numerous fields of mathematics. In 1976, to give an innovative proof and compute the Brouwer fixed point in a convex set, Kellogg et al. [6] presented its constructive proof via a homotopy method for a twice continuous differentiable self-mapping. In 1978, Chow et al. [3] constructed a single and convenient fixed point homotopy for computing Brouwer fixed point for a twice continuous differentiable self-mapping in convex set. In 1996, to remove the convex assumption and numerically solve the fixed point problems under much weaker conditions, Yu and Lin [16] first proved an equivalent condition of the existence for fixed point on non-convex bounded sets with only inequality constraints $X = \{x : g_i(x) \leq 0, i = 1, 2, \dots, m\}$ and constructed an interior point combined homotopy for computing fixed points of a twice continuous differentiable self-mapping as follows:

$$H(\theta, t) = \begin{pmatrix} (1-t)(x - \Psi(x) + \sum_{i=1}^{m} \nabla g_i(x) y_i) + t(x - x^0) \\ Y g(x) - tY^0 g(x^0) \end{pmatrix},$$

where $(x^0, y^0) \in X^0 \times R^m_{++}$, $y_i \geq 0$, $t \in [0, 1]$, $g(x) = (g_1(x), \dots, g_m(x))^T$, Y and Y^0 denote the diagonal matrices whose ith diagonal element are y_i and y_i^0, respectively, and the strict feasible set $X^0 = \{x : g_i(x) < 0, i = 1, 2, \dots, m\}$. Throughout the paper, let R^m, R^m_+ and R^m_{++} denote m-dimension Euclidean space, nonnegative orthant and positive orthant of R^m, respectively.

In 2008, to solve the fixed point problems on the general non-convex sets with equality constraints and inequality constraints, Su and Liu [9] generalized the interior point combined homotopy to compute the fixed point of self-mapping in a broader class of non-convex bounded sets

$$X = \{x : g_i(x) \leq 0, h_j(x) = 0, i = 1, 2, \dots, m, j = 1, 2, \dots, l\}$$

and the homotopy was constructed as follows:

$$H(\theta, t) = \begin{pmatrix} (1-t)(x - \Psi(x) + \sum_{i=1}^{m} \nabla g_i(x) y_i) + \sum_{j=1}^{l} \nabla h_j(x) z_j + t(x - x^0) \\ h(x) \\ Y g(x) - tY^0 g(x^0) \end{pmatrix}, \tag{1.1}$$

where $\theta = (x, y, z) \in R^n \times R^m_+ \times R^l$, $(x^0, y^0) \in X^0 \times R^m_{++}$, $t \in [0, 1]$, and the strict feasible set

$$X^0 = \{x : g_i(x) < 0, , h_j(x) = 0, i = 1, 2, \dots, m, j = 1, 2, \dots, l\}.$$

In 2013, to relax the bounded condition and weaken the normal cone condition, Zhu et al. [21] constructed a modified combined homotopy for computing the fixed point of a self-mapping on the general unbounded non-convex sets with both equality constraints and inequality constraints and the existence and global convergence of the smooth homotopy pathway was proven under much weaker pseudo cone condition. Since the interior point combined homotopy requires that the initial point must be in the original feasible set, to enlarge the chosen scope of initial points, in 2011, Su et al. [11] presented a boundary perturbation interior point homotopy method for solving the fixed point problems on the non-convex bounded sets with only inequality constraints. In 2015, Su and Qian [10] proposed a modified combined homotopy method for computing the fixed point of a self-mapping by a perturbation on the inequality constraints on the general non-convex unbounded sets. In 2015, for solving the fixed point problems of self-mapping, Fan et al. [4] proposed an infeasible interior point homotopy method for enlarging the choice scope of initial points by a perturbation on equality constraints on the general unbounded sets. In 2016, Zhu and Yang [19] proposed a more conveniently constraint shifting homotopy method for computing the fixed point of self-mapping on the non-convex set with only inequality constraints.

To enlarge the chosen scope of initial points by a perturbation simultaneously both inequality constraints and equality constraints on the general non-convex sets, it seems reasonable to construct the homotopy directly instead of (1.1) as follows:

$$H(\theta,t)=\begin{pmatrix}(1-t)(x-\Psi(x)+\sum\limits_{i=1}^{m}\nabla g_i(x)y_i)+\sum\limits_{j=1}^{l}\nabla h_j(x)z_j+t(x-x^0)\\ h(x)-th(x^0)\\ Yg(x)-tY^0g(x^0)\end{pmatrix}. \tag{1.2}$$

Unfortunately, the probability-one regularity of the homotopy (1.2) cannot be ensured. The reason is that the following submatrix may be not full row rank for any $t\in(0,1]$ by taking x^0 and y^0 as variate:

$$\frac{\partial H(\theta,x^0,y^0,t)}{\partial(x,x^0,y^0)}=\begin{pmatrix}\Xi & -tI & 0\\ (\nabla h(x))^T & -t(\nabla h(x^0))^T & 0\\ Y(\nabla g(x))^T & -tY^0(\nabla g(x^0))^T & -t\mathrm{diag}(g(x^0))\end{pmatrix},$$

where $\Xi=(1-t)(I-\nabla\Psi(x)+\sum_{i=1}^{m}\nabla^2 g_i(x)y_i)+\sum_{j=1}^{l}\nabla^2 h_j(x)z_j+tI$, and $\mathrm{diag}(g(x^0))$ denotes the diagonal matrix with its ith diagonal element $g_i(x^0)$.

Referring to the former results, in this article, a constraint shifting homotopy for computing the fixed point of self-mapping on the general non-convex sets by perturbing simultaneously both inequality constraints and equality constraints is constructed and the existence and global convergence of the smooth homotopy pathways is proved under some mild conditions.

In Section 2, an equivalent condition of the existence of fixed point will be given and some lemmas from differential topology which will be used for proving the main result will be presented. In Section 3, a constraint shifting homotopy for solving the fixed point problems is constructed and the existence and global convergence of a smooth path from any given initial point in shifted feasible set to a fixed point of any twice continuous differentiable self-mapping will be proved. In Section 4, some numerical examples will be given to show the feasibility and effectiveness of the proposed method.

2. Preliminaries

Throughout the paper, the general non-convex closed subset $X\in R^n$ is defined as follows:

$$X=\{x\in R^n: g_i(x)\le 0, h_j(x)=0,\quad i=1,2,\dots,m, j=1,2,\dots,l\}. \tag{2.1}$$

In this paper, to enlarge the chosen scope of initial points to almost all of the Euclidean space, we construct the shifted constraint functions $\widehat{g}_i(x,t), i=1,2,\dots,m$ and $\widehat{h}_j(x,t), j=1,2,\dots,l$, which are three times continuous differentiable, and satisfy $\widehat{g}_i(x,0)=g_i(x), i=1,2,\dots,m$ and $\widehat{h}_j(x,0)=h_j(x), j=1,2,\dots,l$, respectively. For convenience, some denotations are given as follows:

$$X(t)=\{x:\widehat{g}_i(x,t)\le 0,\widehat{h}_j(x,t)=0, i=1,2,\dots,m, j=1,2,\dots,l\},$$

$$X^0(t)=\{x:\widehat{g}_i(x,t)< 0,\widehat{h}_j(x,t)=0, i=1,2,\dots,m, j=1,2,\dots,l\},$$

$$\partial X(t)=X(t)\backslash X^0(t),$$

$$I_t(x)=\{i\in\{1,2,\dots,m\}:\widehat{g}_i(x,t)=0\},$$

$$\nabla\tilde{g}(x,t)=\frac{\partial\tilde{g}(x,t)}{\partial x},$$

and

$$\nabla\widehat{h}(x,t) = \frac{\partial\widehat{h}(x,t)}{\partial x}.$$

To prove our main result in the next section, the following assumptions will be used.

Assumption 2.1.

(A1) For all $t \in [0,1]$, $X^0(t) \neq \phi$ and $\cup_{t\in[0,1]}X(t)$ is bounded.

(A2) For all $t \in [0,1]$, $x \in X(t)$, $\sum_{i\in I_t(x)} \nabla\widehat{g}_i(x,t)y_i + \sum_{j=1}^{m} \nabla\widehat{h}_j(x,t)z_j = 0$ for $y_i \in R_+$ and $z_j \in R$ implies $y_i = z_j = 0$, $i \in I_t(x)$, $j = 1,2,\ldots,l$.

(A3) For all $t \in [0,1]$, for any $x \in \partial X(t)$,

$$\{x + \sum_{i\in I_t(x)} \nabla\widehat{g}_i(x,t)y_i + \sum_{j=1}^{m} \nabla\widehat{h}_j(x,t)z_j | y_i \geq 0, z_j \in R\} \cap X(t) = \{x\}.$$

To solve the non-convex fixed point problems of self-mapping via the combined homotopy method, the following theorem which is an equivalent condition of the existence for the fixed point in the general non-convex sets X is important.

Theorem 2.2. *Let X be defined as (2.1), and the constraint functions $g_i(x)$, $i = 1,2,\ldots,m$ and $h_j(x)$, $j = 1,2,\ldots,l$ be C^3 mappings. If the conditions (A1)-(A3) hold, then for any C^2 self-mapping $\Psi : X \to X$, $x \in X$ is a fixed point of the mapping $\Psi(x)$ iff there exists a vector $(y,z) \in R^m_+ \times R^l$, such that (x,y,z) is a solution of the following system:*

$$\begin{array}{l} x - \Psi(x) + \nabla g(x)y + \nabla h(x)z = 0, \\ h(x) = 0, \\ Yg(x) = 0, \quad g(x) \leq 0, \quad y \geq 0. \end{array} \tag{2.2}$$

Proof. When the parameter $t = 0$, the conditions (A1)-(A3) of Assumption 2.1 are the same as the conditions of Lemma 2.1 in [9]. Therefore, the proof is omitted here. □

To prove our main results, the following parameterized Sard theorem will be used and given here. Let $U \subset R^n$ be an open set and $\phi : U \to R^p$ be a C^α $(\alpha > \max\{0, n-p\})$ mapping; we say that $y \in R^p$ is a regular value for ϕ if

$$Range[\partial\phi(x)/\partial x] = R^p, \quad \forall x \in \phi^{-1}(y).$$

Lemma 2.3 ([8]). *Let $V \subset R^n$, $U \subset R^m$ be open sets, and let $\phi : V \times U \to R^k$ be a C^α-mapping, where $\alpha > \max\{0, m-k\}$. If $0 \in R^k$ is a regular value of ϕ, then for almost all $a \in V$, 0 is a regular value of $\phi_a = F(a,\cdot)$.*

3. Main Result

By Theorem 2.2, computing the fixed point of a C^2 self-mapping is equivalent to solve the system (2.2). Hence, in this paper, to solve the system (2.2), for any randomly chosen vector $\tau \in R^n$ and any given vector $\zeta \in R^m_{++}$, a constraint shifting homotopy equation is constructed as follows:

$$H(\theta,t) = \begin{pmatrix} (1-t)(x - \Psi(x) + \nabla\widehat{g}(x,t)y) & + & \nabla\widehat{h}(x,t)z + t(x-x^0) + t(1-t)\tau \\ Y\widehat{g}(x,t) & + & t\zeta \\ & \widehat{h}(x,t) & \end{pmatrix} = 0, \tag{3.1}$$

where $\theta = (x, y, z)^T \in R^n \times R^m_+ \times R^l$, $(x^0, y^0) \in X^0(1) \times R^m_{++}$.

For the homotopy equation (3.1), when $t = 0$, the homotopy equation $H(\theta, 0) = 0$ turns to the system (2.2) and when $t = 1$, the homotopy equation $H(\theta, 1) = 0$ can be written as follows:

$$\begin{pmatrix} \nabla\widehat{h}(x,1)z & + & x - x^0 \\ Y\widehat{g}(x,1) & + & \zeta \\ & \widehat{h}(x,1) & \end{pmatrix} = 0,$$

which has a unique simple solution under Assumption 2.1, and can be proved by the following lemma.

Lemma 3.1. *Let the homotopy equation $H(\theta, t) = 0$ be constructed as* (3.1). *If $g_i(x)$, $i = 1, 2, \ldots, m$ and $h_j(x)$, $j = 1, 2, \ldots, l$ are all C^3 functions, and conditions* (A1)-(A3) *of Assumption* 2.1 *hold, then the homotopy equation $H(\theta, 1) = 0$ has a unique solution*

$$(x, y, z) = (x^0, y^0, z^0) = (x^0, -[\mathrm{diag}(\widehat{g}(x^0, 1))]^{-1}\zeta, 0).$$

Proof. By the condition (A1), for any parameter $t \in [0, 1]$, the set $X^0(t)$ is nonempty. Let $(\bar{\theta}, 1) = (\bar{x}, \bar{y}, \bar{z}, 1)$ be a solution of (3.1), i.e., $H(\bar{\theta}, 1) = 0$. Then, by the fact that the given vector $\zeta \in R^m_{++}$ and the vector $\bar{y} \geq 0$, we can have $\bar{x} \in X^0(1)$. Now, we prove $\bar{x} = x^0$ by the contradiction. If $\bar{x} \neq x^0$, by the first equation $\nabla\widehat{h}(\bar{x}, 1)\bar{z} + \bar{x} - x^0 = 0$ of $H(\bar{\theta}, 1) = 0$ and condition (A2), we get $\bar{z} \neq 0$, and hence $x^0 = \bar{x} + \nabla\widehat{h}(\bar{x}, 1)\bar{z}$, which contradicts with condition (A3). Therefore, $\bar{x} = x^0$ and $\nabla\widehat{h}(\bar{x}, 1)\bar{z} = \nabla\widehat{h}(x^0, 1)\bar{z} = 0$. Moreover, we get $\bar{z} = 0$ from the condition (A2). Finally, by the second equation $\bar{Y}\widehat{g}(\bar{x}, 1) + \zeta = 0$ of $H((\bar{x}, \bar{y}, \bar{z}), 1) = 0$ and $\bar{x} = x^0$, we can obtain $\bar{y} = -[\mathrm{diag}(\widehat{g}(x^0, 1))]^{-1}\zeta$. The proof is complete. □

For any given initial point $\theta^0 \in X^0(1) \times R^m_{++} \times R^l$, the zero-point set of the homotopy equation $H(\theta, t) = 0$ is denoted as follows:

$$H^{-1}_{\theta^0}(0) = \{(\theta, t) \in X^0(1) \times R^m_{++} \times R^l \times (0, 1] : H(\theta, t) = 0\}.$$

Theorem 3.2. *Suppose that X is defined as* (2.1). *If the constraint functions $g_i(x)$, $i = 1, 2, \ldots, m$ and $h_j(x)$, $j = 1, 2, \ldots, l$ are all C^3, and conditions* (A1)-(A3) *hold, then any C^2 self-mapping $\Psi : X \to X$ has a fixed point in X, and for any $\theta^0 \in X(1)^0 \times R^m_{++} \times R^l$, $H^{-1}_{\theta^0}(0)$ contains a smooth curve $\Gamma_{\theta^0} \subset X(1) \times R^n_+ \times R^l \times (0, 1]$, which begins from $(x^0, y^0, z^0, 1)$ and terminates in or approaches to the hyperplane $t = 0$. Moreover, if $(\tilde{x}, \tilde{y}, \tilde{z}, 0)$ is an ending limit point of the smooth curve Γ_{θ^0}, then $\tilde{w} = (\tilde{x}, \tilde{y}, \tilde{z})$ is a solution to system* (2.2) *and the $\tilde{x}$-component is a fixed point of $\Psi(x)$ in X.*

Proof. Taking (x^0, τ) as variate and let $\widehat{H}(\theta, x^0, \tau, t)$ be the same mapping as $H(\theta, t)$, we consider the submatrix of the Jacobian $D\widehat{H}(\theta, x^0, \tau, t)$:

$$\frac{\partial \widehat{H}(\theta, x^0, \tau, t)}{\partial(x^0, y, \tau)} = \begin{pmatrix} -tI & (1-t)\nabla\widehat{g}(x,t) & t(1-t)I \\ 0 & \mathrm{diag}(\widehat{g}(x,t)) & 0 \\ \nabla\widehat{h}(x^0,t)^T & 0 & 0 \end{pmatrix}$$

for all $t \in (0, 1)$. By the fact that $\zeta > 0$ and $Y\widehat{g}(x, t) + t\zeta = 0$, we can obtain that the matrix $\mathrm{diag}(\widehat{g}(x, t))$ is nonsingular. By the condition (A2), since for any $x^0 \in X^0(1)$, the matrix $\nabla\widehat{h}(x^0, t)$ is full column rank, we get that the matrix $\nabla\widehat{h}(x^0, t)^T$ is full row rank. Hence, for any $t \in (0, 1)$, the matrix $\frac{\partial \widehat{H}(\theta, x^0, \tau, t)}{\partial(x^0, y, \tau)}$ is full row rank, which implies that $D\widehat{H}(\theta, x^0, \tau, t)$ is a matrix of full row rank. Since the matrix

$$\frac{\partial H(\theta^0, 1)}{\partial\theta} = \begin{pmatrix} I & 0 & \nabla\widehat{h}(x^0,1) \\ Y^0\nabla\widehat{g}(x^0,1)^T & \mathrm{diag}(\widehat{g}(x^0,1)) & 0 \\ \nabla\widehat{h}(x^0,1)^T & 0 & 0 \end{pmatrix}$$

is also nonsingular, we can get that the matrix $D\widehat{h}(\theta, x^0, \tau, t)$ is full row rank for any $t \in (0, 1]$. Hence, 0 is a regular value of the mapping $\widehat{h}(\theta, x^0, \tau, t)$. Therefore, by Lemma 2.3, for almost all $(x^0, \tau) \in X^0(1) \times R^n$, 0 is a regular value of $H(\theta,t)$. Then, for any given $\theta^0 \in X^0(1)\times R^m_{++}\times R^l$, if 0 is a regular value of $H(\theta,t)$, by the fact that $(\theta^0,1)$ is a solution of $H(\theta,t)=0$, $\frac{\partial H(\theta^0,1)}{\partial\theta}$ is nonsingular and the famous implicit function theorem, we get $H^{-1}_{\theta^0}(0)$ must contain a smooth curve Γ_{θ^0}, which starts from $(x^0,y^0,z^0,1)$, goes into $X^0(1)\times R^m_{++}\times R^l\times(0,1)$ and terminates in the boundary of $X(t)\times R^m_{+}\times R^l\times[0,1]$.

When $t\to 0$, let $(\tilde\theta,\tilde t)$ be an ending limit point of Γ_{θ^0}, only the following five cases may hold:

(i) $(\tilde\theta,\tilde t)\in X(1)\times R^m_+\times R^l\times\{1\}$, $\|(\tilde y,\tilde z)\|<\infty$;

(ii) $(\tilde\theta,\tilde t)\in X(\tilde t)\times R^m_+\times R^l\times[0,1]$, $\|(\tilde y,\tilde z)\|=\infty$;

(iii) $(\tilde\theta,\tilde t)\in X(\tilde t)\times \partial R^m_+\times R^l\times(0,1)$, $\|(\tilde y,\tilde z)\|<\infty$;

(iv) $(\tilde\theta,\tilde t)\in \partial X(\tilde t)\times R^m_{++}\times R^l\times(0,1)$, $\|(\tilde y,\tilde z)\|<\infty$;

(v) $(\tilde\theta,\tilde t)\in X\times R^m_+\times R^l\times\{0\}$, $\|(\tilde y,\tilde z)\|<\infty$.

By Lemma 3.1, we have that θ^0 is the unique simple solution of the homotopy equation $H(\theta,1)=0$. Hence, case (i) is impossible.

Next, we will prove that case (ii) is also impossible by the contradiction. If the case (ii) happens, there must exist a sequence of points $\{(x^k,y^k,z^k,t^k)\}\subset\Gamma_{\theta^0}$ such that $t^k\to\tilde t\in[0,1]$, $x^k\to\tilde x\in X(\tilde t)$ and $\|(y^k,z^k)\|\to\infty$ as $k\to\infty$. By the first equation of the homotopy equation (3.1), we get

$$(1-t^k)(x^k-\Psi(x^k)+\sum_{i=1}^{m} y_i^k\nabla\widehat g_i(x^k,t^k))+\sum_{j=1}^{l}\nabla\widehat h_j(x^k,t^k)z_j^k+t^k(x^k-x^0)+t^k(1-t^k)\tau=0. \tag{3.2}$$

When $k\to\infty$, only two subcases are possible:

(I) $\tilde t=1$;

(II) $\tilde t\in[0,1)$.

Case (I): $\tilde t=1$. Rewrite (3.2) as follows:

$$\begin{aligned}(1-t^k)&\sum_{i\in I_1(\tilde x)} y_i^k\nabla\widehat g_i(x^k,t^k)+\sum_{j=1}^{l}\nabla\widehat h_j(x^k,t^k)z_j^k+x^k-x^0\\ &=(1-t^k)[-\sum_{i\notin I_1(\tilde x)} y_i^k\nabla\widehat g_i(x^k,t^k)-x^k+\Psi(x^k)+x^k-x^0-t^k\tau].\end{aligned} \tag{3.3}$$

If $\|((1-t^k)y^k,z^k)\|<\infty$, we assume $((1-t^k)y^k,z^k)\to(\tilde y,\tilde z)$, then $\tilde y_i=0$ for $i\notin I_1(\tilde x)$ by the second equation of the homotopy equation (3.1). Now, taking limits in the both sides of (3.3) as $k\to\infty$, we have

$$\begin{aligned}\lim_{k\to\infty}&[(1-t^k)\sum_{i\in I_1(\tilde x)} y_i^k\nabla\widehat g_i(x^k,t^k)+\sum_{j=1}^{l}\nabla\widehat h_j(x^k,t^k)z_j^k+x^k-x^0]\\ &=\lim_{k\to\infty}(1-t^k)[-\sum_{i\notin I_1(\tilde x)} y_i^k\nabla\widehat g_i(x^k,t^k)-x^k+\Psi(x^k)+x^k-x^0-t^k\tau]\\ &=\lim_{k\to\infty}[-\sum_{i\notin I_1(\tilde x)}(1-t^k)y_i^k\nabla\widehat g_i(x^k,t^k)]+(1-\tilde t)[\Psi(\tilde x)-x^0-\tilde t\tau]\\ &=[-\sum_{i\notin I_1(\tilde x)}\tilde y_i\nabla\widehat g_i(\tilde x,\tilde t)]+(1-\tilde t)[\Psi(\tilde x)-x^0-\tilde t\tau]\\ &=0.\end{aligned}$$

Hence, we get

$$\begin{aligned} x^0 &= \lim_{k\to\infty}[x^k + (1-t^k)\sum_{i\in I_1(\tilde{x})} y_i^k \nabla\widehat{g}_i(x^k,t^k) + \sum_{j=1}^{l}\nabla\widehat{h}_j(x^k,t^k)z_j^k] \\ &= \tilde{x} + \sum_{j=1}^{l}\nabla\widehat{h}_j(\tilde{x},1)\tilde{z}_j + \lim_{k\to\infty}\sum_{i\in I_1(\tilde{x})}(1-t^k)y_i^k\nabla\widehat{g}_i(x^k,t^k) \\ &= \tilde{x} + \sum_{j=1}^{l}\nabla\widehat{h}_j(\tilde{x},1)\tilde{z}_j + \sum_{i\in I_1(\tilde{x})}\tilde{y}_i\nabla\widehat{g}_i(\tilde{x},1), \end{aligned}$$

which contradicts with the condition (A3). If $\|((1-t^k)y^k, z^k)\| \to \infty$, the proof is the same as the following Case (II).

Case (II): $\tilde{t} \in [0,1)$. Without loss of generality, we assume that $((1-t^k)y^k, z^k)/\|((1-t^k)y^k, z^k)\| \to (\tilde{\alpha},\tilde{\beta})$ with $\|(\tilde{\alpha},\tilde{\beta})\| = 1$ and $\tilde{\alpha}_i = 0$ for $i \notin I_{\tilde{t}}(\tilde{x})$. Since $t \in [0,1]$ and $X(t)$ is bounded, dividing the both sides of (3.2) by $\|((1-t^k)y^k, z^k)\|$ and taking limits as $k \to \infty$, we can get

$$\sum_{i\in I_{\tilde{t}}(\tilde{x})}\tilde{\alpha}_i\nabla\widehat{g}_i(\tilde{x},\tilde{t}) + \sum_{j=1}^{l}\tilde{\beta}_j\nabla\widehat{h}_j(\tilde{x},\tilde{t}) = 0,$$

which contradicts with the condition (A2). Therefore, from the above discussions on subcases (I) and (II), we get that case (ii) is also impossible.

Now, we prove that case (iii) and case (iv) are also impossible. By the second equation of the homotopy equation (3.1), we have $\mathrm{diag}(\widehat{g}(\tilde{x},\tilde{t}))\tilde{y} + \tilde{t}\zeta = 0$. Hence, $\tilde{t} > 0$ and $\tilde{y} \in \partial R_+^m$, i.e., $\tilde{y}_i = 0$ for some $1 \le i \le m$ can not hold simultaneously. So, case (iii) is impossible. If case (iv) holds, then we have that $\tilde{y} > 0$ and $\tilde{t} > 0$. However, since $\mathrm{diag}(\widehat{g}(\tilde{x},\tilde{t}))\tilde{y} + \tilde{t}\zeta = 0$ and the given vector $\zeta \in R_{++}^m$, we get $\mathrm{diag}(\widehat{g}(\tilde{x},\tilde{t})) < 0$, which implies that $\tilde{x} \notin \partial X(\tilde{t})$. Hence, case (iv) is also impossible.

In conclusion, from the above discussions, we have that case (v) is the only possible case. Hence, Γ_{θ^0} must terminate in or approach to the hyperplane at $\tilde{t} = 0$ and $\tilde{\theta} = (\tilde{x},\tilde{y},\tilde{z})$ is a solution to the system (2.2). By Theorem 2.2, $\tilde{x}$ is a fixed point of $\Psi(x)$ in X. The proof is complete. □

4. Numerical test

In this section, we will give some numerical examples to numerically trace the smooth curve Γ_{θ^0}. By Theorem 3.2, the zero-point set $H_{\theta^0}^{-1}$ of the homotopy equation (3.1) determines a smooth curve for any given initial point $\theta^0 \in X^0(1) \times R_{++}^m \times R^l$ as $t \to 0$, one can find a fixed point of a C^2 self-mapping $\Psi(x)$ in X. For the numerical method of tracing the homotopy path Γ_{θ^0}, we can use standard Euler-Newton predictor-corrector procedure (for more details see references, e.g., [1, 16, 18, 20]).

In the following numerical examples, the shifted constraint functions are constructed as:

$$\widehat{g}_i(x,t) = \begin{cases} g_i(x), & \text{if } g_i(x^0) < 0, \\ g_i(x) - t^2(\max\{g_i(x^0)\} + 10), & \text{if } g_i(x^0) \ge 0, \end{cases} \tag{4.1}$$

and $\widehat{h}_j(x,t) = h_j(x) - t^2 h_j(x^0)$.

The parameters in homotopy equation (3.1) are set as $\zeta_i = 10$ and $\tau = \mathrm{randn}(n,1)$, an n vector with random entries drawn from a normal distribution with mean zero and standard deviation one.

The computations are performed on a computer running the software Matlab R2014a on Microsoft Windows 7 Professional with Intel(R) 3.20GHz processor and 4.00 GB megabytes of memory. And the termination tolerance is set as $\epsilon = 10^{-6}$.

In the following tables, let CPU denote the computer time, IT denote the iteration step which is the summation of the predictor step and the corrector step in the computing process, and $\tilde{x}$ denote the fixed point of $F(X) \subseteq X$.

Example 4.1. To find a fixed point of self-mapping:

$$\Psi(x) = (\frac{1}{2}x_1 + \frac{1}{25}x_2, x_1^2 + \frac{1}{4}x_2)^T,$$

in the set

$$X = \{(x_1, x_2) \in R^2 : -x_1 - 5 \le 0, x_1 - 5 \le 0, x_2 - 50 \le 0, -x_1^2 + \frac{1}{2}x_2^2 = 0\}.$$

In this example, the initial points are chosen as $x_1^0 = (2, 2)$, $x_2^0 = (5, 3)$, $x_3^0 = (-10, -10)$ and $x_4^0 = (-5, 20)$, which are not interior points in the original feasible set. Hence, the combined homotopy interior point method in [9, 18] fails. But, by the homotopy equation (3.1), we can get the unique fixed point $\tilde{x} = (0, 0)$ of the self-mapping $\Psi(X) \subseteq X$ as $t \to 0$. The detailed numerical results are listed in Table 1.

Table 1: The numerical results of Example 4.1.

$x^{(0)}$	CPU	IT	$\tilde{x}$
(2,2)	0.0624	24	$10^{-10}\times$(-3.1075, -1.1740)
(5,3)	0.2652	85	$10^{-7}\times$(-1.1578, -5.9051)
(-10,-10)	0.2184	86	$10^{-10}\times$(4.3591, -1.7146)
(-5,20)	0.1560	52	$10^{-8}\times$(-0.0343, -1.6570)

Example 4.2. To find a fixed point of self-mapping:

$$\Psi(x) = (x_1, -x_2)^T,$$

in the set

$$X = \{(x_1, x_2) \in R^2 : -x_1 - 5 \le 0, x_2 - 10 \le 0, -x_2 - 10 \le 0, x_1 - x_2^2 - 5 = 0\}.$$

In this example, the initial points are chosen as $x_1^0 = (1, 1)$, $x_2^0 = (10, 5)$, $x_3^0 = (20, -20)$ and $x_4^0 = (-20, 10)$, which are not interior points in the original feasible set. Hence, the combined homotopy interior point method in [9, 18] fails. But, by the homotopy equation (3.1), we can get the unique fixed point $\tilde{x} = (5, 0)$ of the self-mapping $\Psi(X) \subseteq X$ as $t \to 0$. The detailed numerical results are listed in Table 2.

Table 2: The numerical results of Example 4.2.

$x^{(0)}$	CPU	IT	$\tilde{x}$
(1,1)	0.0624	9	(5, 4.3988$\times 10^{-9}$)
(10,5)	0.0312	13	(5, 6.2204$\times 10^{-13}$)
(20,-20)	0.4524	80	(5, -1.7418$\times 10^{-11}$)
(-20,10)	0.1404	50	(5, -4.7762$\times 10^{-16}$)

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