Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4897–4908
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Block methods for a convex feasibility problem in a Banach space

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Communicated by X. Qin

Abstract

In this paper, a convex feasibility problem is investigated based on a block method. Strong convergence theorems for common solutions of equilibrium problems and generalized asymptotically quasi-ϕ-nonexpansive mappings are established in a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. The results obtained in this paper unify and improve many corresponding results announced recently.

Keywords: Banach space, block method, equilibrium problem, convex feasibility problem, variational inequality.
2010 MSC: 47H05, 90C33.

1. Introduction

The theory of fixed points of nonlinear operators is an important branch of modern mathematics as a bridge between nonlinear functional analysis and convex optimization theory. Many nonlinear problems arising in economics, medicine, engineering, and physics can be studied based on fixed point methods (see [8, 12, 27], and the references therein).

Recently, convex feasibility problems have been intensively investigated based on iterative techniques. The so called convex feasibility problems which capture lots of applications in various disciplines such as image restoration, and radiation therapy treatment planning are to find a special point in the intersection of common fixed point sets, which is a convex set of a (countable or uncountable) family of nonlinear operators.

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Received 2016-07-09

Mean-valued iterative methods are efficient for studying the fixed points of Lipschitz continuous nonlinear operators. However, in the framework of infinite-dimensional Hilbert spaces, they are only weakly convergent (see [13] and the references therein). In many modern disciplines, including image recovery, economics, control theory, and quantum physics, problems arise in the framework of infinite dimension spaces. In such nonlinear problems, the strong convergence is often much more desirable than the weak convergence. To guarantee the strong convergence of mean-valued iteration algorithms, many authors use different regularization methods (see [4, 5, 7, 9–11, 20, 29]). The projection method which was first introduced by Haugazeau [15] has been considered for the approximation of fixed points of nonexpansive mappings. The advantage of projection methods is that the strong convergence of iterative sequences can be guaranteed without any compact assumptions imposed on mappings or spaces.

In this paper, we study on the generalized asymptotically-ϕ-nonexpansive mappings and equilibrium problems in the terminology of Blum and Oettli [6] based on a block method. The equilibrium problem includes many important problems in nonlinear functional analysis and convex optimization such as the Nash equilibrium problem, variational inequalities, complementarity problems, saddle point problems and game theory. Strong convergence theorems of common solutions are established with the aid of a generalized projection in a Banach space. The results obtained in this paper mainly unify and improve the corresponding results in [14, 16–19, 22, 23, 25].

2. Preliminaries

Let E be a real Banach space and E^* the dual space of E. Let B_E be the unit sphere of E. Recall that E is said to be a strictly convex space iff $\|x + y\| < 2$ for all $x, y \in B_E$ and $x \neq y$.

Recall that E is said to have a Gâteaux differentiable norm iff $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in B_E$. In this case, we also say that E is smooth. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in B_E$, the limit is attained uniformly for all $x \in B_E$. E is also said to have a uniformly Fréchet differentiable norm iff the above limit is attained uniformly for $x, y \in B_E$. In this case, we say that E is uniformly smooth. It is known that if E is uniformly smooth, then duality mapping J is uniformly norm-to-norm continuous on every bounded subset of E. It is also known that E^* is uniformly convex if and only if E is uniformly smooth.

In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

Recall that E has the Kadec-Klee (KK) property if $\lim_{n\to\infty} \|x_n - x\| = 0$ as $n \to \infty$, for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$ as $n \to \infty$. It is known that every uniformly convex Banach space has the KK property.

Let C be a nonempty closed and convex subset of E and let $G : C \times C \to \mathbb{R}$ be a function. Recall that in equilibrium problem, the aim is to find $\bar{x} \in C$ such that $G(\bar{x}\, y) \geq 0$ for all $y \in C$. We use $S(G)$ to denote the solution set of the equilibrium problem. That is, $S(G) = \{x \in C : G(x, y) \geq 0\ \forall y \in C\}$.

In order to study the equilibrium problem, we assume that G satisfies the following conditions:

(A1) $G(a, a) \equiv 0,\ \forall a \in C$;

(A2) $G(b, a) + G(a, b) \leq 0, \forall a, b \in C$;

(A3) $G(a, b) \geq \limsup_{t\downarrow 0} G(tc + (1 - t)a, b)\,, \forall a, b, c \in C$;

(A4) $b \mapsto G(a, b)$ is convex and weakly lower semi-continuous for all $a \in C$.

Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of mapping T. T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x'$ and $\lim_{n\to\infty} Tx_n = y'$, then $Tx' = y'$.

Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{f^* \in E^* : \|x\|^2 = \langle x, f^* \rangle = \|f^*\|^2\}.$$

Next, we assume that E is a smooth Banach space which means mapping J is single-valued and study the functional

$$\phi(x,y) := \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle \quad \forall x,y \in E.$$

In [3], Alber studied a new mapping Π_C in a Banach space E which is an analogue of P_C, the metric projection, in Hilbert spaces. Recall that the generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem $\phi(\bar{x},x) = \min_{y\in C}\phi(y,x)$. It is obvious from the definition of function ϕ that $(\|x\| - \|y\|)^2 \le \phi(x,y)$ for all $x,y \in E$.

Recall that a point p is said to be an asymptotic fixed point of mapping T if and only if subset C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. In this paper, we use $\widetilde{F}(T)$ to stand for the asymptotic fixed point set. Let K be a bounded subset of C. Recall that T is said to be uniformly asymptotically regular on C if and only if $\limsup_{n\to\infty}\sup_{x\in K}\{\|T^n x - T^{n+1}x\|\} = 0$.

T is said to be relatively nonexpansive iff

$$\phi(p,Tx) \le \phi(p,x) \quad \forall x \in C,\ \forall p \in \widetilde{F}(T) = F(T) \neq \emptyset.$$

T is said to be relatively asymptotically nonexpansive iff

$$\phi(p,T^n x) \le (\mu_n + 1)\phi(p,x) \quad \forall x \in C,\ \forall p \in \widetilde{F}(T) = F(T) \neq \emptyset \quad \forall n \ge 1,$$

where $\{\mu_n\} \subset [0,\infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

T is said to be quasi-ϕ-nonexpansive iff

$$\phi(p,Tx) \le \phi(p,x) \quad \forall x \in C,\ \ \forall p \in F(T) \neq \emptyset.$$

T is said to be asymptotically quasi-ϕ-nonexpansive iff there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\phi(p,T^n x) \le (\mu_n + 1)\phi(p,x) \quad \forall x \in C,\ \ \forall p \in F(T) \neq \emptyset,\ \forall n \ge 1.$$

Remark 2.1. The class of relatively asymptotically nonexpansive mappings, which was considered in [1] covers the class of relatively nonexpansive mappings. The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings cover the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings, respectively. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive mappings do not require the strong restriction that the fixed point set equals the asymptotic fixed point set, respectively (see [23, 24] and the references therein).

Remark 2.2. We note that the class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are generalizations of the class of quasi-nonexpansive mappings and the class of asymptotically quasi-nonexpansive mappings in Banach spaces, respectively, since the $\sqrt{\phi(x,y)} = \|x-y\|$.

T is said to be a generalized asymptotically quasi-ϕ-nonexpansive mapping if there exist two nonnegative sequences $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$, and $\{\xi_n\} \subset [0,\infty)$ with $\xi_n \to 0$ as $n \to \infty$ such that

$$\phi(p,T^n x) \le (\mu_n + 1)\phi(p,x) + \xi_n, \quad \forall x \in C, \forall p \in F(T) \neq \emptyset,\ \forall n \ge 1.$$

Remark 2.3. The class of generalized asymptotically quasi-ϕ-nonexpansive mappings, which was introduced in [21] is a generalization of the class of generalized asymptotically quasi-nonexpansive mappings [2] in the framework of Banach spaces. Common fixed points of generalized asymptotically quasi-nonexpansive mappings were investigated via the implicit iterations in [2].

The following lemmas play an important role in this paper.

Lemma 2.4 ([3]). *Let E be a strictly convex, reflexive, and smooth Banach space and let C be a nonempty, closed, and convex subset of E. Let $x \in E$. Then*

$$\phi(y, \Pi_C x) \leq \phi(y, x) - \phi(\Pi_C x, x), \quad \forall y \in C,$$

and $\langle y - x_0, Jx - Jx_0 \rangle \leq 0$ for all $y \in C$, if and only if $x_0 = \Pi_C x$.

Lemma 2.5 ([27]). *Let r be a positive real number and let E be uniformly convex. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|(1-t)y + ta\|^2 + t(1-t)g(\|b-a\|) \leq t\|a\|^2 + (1-t)\|b\|^2$$

for all $a, b \in B^r := \{a \in E : \|a\| \leq r\}$ and $t \in [0, 1]$.

Lemma 2.6 ([23, 28]). *Let E be a strictly convex, smooth, and reflexive Banach space and let C be a closed convex subset of E. Let G be a function, which satisfies* (A1)-(A4), *from $C \times C$ to $\mathbb{R}$. Let $x \in E$ and let $r > 0$. Define a mapping $W_{G,r} : E \to C$ by*

$$W^{G,r}x = \{z \in C : rG(z, y) + \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C\}.$$

Then, there exists $z \in C$ such that $rG(z, y) + \langle z - y, Jz - Jx \rangle \leq 0$ for all $y \in C$, and the following conclusions hold:

(1) *$W^{G,r}$ is single-valued quasi-ϕ-nonexpansive and*

$$\langle W^{G,r}x - W^{G,r}y, JW^{G,r}x - JW^{G,r}y \rangle \leq \langle W^{G,r}x - W^{G,r}y, Jx - Jy \rangle$$

for all $x, y \in E$;

(2) *$F(W^{G,r}) = S(G)$ is closed and convex;*

(3) *$\phi(q, W^{G,r}x) + \phi(W^{G,r}x, x) \leq \phi(q, x)$, $\forall q \in F(W^{G,r})$.*

3. Main results

Theorem 3.1. *Let E be a strictly convex and uniformly smooth Banach space which also has the KK property. Let C be a nonempty closed and convex subset of E. Let G_i be a bifunction satisfying conditions (A1)-(A4) and let T_i be a generalized asymptotically quasi-ϕ-nonexpansive mapping on C for every $i \in \Lambda$, where Λ is an arbitrary index set. Assume that T_i is closed and uniformly asymptotically regular on C for every $i \in \Lambda$ and $\cap_{i\in\Lambda} F(T_i) \bigcap \cap_{i\in\Lambda} S(G_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following process. $x_0 \in E$ chosen arbitrarily and*

$$\begin{cases} C_{(1,i)} = C, \quad \forall i \in \Lambda, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, \ x_1 = \Pi_{C_1} x_0, \\ Jy_{(n,i)} = (1 - \alpha_{(n,i)}) JT_i^n x_n + \alpha_{(n,i)} Jx_n, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \mu_{(n,i)} M_{(n,i)} + \phi(z, x_n) + \xi_{(n,i)} \geq \phi(z, u_{(n,i)})\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $u_{(n,i)} \in C_n$ such that $r_{(n,i)} G_i(u_{(n,i)}, y) + \langle u_{(n,i)} - y, Ju_{(n,i)} - Jy_{(n,i)} \rangle \leq 0$ for all $y \in C_n$, $M_{(n,i)} = \sup\{\phi(p, x_n) : p \in \cap_{i\in\Lambda} F(T_i) \bigcap \cap_{i\in\Lambda} S(G_i)\}$, $\{\alpha_{(n,i)}\}$ is a real sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_{(n,i)}$ $(\alpha_{(n,i)} - 1) < 0$, and $\{r_{(n,i)}\}$ is a real sequence in $[r_i, \infty)$, where $\{r_i\}$ is a positive real number sequence. Then $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Lambda} F(T_i) \bigcap \cap_{i\in\Lambda} S(G_i)} x_1$.

Proof. Since T_i is closed, we find that $\cap_{i\in\Lambda}F(T_i)$ is also closed. Let $p_{(1,i)}, p_{(2,i)} \in F(T_i)$, and $p_i = (1-t_i)p_{(2,i)} + t_i p_{(1,i)}$, where $t_i \in (0,1)$, for every $i \in \Lambda$. Note that $\xi_{(n,i)} + (1+\mu_{(n,i)})\phi(p_{(1,i)}, p_i) \geq \phi(p_{(1,i)}, T_i^n p_i)$, and $\xi_{(n,i)} + (1+\mu_{(n,i)})\phi(p_{(2,i)}, p_i) \geq \phi(p_{(2,i)}, T_i^n p_i)$. It follows that

$$\phi(p_{(1,i)}, T_i^n p_i) - 2\langle p_i - p_{(1,i)}, Jp_i - JT_i^n p_i\rangle = \phi(p_{(1,i)}, p_i) + \phi(p_i, T_i^n p_i),$$

and

$$\phi(p_{(2,i)}, T_i^n p_i) - 2\langle p_i - p_{(2,i)}, Jp_i - JT_i^n p_i\rangle = \phi(p_{(2,i)}, p_i) + \phi(p_i, T_i^n p_i).$$

Hence, we have

$$\phi(p_i, T_i^n p_i) \leq 2\langle p_i - p_{(1,i)}, Jp_i - J(T_i^n p_i)\rangle + \xi_{(n,i)} + \mu_{(n,i)}\phi(p_{(1,i)}, p_i) \tag{3.1}$$

and

$$\phi(p_i, T_i^n p_i) \leq 2\langle p_i - p_{(2,i)}, Jp_i - J(T_i^n p_i)\rangle + \xi_{(n,i)} + \mu_{(n,i)}\phi(p_{(2,i)}, p_i). \tag{3.2}$$

Multiplying t_i and $(1-t_i)$ on the both sides of (3.1) and (3.2), respectively, yields that

$$\phi(p_i, T_i^n p_i) \leq (1-t_i)\mu_{(n,i)}\phi(p_{(2,i)}, p_i) + \xi_{(n,i)} + t_i\mu_{(n,i)}\phi(p_{(1,i)}, p_i).$$

It follows that $\lim_{n\to\infty}\phi(p_i, T_i^n p_i) = 0$. This implies that

$$\|p_i\| = \lim_{n\to\infty}\|T_i^n p_i\|. \tag{3.3}$$

It follows that

$$\|Jp_i\| = \lim_{n\to\infty}\|J(T_i^n p_i)\|. \tag{3.4}$$

Since E^* is reflexive, we may assume that $J(T_i^n p_i)$ converges weakly to $g^{*,i}$. In view of the reflexivity of E, we have $E^* = J(E)$. This shows that there exists an element $g^i \in E$ such that $g^{*,i} = Jg^i$. It follows that $\phi(p_i, T_i^n p_i) + 2\langle p_i, J(T_i^n p_i)\rangle = \|p_i\|^2 + \|J(T_i^n p_i)\|^2$. Taking limit infimum as $n \to \infty$ on the both sides of the equality above, we obtain that

$$0 \geq \|p_i\|^2 + \|g^{*,i}\|^2 - 2\langle p_i, g^{*,i}\rangle = \|p_i\|^2 + \|g^i\|^2 - 2\langle p_i, Jg^i\rangle = \phi(p_i, g^i) \geq 0.$$

This implies that $g^i = p_i$, that is, $Jp_i = g^{*,i}$. It follows that $J(T_i^n p_i) \rightharpoonup Jp_i \in E^*$. In view of the KK property of E^*, we obtain from (3.4) that $\lim_{n\to\infty}\|Jp_i - J(T_i^n p_i)\| = 0$. It follows that $T_i^n p_i \rightharpoonup p_i$. Using the KK property of E, one finds from (3.3) that $T_i^n p_i \to p_i$ as $n \to \infty$. It follows that $T_i T_i^n p_i = T_i^{n+1} p_i \to p_i$, as $n \to \infty$. Since T_i is closed, one sees that $p_i \in F(T_i)$, for every $i \in \Lambda$. This proves, for every $i \in \Lambda$, that $F(T_i)$ is convex. This in turn implies that $\cap_{i\in\Lambda}F(T_i)$ is a convex set. Using Lemma 2.6, one obtains that $\cap_{i\in\Lambda}S(G_i)$ is closed and convex. This proves that $\Pi_{\cap_{i\in\Lambda}S(G_i)\bigcap\cap_{i\in\Lambda}F(T_i)}x$ is well-defined, for any element in E.

Next, we prove that C_n is convex and closed. It is obvious that $C_{(1,i)} = C$ is convex and closed. Assume that $C_{(m,i)}$ is convex and closed for some $m \geq 1$. Let $z_1, z_2 \in C_{(m+1,i)}$, we see that $z_1, z_2 \in C_{(m,i)}$. It follows that $z = tz_1 + (1-t)z_2 \in C_{(m,i)}$, where $t \in (0,1)$. Notice that $\phi(z_1, u_{(m,i)}) - \phi(z_1, x_m) \leq \xi_{(m,i)} + \mu_{(m,i)}M_{(m,i)}$, and $\phi(z_2, u_{(m,i)}) - \phi(z_2, x_m) \leq \xi_{(m,i)} + \mu_{(m,i)}M_{(m,i)}$, Hence, one has

$$2\langle z_1, Jx_m - Ju_{(m,i)}\rangle - \|x_m\|^2 + \|u_{(m,i)}\|^2 \leq \xi_{(m,i)} + \mu_{(m,i)}M_{(m,i)}$$

and

$$2\langle z_2, Jx_k - Ju_{(m,i)}\rangle - \|x_m\|^2 + \|u_{(m,i)}\|^2 \leq \xi_{(m,i)} + \mu_{(m,i)}M_{(m,i)}.$$

This finds $\phi(z, x_m) + \xi_{(m,i)} + \mu_{(m,i)}M_{(m,i)} \geq \phi(z, u_{(m,i)})$, where $z \in C_{(m,i)}$. This shows that $C_{(m+1,i)}$ is closed and convex. Hence, $C_n = \cap_{i\in\Lambda}C_{(n,i)}$ is a convex and closed set. This proves that $\Pi_{C_{n+1}}x_1$ is well defined.

Now, we are in a position to prove that $\cap_{i\in\Lambda}F(T_i)\bigcap\cap_{i\in\Lambda}S(G_i) \subset C_n$.

Here, $\cap_{i\in\Lambda}F(T_i)\bigcap\cap_{i\in\Lambda}S(G_i)\subset C_1=C$ is clear. Suppose that $\cap_{i\in\Lambda}F(T_i)\bigcap\cap_{i\in\Lambda}S(G_i)\subset C_{(m,i)}$ for some positive integer m. For any $w\in\cap_{i\in\Lambda}F(T_i)\bigcap\cap_{i\in\Lambda}S(G_i)\subset C_{(m,i)}$, we see that

$$\begin{aligned}\phi(w,u_{(m,i)}) &\le \phi(w,y_{(m,i)})\\ &= \|w\|^2+\|\alpha_{(m,i)}Jx_m+(1-\alpha_{(m,i)})JT_i^m x_m\|^2-2\langle w,\alpha_{(m,i)}Jx_m+(1-\alpha_{(m,i)})JT_i^m x_m\rangle\\ &\le \|w\|^2-2(1-\alpha_{(m,i)})\langle w,JT_i^m x_m\rangle-2\alpha_{(m,i)}\langle w,Jx_m\rangle\\ &\quad+(1-\alpha_{(m,i)})\|T_i^m x_m\|^2+\alpha_{(m,i)}\|x_m\|^2\\ &\le (1-\alpha_{(m,i)})\phi(w,x_m)+\alpha_{(m,i)}\phi(w,x_m)+(1-\alpha_{(m,i)})\mu_{(m,i)}\phi(w,x_m)+(1-\alpha_{(m,i)})\xi_{(m,i)}\\ &\le \phi(w,x_m)+\xi_{(m,i)}+\mu_{(m,i)}\phi(w,x_m),\end{aligned}$$

which shows that $w\in C_{(m+1,i)}$. This implies that $\cap_{i\in\Lambda}S(G_i)\bigcap\cap_{i\in\Lambda}F(T_i)\subset C_{(n,i)}$. Hence, we find that $\cap_{i\in\Lambda}S(G_i)\bigcap\cap_{i\in\Lambda}F(T_i)\subset\cap_{i\in\Lambda}C_{(n,i)}$. Using Lemma 2.4, one has $\langle z-x_n,Jx_1-Jx_n\rangle\le 0$ for any $z\in C_n$. It follows that

$$\langle w-x_n,Jx_1-Jx_n\rangle\le 0\quad \forall w\in\cap_{i\in\Lambda}S(G_i)\bigcap\cap_{i\in\Lambda}F(T_i). \tag{3.5}$$

Using Lemma 2.4 yields that $\phi(x_n,x_1)\le\phi(\Pi_{\cap_{i\in\Lambda}F(T_i)\bigcap\cap_{i\in\Lambda}S(G_i)}x_1,x_1)$, which shows that $\{\phi(x_n,x_1)\}$ is bounded. Hence, $\{x_n\}$ is also bounded. Since E is reflexive, we may assume that $x_n\rightharpoonup\bar{x}$. Since C_n is closed and convex, this yields $\bar{x}\in C_n$. Therefore, $\phi(x_n,x_1)\le\phi(\bar{x},x_1)$. Since the norm is weakly lower semicontinuous, we have

$$\phi(\bar{x},x_1)\le\liminf_{n\to\infty}(\|x_n\|^2+\|x_1\|^2-2\langle x_n,Jx_1\rangle)=\liminf_{n\to\infty}\phi(x_n,x_1)\le\phi(\bar{x},x_1).$$

It follows that $\lim_{n\to\infty}\phi(x_n,x_1)=\phi(\bar{x},x_1)$. Hence, $\lim_{n\to\infty}\|x_n\|=\|\bar{x}\|$. Using the KK property of the space, one obtains that x_n converges strongly to $\bar{x}$ as $n\to\infty$. On the other hand, we find that $\phi(x_{n+1},x_1)\ge\phi(x_n,x_1)$, which shows that $\{\phi(x_n,x_1)\}$ is nondecreasing. Therefore, $\lim_{n\to\infty}\phi(x_n,x_1)$ exists. It follows that $\phi(x_{n+1},x_1)-\phi(x_n,x_1)\ge\phi(x_{n+1},x_n)$. Therefore, we have $\lim_{n\to\infty}\phi(x_{n+1},x_n)=0$. Since $x_{n+1}\in C_{n+1}$, we have $\phi(x_{n+1},x_n)+\xi_{(n,i)}+\mu_{(n,i)}M_{(n,i)}\ge\phi(x_{n+1},u_{(n,i)})\ge 0$. It follows that $\lim_{n\to\infty}\phi(x_{n+1},u_{(n,i)})=0$. This yields that $\lim_{n\to\infty}(\|u_{(n,i)}\|-\|x_{n+1}\|)=0$. Therefore, $\lim_{n\to\infty}\|u_{(n,i)}\|=\|\bar{x}\|$. That is, $\lim_{n\to\infty}\|Ju_{(n,i)}\|=\lim_{n\to\infty}\|u_{(n,i)}\|=\|\bar{x}\|=\|J\bar{x}\|$. This implies that $\{Ju_{(n,i)}\}$ is bounded. Assume that $Ju_{(n,i)}$ converges weakly to $u^{(*,i)}\in E^*$. In view of the reflexivity of E, we see that $J(E)=E^*$. This shows that there exists an element $u^i\in E$ such that $Ju^i=u^{(*,i)}$. It follows that $\phi(x_{n+1},u_{(n,i)})+2\langle x_{n+1},Ju_{(n,i)}\rangle=\|x_{n+1}\|^2+\|Ju_{(n,i)}\|^2$. Taking limit infimum as $n\to\infty$ yields that

$$0\ge\|\bar{x}\|^2-2\langle\bar{x},u^{(*,i)}\rangle+\|u^{(*,i)}\|^2=\|\bar{x}\|^2+\|Ju^i\|^2-2\langle\bar{x},Ju^i\rangle=\phi(\bar{x},u^i).$$

That is, $\bar{x}=u^i$, which in turn implies that $J\bar{x}=u^{(*,i)}$. Hence, $Ju_{(n,i)}\rightharpoonup J\bar{x}\in E^*$. Since E^* is uniformly convex, then it has the KK property and we obtain that $\lim_{n\to\infty}Ju_{(n,i)}=J\bar{x}$. Since $J^{-1}:E^*\to E$ is demicontinuous and E has the KK property, one gets that $u_{(n,i)}\to\bar{x}$, as $n\to\infty$. This implies $\lim_{n\to\infty}\|x_n-u_{(n,i)}\|=0$. Hence, we have $\lim_{n\to\infty}(\phi(w,x_n)-\phi(w,u_{(n,i)}))=0$. Since E^* is uniformly convex, we find from Lemma 2.5 that

$$\begin{aligned}\phi(w,u_{(n,i)}) &\le \|w\|^2+\|\alpha_{(n,i)}Jx_n+(1-\alpha_{(n,i)})JT_i^n x_n\|^2-2\langle w,(1-\alpha_{(n,i)})JT_i^n x_n+\alpha_{(n,i)}Jx_n\rangle\\ &\le \|w\|^2-2(1-\alpha_{(n,i)})\langle w,JT_i^n x_n\rangle-2\alpha_{(n,i)}\langle w,Jx_n\rangle\\ &\quad-\alpha_{(n,i)}(1-\alpha_{(n,i)})g(\|Jx_n-JT_i^n x_n\|)+\alpha_{(n,i)}\|x_n\|^2+(1-\alpha_{(n,i)})\|T_i^n x_n\|^2\\ &\le \phi(w,x_n)+(1-\alpha_{(n,i)})\mu_{(n,i)}\phi(w,x_n)-\alpha_{(n,i)}(1-\alpha_{(n,i)})g(\|Jx_n-JT_i^n x_n\|)+\xi_{(n,i)}\\ &\le \phi(w,x_n)+\mu_{(n,i)}M_{(n,i)}-\alpha_{(n,i)}(1-\alpha_{(n,i)})g(\|Jx_n-JT_i^n x_n\|)+\xi_{(n,i)}.\end{aligned} \tag{3.6}$$

It follows that $\lim_{n\to\infty}\|Jx_n-JT_i^n x_n\|=0$. This implies that $\lim_{n\to\infty}\|JT_i^n x_n-J\bar{x}\|=0$. Since $J^{-1}:E^*\to E$ is demicontinuous, one has $T_i^n x_n\rightharpoonup\bar{x}$. Hence, one has $\lim_{n\to\infty}\|T_i^n x_n\|=\|\bar{x}\|$. Since E has the KK property, we obtain $\lim_{n\to\infty}\|\bar{x}-T_i^n x_n\|=0$. Since each T_i is uniformly asymptotically regular, one has $\lim_{n\to\infty}\|\bar{x}-T_i^{n+1}x_n\|=0$. That is, $T_i(T_i^n x_n)\to\bar{x}$. Using the closedness of T_i, we find $\bar{x}=T_i\bar{x}$ for each $i\in\Lambda$. This proves $\bar{x}\in\cap_{i\in\Lambda}F(T_i)$.

Next, we show that $\bar{x}\in\cap_{i\in\Lambda}S(G_i)$. It follows from (3.6) that $\lim_{n\to\infty}\phi(u_{(n,i)},y_{(n,i)})=0$. Hence, we have $\lim_{n\to\infty}(\|u_{(n,i)}\|-\|y_{(n,i)}\|)=0$. Since $u_{(n,i)}\to\bar{x}$ as $n\to\infty$, we find that $\|y_{(n,i)}\|\to\|\bar{x}\|$ as $n\to\infty$, that is, $\lim_{n\to\infty}\|Jy_{(n,i)}\|=\|J\bar{x}\|$. This shows that $\{Jy_{(n,i)}\}$ is bounded. Since E is uniformly smooth, one sees that E^* is reflexive. We may assume that $Jy_{(n,i)}\rightharpoonup y^{(*,i)}\in E^*$. There exists $y^i\in E$ such that $Jy^i=y^{(*,i)}$. It follows that $\phi(u_{(n,i)},y_{(n,i)})+2\langle u_{(n,i)},Jy_{(n,i)}\rangle=\|u_{(n,i)}\|^2+\|Jy_{(n,i)}\|^2$. Therefore, we have

$$0\geq\|\bar{x}\|^2+\|y^{(*,i)}\|^2-2\langle\bar{x},y^{(*,i)}\rangle=\|\bar{x}\|^2+\|y^i\|^2-2\langle\bar{x},Jy^i\rangle=\phi(\bar{x},y^i).$$

That is, $\bar{x}=y^i$. Hence, we have $y^{(*,i)}=J\bar{x}$. It follows that $Jy_{(n,i)}\rightharpoonup J\bar{x}\in E^*$. Since E^* is uniformly convex, it has the KK property, we obtain that $Jy_{(n,i)}-J\bar{x}\to 0$ as $n\to\infty$. Using the fact that $J^{-1}:E^*\to E$ is demicontinuous, we see that $y_{(n,i)}\rightharpoonup\bar{x}$. Using the KK property, we obtain that $y_{(n,i)}\to\bar{x}$ as $n\to\infty$. Since E is uniformly smooth, $\lim_{n\to\infty}\|Jy_{(n,i)}-Ju_{(n,i)}\|=0$. Since G_i is monotone, we find that

$$r_{(n,i)}G_i(y,u_{(n,i)})\leq\|y-u_{(n,i)}\|\|Ju_{(n,i)}-Jy_{(n,i)}\|,\quad\forall y\in C_n.$$

Therefore, one sees $G_i(y,\bar{x})\leq 0$ for all $y\in C$. For $y\in C$ and $0<t_i<1$, define $y_{(t,i)}=(1-t_i)\bar{x}+t_i y$. This implies that $0\geq G_i(y_{(t,i)},\bar{x})$. Hence, we have $0=G_i(y_{(t,i)},y_{(t,i)})\leq t_iG_i(y_{(t,i)},y)$. It follows that $G_i(\bar{x},y)\geq 0$ for all $y\in C$. This implies that $\bar{x}\in S(G_i)$ for every $i\in\Lambda$.

Finally, we prove $\bar{x}=\Pi_{\cap_{i\in\Lambda}F(T_i)\bigcap\cap_{i\in\Lambda}S(G_i)}x_1$. Letting $n\to\infty$ in (3.5), we arrive at $\langle\bar{x}-w,Jx_1-J\bar{x}\rangle\geq 0$ for $w\in\cap_{i\in\Lambda}F(T_i)\bigcap\cap_{i\in\Lambda}S(G_i)$. Using Lemma 2.4, we find that $\bar{x}=\Pi_{\cap_{i\in\Lambda}F(T_i)\bigcap\cap_{i\in\Lambda}S(G_i)}x_1$. This completes the proof. □

Remark 3.2. Theorem 3.1 mainly improves the corresponding results in Hao [14], Qin et al. [22] and unifies the corresponding results in [14, 16–19, 22, 23, 25]. The framework of the space is applicable to L^p, where $p>1$.

For the class of asymptotically quasi-ϕ-nonexpansive mappings, we have the following result.

Corollary 3.3. *Let E be a strictly convex and uniformly smooth Banach space which also has the KK property. Let C be a nonempty closed and convex subset of E. Let G_i be a bifunction satisfying conditions (A1)-(A4) and let $T_i:C\to C$ be asymptotically quasi-ϕ-nonexpansive for every $i\in\Lambda$, where Λ is an arbitrary index set. Assume that T_i is closed and uniformly asymptotically regular on C for every $i\in\Lambda$. Assume that $\cap_{i\in\Lambda}S(G_i)\bigcap\cap_{i\in\Lambda}F(T_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following process. $x_0\in E$ chosen arbitrarily and*

$$\begin{cases}C_{(1,i)}=C,\quad\forall i\in\Lambda,\\ C_1=\cap_{i\in\Lambda}C_{(1,i)},\ x_1=\Pi_{C_1}x_0,\\ Jy_{(n,i)}=\alpha_{(n,i)}Jx_n+(1-\alpha_{(n,i)})JT_i^n x_n,\\ C_{(n+1,i)}=\{z\in C_{(n,i)}:\phi(z,x_n)+\mu_{(n,i)}M_{(n,i)}\geq\phi(z,u_{(n,i)})\},\\ C_{n+1}=\cap_{i\in\Lambda}C_{(n+1,i)},x_{n+1}=\Pi_{C_{n+1}}x_1,\end{cases}$$

where $u_{(n,i)}\in C_n$ such that $r_{(n,i)}G_i(u_{(n,i)},y)+\langle y-u_{(n,i)},Ju_{(n,i)}-Jy_{(n,i)}\rangle\geq 0$ for all $y\in C_n$, $M_{(n,i)}=$ $\sup\{\phi(p,x_n):p\in\cap_{i\in\Lambda}S(G_i)\bigcap\cap_{i\in\Lambda}F(T_i)\}$, $\{\alpha_{(n,i)}\}$ is a real sequence in $(0,1)$ such that $\liminf_{n\to\infty}\alpha_{(n,i)}$ $(\alpha_{(n,i)}-1)<0$, and $\{r_{(n,i)}\}$ is a real sequence in $[r_i,\infty)$, where $\{r_i\}$ is a positive real number sequence. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Lambda}S(G_i)\bigcap\cap_{i\in\Lambda}F(T_i)}x_1$.

Remark 3.4. Corollary 3.3 improves the corresponding results in Kim [17]. To be more clear, we have the following: a single mapping was extended to a family of mappings and a single bifunction was extended to a family of bifunctions, respectively.

For the class of quasi-ϕ-nonexpansive mappings, the boundedness of the common solution set is not required. Indeed, we have the following result.

Corollary 3.5. *Let E be a strictly convex and uniformly smooth Banach space which also has the KK property. Let C be a nonempty closed and convex subset of E. Let G_i be a bifunction satisfying (A1)-(A4) and let $T_i : C \to C$ be a quasi-ϕ-nonexpansive for every $i \in \Lambda$, where Λ is an arbitrary index set. Assume that T_i is closed for every $i \in \Lambda$. Assume that $\cap_{i\in\Lambda} S(G_i) \bigcap \cap_{i\in\Lambda} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following process. $x_0 \in E$ chosen arbitrarily and*

$$\begin{cases} C_{(1,i)} = C, \ \ \forall i \in \Lambda, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, \ x_1 = \Pi_{C_1} x_0, \\ Jy_{(n,i)} = \alpha_{(n,i)} Jx_n + (1 - \alpha_{(n,i)}) JT_i x_n, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \phi(z, x_n) \geq \phi(z, u_{(n,i)})\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $u_{(n,i)} \in C_n$ such that $r_{(n,i)} G_i(u_{(n,i)}, y) + \langle y - u_{(n,i)}, Ju_{(n,i)} - Jy_{(n,i)}\rangle \geq 0$ for all $y \in C_n$, $\{\alpha_{(n,i)}\}$ is a real sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_{(n,i)}(\alpha_{(n,i)} - 1) < 0$, and $\{r_{(n,i)}\}$ is a real sequence in $[r_i, \infty)$, where $\{r_i\}$ is a positive real number sequence. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Lambda} S(G_i) \bigcap \cap_{i\in\Lambda} F(T_i)} x_1$.

Remark 3.6. Corollary 3.5 improves the corresponding results in Qin et al. [25]. A single equilibrium problem was extended to a family of equilibrium problems and a pair of mappings was extended to a family of mappings. Both bifunctions and mappings are uncountable infinite families. The projection studied in this paper is also different from [25], which is only valid for a countable infinite family mappings. Since every uniformly convex Banach space is a strictly convex Banach space which also has the KK property, we see that Corollary 3.5 is still valid in uniformly smooth and uniformly convex Banach spaces. Corollary 3.5 improves the corresponding results in Qin et al. [24].

4. Applications

In this section, we give some deduced results in the framework of Hilbert spaces. We also consider solutions of a family of variational inequalities and common minimizers of a family of proper, lower semi-continuous, and convex functionals.

Theorem 4.1. *Let E be a Hilbert space and let C be a nonempty closed and convex subset of E. Let G_i be a bifunction satisfying (A1)-(A4) and let $T_i : C \to C$ be a generalized asymptotically quasi-nonexpansive mapping for every $i \in \Lambda$, where Λ is an arbitrary index set. Assume that T_i is closed and uniformly asymptotically regular on C for every $i \in \Lambda$. Assume that $\cap_{i\in\Lambda} S(G_i) \bigcap \cap_{i\in\Lambda} F(T_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following process. $x_0 \in E$ chosen arbitrarily and*

$$\begin{cases} C_{(1,i)} = C, \ \ \forall i \in \Lambda, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, \ x_1 = Proj_{C_1} x_0, \\ y_{(n,i)} = \alpha_{(n,i)} x_n + (1 - \alpha_{(n,i)}) T_i^n x_n, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \|z - x_n\|^2 + \xi_{(n,i)} + \mu_{(n,i)} M_{(n,i)} \geq \|z - u_{(n,i)}\|^2\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where $u_{(n,i)} \in C_n$ such that $\langle y - u_{(n,i)}, u_{(n,i)} - y_{(n,i)}\rangle + r_{(n,i)} G_i(u_{(n,i)}, y) \geq 0$ for all $y \in C_n$, $M_{(n,i)} = \sup\{\|p - x_n\|^2 : p \in \cap_{i\in\Lambda} S(G_i) \bigcap \cap_{i\in\Lambda} F(T_i)\}$, $\{\alpha_{(n,i)}\}$ is a real sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_{(n,i)}(\alpha_{(n,i)} - 1) < 0$, $\{r_{(n,i)}\}$ is a real sequence in $[r_i, \infty)$, where $\{r_i\}$ is a positive real number sequence and $Proj$ is the metric projection. Then the sequence $\{x_n\}$ converges strongly to $Proj_{\cap_{i\in\Lambda} S(G_i) \bigcap \cap_{i\in\Lambda} F(T_i)} x_1$.

Proof. In the framework of Hilbert spaces, we see that $\sqrt{\phi(x,y)} = \|x-y\|$ for all $x, y \in E$. The generalized projection is reduced to the metric projection and the generalized asymptotically-ϕ-nonexpansive mapping is reduced to the generalized asymptotically quasi-nonexpansive mapping. Using Theorem 3.1, we find the desired conclusion immediately. □

For the class of quasi-nonexpansive mappings, we have the following result.

Corollary 4.2. *Let E be a Hilbert space. Let C be a nonempty closed and convex subset of E. Let G_i be a bifunction satisfying (A1)-(A4) and let $T_i : C \to C$ be a quasi-nonexpansive mapping for every $i \in \Lambda$, where Λ is an arbitrary index set. Assume that T_i is closed for every $i \in \Lambda$. Assume that $\cap_{i\in\Lambda} S(G_i) \bigcap \cap_{i\in\Lambda} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following process. $x_0 \in E$ chosen arbitrarily and*

$$\begin{cases} C_{(1,i)} = C, \ \ \forall i \in \Lambda, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, \ x_1 = Proj_{C_1} x_0, \\ y_{(n,i)} = \alpha_{(n,i)} x_n + (1-\alpha_{(n,i)}) T_i x_n, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \|z - x_n\| \geq \|z - u_{(n,i)}\|\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, \ x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where $u_{(n,i)} \in C_n$ such that $r_{(n,i)} G_i(u_{(n,i)}, y) + \langle y - u_{(n,i)}, u_{(n,i)} - y_{(n,i)}\rangle \geq 0$ for all $y \in C_n$, $M_{(n,i)} = \sup\{\|p - x_n\|^2 : p \in \cap_{i\in\Lambda} S(G_i) \bigcap \cap_{i\in\Lambda} F(T_i)\}$, $\{\alpha_{(n,i)}\}$ is a real sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_{(n,i)}(\alpha_{(n,i)} - 1) < 0$, $\{r_{(n,i)}\}$ is a real sequence in $[r_i, \infty)$, where $\{r_i\}$ is a positive real number sequence and $Proj$ is the metric projection. Then the sequence $\{x_n\}$ converges strongly to $Proj_{\cap_{i\in\Lambda} S(G_i) \bigcap \cap_{i\in\Lambda} F(T_i)} x_1$.

Let $A : C \to E^*$ be a single valued monotone operator which is continuous along each line segment in C with respect to the weak* topology of E^* (hemicontinuous). Recall the following variational inequality. Finding a point $x \in C$ such that $\langle x - y, Ax\rangle \leq 0$ for all $y \in C$. The symbol $N_C(x)$ stands for the normal cone for C at a point $x \in C$; that is, $N_C(x) = \{x^* \in E^* : \langle x - y, x^*\rangle \geq 0, \ \forall y \in C\}$. From now on, we use $VI(C, A)$ to denote the solution set of the variational inequality.

Theorem 4.3. *Let E be a strictly convex and uniformly smooth Banach space which also has the KK property. Let C be a nonempty closed and convex subset of E. Let Λ be an index set and let $A_i : C \to E^*$ be a single valued, monotone, and hemicontinuous operator. Let G_i be a bifunction satisfying (A1)-(A4). Assume that $\cap_{i\in\Lambda} S(G_i) \bigcap \cap_{i\in\Lambda} VI(C, A_i)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following process. $x_0 \in E$ chosen arbitrarily and*

$$\begin{cases} C_{1,i} = C, \ \ \forall i \in \Lambda, \\ C_1 = \cap_{i\in\Delta} C_{(1,i)}, \ x_1 = \Pi_{C_1} x_0, \\ z_{(n,i)} = VI(C, A_i + \frac{1}{r_i}(J - Jx_n)), \\ Jy_{(n,i)} = (1-\alpha_{(n,i)}) Jz_{n,i} + \alpha_{(n,i)} Jx_n, \quad n \geq 1, \\ C_{(n+1,i)} = \{w \in C_{(n,i)} : \phi(w, x_n) \geq \phi(w, u_{n,i})\}, \\ C_{n+1} = \cap_{i\in\Delta} C_{(n+1,i)}, \ x_{n+1} = \Pi_{C_{n+1}} x_0 \quad \forall n \geq 1, \end{cases}$$

where $u_{(n,i)} \in C_n$ such that $r_{(n,i)} G_i(u_{(n,i)}, y) + \langle y - u_{(n,i)}, Ju_{(n,i)} - Jy_{(n,i)}\rangle \geq 0$ for all $y \in C_n$, $M_{(n,i)} = \sup\{\phi(p, x_n) : p \in \cap_{i\in\Lambda} S(G_i) \bigcap \cap_{i\in\Lambda} VI(C, A_i)\}$, $\{\alpha_{(n,i)}\}$ is a real sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_{(n,i)} (\alpha_{(n,i)} - 1) < 0$, and $\{r_{(n,i)}\}$ is a real sequence in $[r_i, \infty)$, where $\{r_i\}$ is a positive real number sequence. Then $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Lambda} S(G_i) \bigcap \cap_{i\in\Lambda} VI(C, A_i)} x_1$.

Proof. Define a mapping $T_i \subset E \times E^*$ by

$$T_i x = \begin{cases} \emptyset, & x \notin C, \\ A_i x + N_C x, & x \in C. \end{cases}$$

Hence, T is maximal monotone and $T_i^{-1}(0) = VI(C, A_i)$ (see [26]). For each $r_i > 0$, and $x \in E$, we see that there exists a unique x_{r_i} in the domain of T_i such that $Jx \in Jx_{r_i} + r_iT_i(x_{r_i})$, where $x_{r_i} = (J + r_iT_i)^{-1}Jx$. Notice that

$$z_{n,i} = VI(C, \frac{1}{r_i}(J - Jx_n) + A_i),$$

which is equivalent to

$$\langle z_{n,i} - y, A_i z_{n,i} + \frac{1}{r_i}(Jz_{n,i} - Jx_n)\rangle \leq 0, \quad \forall y \in C,$$

that is,

$$\frac{1}{r_i}\big(Jx_n - Jz_{n,i}\big) \in N_C(z_{n,i}) + A_i z_{n,i}.$$

This implies that $z_{n,i} = (J+r_iT_i)^{-1}Jx_n$. From [23], we find that $(J+r_iT_i)^{-1}J$ is closed quasi-ϕ-nonexpansive with $F((J + r_iT_i)^{-1}J) = T_i^{-1}(0)$. Using Theorem 3.1, we find the desired conclusion immediately. □

Example 4.4. Let $E = \mathbb{R}$ and $C = [0, 1]$. Define a mapping Q by

$$Qx = \begin{cases} 0, & x \in (\frac{1}{3}, 1], \\ \frac{1}{3}x, & x \in [0, \frac{1}{3}]. \end{cases}$$

Then Q is a generalized asymptotically-ϕ-nonexpansive mapping with a unique fixed pint 0. We also have the following:

$$\phi(Q^nx, Q^ny) = |Q^nx - Q^ny|^2 = \frac{1}{3^{2n}}|x - y|^2 \leq |x - y|^2 = \phi(x, y)x, y \in [0, \frac{1}{3}],$$

$$\phi(Q^nx, Q^ny) = |Q^nx - Q^ny|^2 = 0 \leq |x - y|^2 = \phi(x, y)\,\forall x, y \in (\frac{1}{3}, 1],$$

$$\begin{aligned}\phi(Q^nx, Q^ny) &= |Q^nx - Q^ny|^2 \\ &= |\frac{1}{3^n}x - 0|^2 \\ &\leq (|\frac{1}{3^n}x - \frac{1}{3^n}y| + |\frac{1}{3^n}y|)^2 \\ &\leq (\frac{1}{3^n}|x - y| + \frac{1}{3^n})^2 \\ &\leq |x - y|^2 + \xi_n,\end{aligned}$$

where $\xi_n = \frac{1}{3^{2n}} + \frac{2}{3^n}$ for any $x \in [0, \frac{1}{3}], y \in (\frac{1}{3}, 1]$, and

$$\begin{aligned}\phi(Q^nx, Q^ny) &= |Q^nx - Q^ny|^2 \\ &= |0 - \frac{1}{3^n}y|^2 \\ &\leq (|\frac{1}{3^n}y - \frac{1}{3^n}x| + |\frac{1}{3^n}x|)^2 \\ &\leq (\frac{1}{3^n}|y - x| + \frac{1}{3^n})^2 \\ &\leq |y - x|^2 + \xi_n,\end{aligned}$$

where $\xi_n = \frac{1}{3^{2n}} + \frac{2}{3^n}$ for any $x \in (\frac{1}{3}, 1], y \in [0, \frac{1}{3}]$. This implies that $\phi(Q^nx, Q^ny) \leq (1 + \mu_n)\phi(x, y) + \xi_n$. This shows that Q is a generalized asymptotically ϕ-nonexpansive mapping with a unique fixed point 0. Let $G : [0, 1] \times [0, 1] \to \mathbb{R}$ be defined as

$$G(x, y) := -\frac{3}{4}x^2 - \frac{\sqrt{3}}{2}\sin x + \frac{3}{4}y^2 + \frac{\sqrt{3}}{2}\sin y.$$

It is easy to check that

(1) $G(b,a) + G(a,b) \leq 0, \quad \forall a,b \in [0,1]$;

(2) $G(a,a) \equiv 0, \quad \forall a \in [0,1]$;

(3) $b \mapsto G(a,b)$ is convex and weakly lower semi-continuous for each $a \in [0,1]$;

(4) $G(a,b) \geq \limsup_{t\downarrow 0} G(tc + (1-t)a, b), \ \forall a,b,c \in [0,1]$;

and the common solution set of the equilibrium and the fixed point problems is not empty.

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