



# Fixed point results and its applications to the systems of non-linear integral and differential equations of arbitrary order

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## Abstract

In this manuscript, common fixed point results for self-mappings satisfying generalized weak integral type contraction in the setting of G-metric space are established. Using the derived results, some applications to the systems of non-linear integral and fractional differential equations are also discussed. ©2016 All rights reserved.

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## 1. Introduction and preliminaries

Throughout the paper  $\mathbb{R}^+$ ,  $\mathbb{N}$ ,  $\mathbb{N}_0$  will denote the set of all non-negative real numbers, the set of positive integers, the set of non-negative integers respectively and  $\Phi = \varphi$  such that  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , where  $\varphi$  is Lebesgue integrable, summable on each compact subset of  $\mathbb{R}^+$  and  $\int_0^\epsilon \varphi(t)dt > 0, \forall \epsilon > 0$ .

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Banach [5] established a theorem known as Banach contraction principle. Banach contraction principle states that “Any contraction mapping in a complete metric space has a unique fixed point”. After that many researchers generalized this principle in many directions using different contractive type conditions. Alber and Guerre-Delabriere [3], gave the concept of weak contraction and studied the existence of fixed points for self-map in Hilbert spaces. The concept of weak contraction has been extended by Rhoades to metric spaces who also defined  $\phi$ -weak contraction as follows:

A self-map  $T$  on metric space  $(X, d)$  is said to be  $\phi$ -weak contraction if there exists a map  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $\phi(0) = 0$  and  $\phi(t) > 0$  for all  $t > 0$  such that

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \quad \forall x, y \in X.$$

In [21], Rhoades proved the following theorem.

**Theorem 1.1.** *Weak contractive self-map in a complete metric space has a unique fixed point.*

Dutta and Choudhury [9] generalized the concept of weak contraction as a  $(\psi, \phi)$ -weak contraction and established the following result.

**Theorem 1.2.** *Let  $T$  be a self-map on complete metric space  $(X, d)$  satisfying the following inequality*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)), \quad \forall x, y \in X,$$

where  $\psi, \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is monotonic non-decreasing and continuous function such that  $\psi(0) = 0 = \phi(0)$ ,  $\psi(t) > 0$  and  $\phi(t) > 0$  for  $t > 0$ . Then  $T$  has a unique fixed point.

Mustafa and Sims [20], introduced a new concept of generalized metric space, named as G-metric space. In such spaces every triplet of elements are assigned to a non-negative real number, based on the notion of G-metric spaces after that many researchers extended the known contractions in G-metric space. One of these is  $(\psi, \phi)$ -weak contraction (see [6, 8, 10, 18, 19, 23]). Aage and Salunke[1] proved the following result for weak contraction in G-metric space.

**Theorem 1.3.** *Let  $(X, G)$  be a complete G-metric space and let  $T : X \rightarrow X$  be a mapping satisfying*

$$G(Tx, Ty, Tz) \leq G(x, y, z) - \phi(G(x, y, z))$$

for all  $x, y, z \in X$ . If  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a continuous and non-decreasing function with  $\phi^{-1}(0) = 0$ ,  $\phi(t) > 0$  for all  $t \in (0, \infty)$ , then  $T$  has a unique fixed point in  $X$ .

Mohanta [16] proved the following result in G-metric space.

**Theorem 1.4.** *Let  $(X, G)$  be a G-metric space and  $\psi, \phi$  be altering distance functions. Let the mappings  $T, f : X \rightarrow X$  satisfy*

$$\psi(G(Tx, Ty, Tz)) \leq \psi(G(fx, fy, fz)) - \phi(G(fx, fy, fz)), \quad \forall x, y, z \in X,$$

where,  $T(X) \subset F(X)$  and  $F(X)$  is a complete subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique common fixed point in  $X$ .

Branciari [7] introduced the concept of integral type contraction and proved the following famous Banach contraction theorem:

**Theorem 1.5.** *If  $T$  is a self-map of a complete metric space  $(X, d)$  such that for all  $x, y \in X$*

$$\int_0^{d(Tx, Ty)} \varphi(t) dt \leq \eta \int_0^{d(x, y)} \varphi(t) dt, \quad \eta \in (0, 1),$$

where,  $\varphi \in \Phi$ , then  $T$  has a unique fixed point.

This result was more generalized and extended by many authors via using different integral type contraction for the study of fixed point, common fixed point and coincidence point in the setting of different spaces. Among these, some are as follows:

**Theorem 1.6** ([15]). *Let  $T$  be  $\psi_\phi$ -weakly contractive self-map on complete metric space  $(X, d)$  and  $\varphi \in \Phi$  satisfying*

$$\psi\left(\int_0^{d(Tx, Ty)} \varphi(t) dt\right) \leq \psi\left(\int_0^{d(x, y)} \varphi(t) dt\right) - \phi\left(\int_0^{d(x, y)} \varphi(t) dt\right),$$

where  $x, y \in X$ ,  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is continuous and non-decreasing function,  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a lower semi continuous and non-decreasing function such that  $\psi(t) = 0 = \phi(t)$  if and only if  $t = 0$ . Then  $T$  has a unique fixed point.

Ayadi [4] proved the following common fixed point theorem for integral type contraction in generalized metric spaces.

**Theorem 1.7.** *Let  $(X, G)$  be a complete  $G$ -metric space and  $f, g : X \rightarrow X$  be mappings such that*

$$\int_0^{G(fx, fy, fz)} \varphi(t) dt \leq \alpha \int_0^{G(gx, gy, gz)} \varphi(t) dt, \quad \forall x, y, z \in X,$$

where  $\alpha \in [0, 1)$  and  $\varphi \in \Phi$ . If  $f(X) \subset g(X)$  and  $g(X)$  is a complete subspace of  $X$ . Then  $f$  and  $g$  have a unique point of coincidence in  $X$ . Moreover, if  $f$  and  $g$  are weakly compatible, then  $f$  and  $g$  have a unique common fixed point.

In the current work we derive some fixed point results for  $(\psi, \phi)$ -weak integral type contraction in complete  $G$ -metric space. In addition application to the system of non-linear integral and fractional differential equations are also discussed.

**Definition 1.8** ([20]). Let  $X$  be a non-empty set and let  $G : X \times X \times X \rightarrow \mathbb{R}^+$  be a function satisfying the conditions:

1.  $G(x, y, z) = 0$  implies that  $x = y = z$ ,  $\forall x, y, z \in X$ ;
2.  $0 < G(x, x, y)$ ,  $\forall x, y \in X$  with  $x \neq y$ ;
3.  $G(x, x, y) \leq G(x, y, z)$ ,  $\forall x, y, z \in X$  with  $y \neq z$ ;
4.  $G(x, y, z) = G(P\{x, y, z\})$ , where  $P$  is an arbitrary permutation of  $x, y, z$  (symmetry in three variables);
5.  $G(x, y, z) \leq G(x, k, k) + G(k, y, z)$ ,  $\forall x, y, z, k \in X$ .

Then it is a  $G$ -metric on  $X$  and the pair  $(X, G)$  is called  $G$ -metric space.

**Proposition 1.9** ([20]). *Let  $(X, G)$  be a  $G$ -metric space. The following are equivalent:*

1.  $(x_n)$  is  $G$ -convergent to  $x$ ;
2.  $G(x_n, x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ ;
3.  $G(x_n, x, x) \rightarrow 0$  as  $n \rightarrow \infty$ ;
4.  $G(x_n, x_m, x) \rightarrow 0$  as  $n, m \rightarrow \infty$ .

The following definitions can be found in [20].

**Definition 1.10.** Let  $(X, G)$  be a  $G$ -metric space and let  $x_n$  be a sequence in  $X$ . A point  $x \in X$  is said to be the limit of the sequence  $x_n$  if

$$\lim_{n, m \rightarrow \infty} G(x_n, x_m, x) = 0,$$

and the sequence  $x_n$  is said to be  $G$ -convergent to  $X$ .

**Definition 1.11.** A sequence  $x_n$  is called a  $G$ -Cauchy sequence if for every  $\epsilon > 0$ , there is a positive integer  $\mathbb{N}$  such that  $G(x_n, x_m, x_l) < \epsilon$  for all  $n, m, l > \mathbb{N}$ .

**Definition 1.12.** A metric space  $(X, G)$  is said to be  $G$ -complete (or a complete  $G$ -metric space) if every  $G$ -Cauchy sequence in  $(X, G)$  is  $G$ -convergent in  $X$ .

**Definition 1.13** ([12]). The function  $X : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is called a sub-additive integrable function if for any  $a, b \in \mathbb{R}^+$  we have,

$$\int_0^{a+b} X(t)dt = \int_0^a X(t)dt + \int_0^b X(t)dt.$$

**Proposition 1.14** ([24]). Let  $f$  and  $g$  be weakly compatible self-mappings on a set  $X$ . If  $f$  and  $g$  have a unique point of coincidence  $\xi = f\psi = g\psi$ , then  $\xi$  is the unique common fixed point of  $f$  and  $g$ .

**Definition 1.15** ([11]). A mapping  $\varsigma : [0, \infty) \rightarrow [0, \infty)$  is called an altering distance function if the following condition are satisfied:

- $\varsigma$  is continuous and non-decreasing
- $\varsigma(t) = 0$  if and only if  $t = 0$ .

**Definition 1.16** ([24]). Let  $S$  and  $T$  be self-mappings on a non-empty set  $X$ .

1. A point  $x \in X$  is said to be a fixed point of  $T$  if  $Tx = x$ .
2. A point  $x \in X$  is said to be a coincidence point of  $S$  and  $T$  if  $Sx = Tx$  and we shall called  $w = Sx = Tx$  that a point of coincidence of  $S$  and  $T$ .
3. A point  $x \in X$  is said to be a common fixed point of  $S$  and  $T$  if  $x = Sx = Tx$ .

**Definition 1.17** ([2]). Let  $X$  be a non-empty set and  $T, f : X \rightarrow X$ . The mappings  $T, f$  are said to be weakly compatible if they commute at their coincidence point (i.e.,  $Tfx = fTx$  whenever  $Tx = fx$ ).

**Lemma 1.18** ([13, 14]). Let  $\varphi \in \Phi$  and  $\{r_n\}_{n \in \mathbb{N}}$  is a non-negative sequence with  $\lim_{n \rightarrow \infty} r_n = a$ . Then

$$\lim_{n \rightarrow \infty} \int_0^{r_n} \varphi(t)dt = \int_0^a \varphi(t)dt.$$

**Lemma 1.19** ([13, 14]). Let  $\varphi \in \Phi$  and  $\{r_n\}_{n \in \mathbb{N}}$  is a non-negative sequence. Then

$$\lim_{n \rightarrow \infty} \int_0^{r_n} \varphi(t)dt = 0 \Leftrightarrow \lim_{n \rightarrow \infty} r_n = 0.$$

## 2. Main results

**Theorem 2.1.** Let  $(X, G)$  be a  $G$ -metric space and  $\psi, \phi$  be altering distance functions. Let the mappings  $T, f : X \rightarrow X$  satisfy

$$\psi\left(\int_0^{G(Tx, Ty, Tz)} \varphi(t)dt\right) \leq \psi\left(\int_0^{G(fx, fy, fz)} \varphi(t)dt\right) - \phi\left(\int_0^{G(fx, fy, fz)} \varphi(t)dt\right) \quad (2.1)$$

for all  $x, y, z \in X$ , where,  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is a non-negative mapping which is sub-additive integrable on each compact subset of  $[0, +\infty)$  such that  $\int_0^\epsilon \phi(t)dt > 0$  for each  $\epsilon > 0$ .  $T(X) \subset F(X)$  and  $F(X)$  is a complete  $G$ -metric subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique common fixed point in  $X$ .

*Proof.* Let  $x_0$  be an arbitrary point in  $X$ . Since  $T(X) \subset F(X)$ , choose  $x_1 \in X$  such that  $f(x_1) = T(x_0)$ , choose  $x_2 \in X$  such that  $f(x_2) = T(x_1)$  and  $y_n = f(x_{n+1}) = T(x_n)$  for any  $n \in N$ .

Let

$$G(n) = G(f(x_n), f(x_{n+1}), f(x_{n+1})).$$

From (2.1), we have

$$\begin{aligned} \psi\left(\int_0^{G(f(x_n), f(x_{n+1}), f(x_{n+1}))} \varphi(t) dt\right) &= \psi\left(\int_0^{G(T(x_{n-1}), T(x_n), T(x_n))} \varphi(t) dt\right) \\ &\leq \psi\left(\int_0^{G(f(x_{n-1}), f(x_n), f(x_n))} \varphi(t) dt\right) - \phi\left(\int_0^{G(f(x_{n-1}), f(x_n), f(x_n))} \varphi(t) dt\right), \end{aligned} \tag{2.2}$$

which implies that

$$\psi\left(\int_0^{G(f(x_n), f(x_{n+1}), f(x_{n+1}))} \varphi(t) dt\right) \leq \psi\left(\int_0^{G(f(x_{n-1}), f(x_n), f(x_n))} \varphi(t) dt\right).$$

Hence, we get

$$\int_0^{G(n)} \varphi(t) dt \leq \int_0^{G(n-1)} \varphi(t) dt,$$

and

$$\begin{aligned} \psi\left(\int_0^{G(f(x_{n-1}), f(x_n), f(x_n))} \varphi(t) dt\right) &= \psi\left(\int_0^{G(T(x_{n-2}), T(x_{n-1}), T(x_{n-1}))} \varphi(t) dt\right) \\ &\leq \psi\left(\int_0^{G(f(x_{n-2}), f(x_{n-1}), f(x_{n-1}))} \varphi(t) dt\right) \\ &\quad - \phi\left(\int_0^{G(f(x_{n-2}), f(x_{n-1}), f(x_{n-1}))} \varphi(t) dt\right). \end{aligned}$$

Thus, we have

$$\int_0^{G(n-1)} \varphi(t) dt \leq \int_0^{G(n-2)} \varphi(t) dt.$$

In this way we get

$$0 < \int_0^{G(n)} \varphi(t) dt \leq \int_0^{G(n-1)} \varphi(t) dt \leq \int_0^{G(n-2)} \varphi(t) dt \leq \dots$$

Thus, the sequence  $G(n)$  is non-increasing and bounded from below. Hence it converges to some  $l \geq 0$  such that

$$\lim_{n \rightarrow \infty} \int_0^{G(n)} \varphi(t) dt = l. \tag{2.3}$$

We show that  $l = 0$ , otherwise if  $l > 0$  then by taking  $n \rightarrow \infty$  in (2.2) and by using (2.3), we have

$$\psi(l) \leq \psi(l) - \phi(l),$$

which is contradiction unless  $l = 0$ . Hence

$$\lim_{n \rightarrow \infty} \int_0^{G(n)} \varphi(t) dt = 0. \tag{2.4}$$

For  $m, n \in N, n < m$ , by using rectangular inequality, we have

$$\begin{aligned} \int_0^{G(f(x_n), f(x_m), f(x_m))} \varphi(t) \\ \leq \int_0^{G(f(x_n), f(x_{n+1}), f(x_{n+1})) + G(f(x_{n+1}), f(x_{n+2}), f(x_{n+2})) + \dots + G(f(x_{m-1}), f(x_m), f(x_m))} \varphi(t) dt. \end{aligned}$$

By taking limit  $n \rightarrow \infty$  and using sub-additivity property and (2.4), we have

$$\lim_{n \rightarrow \infty} \int_0^{G(f(x_n), f(x_m), f(x_m))} \varphi(t) dt = 0 \text{ as } m, n \rightarrow \infty.$$

By using rectangular property of  $G$ -metric space, we have

$$\int_0^{G(f(x_n), f(x_m), f(x_l))} \varphi(t) dt \leq \int_0^{G(f(x_n), f(x_m), f(x_m)) + G(f(x_l), f(x_m), f(x_m))} \varphi(t) dt.$$

For  $m, n, l \rightarrow \infty$ , we get

$$\int_0^{G(f(x_n), f(x_m), f(x_l))} \varphi(t) dt \rightarrow 0.$$

By using Lemma 1.19, we get

$$G(f(x_n), f(x_m), f(x_l)) \rightarrow 0.$$

This shows that  $f(x_n)$  is a  $G$ -Cauchy sequence in  $f(X)$ . Since  $f(X)$  is  $G$ -Complete, there exist  $u_1, v_1 \in X$  such that  $f(x_n) \rightarrow v_1 = fu_1$ . By Proposition 1.9 we have

$$\lim_{n \rightarrow \infty} G(f(x_n), f(u_1), f(u_1)) = 0,$$

and by using (2.1), we have

$$\begin{aligned} \psi\left(\int_0^{G(f(x_{n+1}), T(u_1), T(u_1))} \varphi(t) dt\right) &= \psi\left(\int_0^{G(T(x_n), T(u_1), T(u_1))} \varphi(t) dt\right) \\ &\leq \psi\left(\int_0^{G(f(x_n), f(u_1), f(u_1))} \varphi(t) dt\right) - \phi\left(\int_0^{G(f(x_n), f(u_1), f(u_1))} \varphi(t) dt\right). \end{aligned}$$

By taking limit  $n \rightarrow \infty$  and by using Lemma 1.19

$$\lim_{n \rightarrow \infty} \psi\left(\int_0^{G(f(x_n), T(u_1), T(u_1))} \varphi(t) dt\right) = 0.$$

We have

$$\psi\left(\int_0^{G(f(u_1), T(u_1), T(u_1))} \varphi(t) dt\right) = 0.$$

Since  $\psi$  is altering distance function, therefore

$$\int_0^{G(f(u_1), T(u_1), T(u_1))} \varphi(t) dt = 0.$$

and

$$f(u_1) = T(u_1) = v_1 \text{ (say)}. \tag{2.5}$$

Hence  $v_1$  is a point of coincidence.

Next we show that the point of coincidence is unique. Suppose there exists another point of coincidence  $t_1 \in X$  such that  $fx = Tx = t_1$  for  $x \in X$ .

$$\begin{aligned} \psi\left(\int_0^{G(v_1, t_1, t_1)} \varphi(t) dt\right) &= \psi\left(\int_0^{G(T(u_1), T(x), T(x))} \varphi(t) dt\right) \\ &\leq \psi\left(\int_0^{G(f(u_1), f(x), f(x))} \varphi(t) dt\right) - \phi\left(\int_0^{G(f(u_1), f(x), f(x))} \varphi(t) dt\right) \\ &= \psi\left(\int_0^{G(v_1, t_1, t_1)} \varphi(t) dt\right) - \phi\left(\int_0^{G(v_1, t_1, t_1)} \varphi(t) dt\right), \end{aligned}$$

which is contradiction unless  $G(v_1, t_1, t_1) = 0$ . Hence  $v_1 = t_1$ . Thus the point of coincidence is unique. If  $T$  and  $f$  are weakly compatible, then by Proposition 1.14  $T$  and  $f$  have a unique common fixed point in  $X$ .  $\square$

The above theorem yields the following corollaries.

**Corollary 2.2.** *Let  $(X, G)$  be a  $G$ -metric space and  $\phi$  be a altering distance function. Let the mappings  $T, f : X \rightarrow X$  satisfy*

$$\int_0^{G(Tx, Ty, Tz)} \varphi(t) dt \leq \int_0^{G(fx, fy, fz)} \varphi(t) dt - \phi\left(\int_0^{G(fx, fy, fz)} \varphi(t) dt\right), \quad \forall x, y, z \in X,$$

where,  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is a non-negative mapping which is sub-additive integrable on each compact subset of  $[0, +\infty)$  such that  $\int_0^\epsilon \phi(t) dt > 0$  for each  $\epsilon > 0$ .  $T(X) \subset F(X)$  and  $F(X)$  is a complete  $G$ -metric subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique common fixed point in  $X$ .

*Proof.* The proof follows by taking  $\psi(t) = t$  in Theorem 2.1. □

**Corollary 2.3.** *Let  $(X, G)$  be a  $G$ -metric space and  $T : X \rightarrow X$  be a mapping satisfy*

$$G(Tx, Ty, Tz) \leq c(G(x, y, z)), \quad \forall x, y, z \in X,$$

where,  $0 \leq c < 1$ , then  $T$  has a unique fixed point in  $X$ .

*Proof.* The proof follows by taking  $\varphi(t) = 1$ ,  $\psi(t) = t$ ,  $fx = I$  and  $\phi(t) = (1 - c)t$ , where  $0 \leq c < 1$  in Theorem 2.1. □

**Corollary 2.4.** *Let  $(X, G)$  be a  $G$ -metric space and  $T : X \rightarrow X$  be a mapping satisfy*

$$G(Tx, Ty, Ty) \leq c(G(x, y, y)), \quad \forall x, y, z \in X,$$

where,  $0 \leq c < 1$ , then  $T$  has a unique fixed point in  $X$ .

*Proof.* The proof follows by taking  $\varphi(t) = 1$ ,  $\psi(t) = t$ ,  $fx = I$ ,  $y = z$  and  $\phi(t) = (1 - c)t$ , where  $0 \leq c < 1$  in Theorem 2.1. □

**Corollary 2.5.** *Let  $(X, G)$  be a  $G$ -metric space and  $\psi, \phi$  be altering distance functions. Let the mappings  $T, f : X \rightarrow X$  satisfy*

$$\psi(G(Tx, Ty, Tz)) \leq \psi(G(fx, fy, fz)) - \phi(G(fx, fy, fz)), \quad \forall x, y, z \in X,$$

where,  $T(X) \subset F(X)$  and  $F(X)$  is a complete  $G$ -metric subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique common fixed point in  $X$ .

*Proof.* The proof follows by taking  $\varphi(t) = 1$  in Theorem 2.1. □

**Corollary 2.6.** *Let  $(X, G)$  be a  $G$ -metric space and  $\psi, \phi$  be altering distance functions. Let the mapping  $T : X \rightarrow X$  satisfy*

$$\psi(G(Tx, Ty, Tz)) \leq \psi(G(x, y, z)) - \phi(G(x, y, z)), \quad \forall x, y, z \in X.$$

Then  $T$  has a unique common fixed point in  $X$ .

*Proof.* The proof follows by taking  $\varphi(t) = 1$  and  $fx = I$  in Theorem 2.1. □

**Corollary 2.7.** *Let  $(X, G)$  be a  $G$ -metric space and  $\psi, \phi$  be altering distance functions. Let the mapping  $T : X \rightarrow X$  satisfy*

$$G(Tx, Ty, Tz) \leq G(x, y, z) - \phi(G(x, y, z)), \quad \forall x, y, z \in X.$$

Then  $T$  has a unique common fixed point in  $X$ .

*Proof.* The proof follows by taking  $\varphi(t) = 1$ ,  $\psi(t) = I$  and  $fx = I$  in Theorem 2.1. □

**Corollary 2.8.** *Let  $(X, G)$  be a  $G$ -metric space. Let the mappings  $T, f : X \rightarrow X$  satisfy*

$$\int_0^{G(Tx, Ty, Tz)} \varphi(t) dt \leq k \int_0^{G(fx, fy, fz)} \varphi(t) dt, \quad \forall x, y, z \in X,$$

where  $0 \leq k < 1$  and  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is a non-negative mapping which is sub-additive integrable on each compact subset of  $[0, +\infty)$  such that  $\int_0^\epsilon \phi(t) dt > 0$  for each  $\epsilon > 0$ .  $T(X) \subset F(X)$  and  $F(X)$  is a complete  $G$ -metric subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique common fixed point in  $X$ .

*Proof.* The proof follows by taking  $\psi(t) = t$  and  $\phi(t) = (1 - k)t$ , where  $0 \leq k < 1$  in Theorem 2.1. □

**Corollary 2.9.** *Let  $(X, G)$  be a  $G$ -metric space and let the mappings  $T, f : X \rightarrow X$  satisfy*

$$G(Tx, Ty, Tz) \leq G(fx, fy, fz), \quad \forall x, y, z \in X,$$

where,  $T(X) \subset F(X)$  and  $F(X)$  is a complete  $G$ -metric subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique common fixed point in  $X$ .

*Proof.* The proof follows by taking  $\varphi(t) = 1$ ,  $\psi(t) = t$  and  $\phi(t) = 0$  in Theorem 2.1. □

**Theorem 2.10.** *Let  $(X, G)$  be a complete  $G$ - metric space and  $\psi, \phi$  be a altering distance function. Let the mappings  $f, g, h : X \rightarrow X$  satisfy*

$$\psi\left(\int_0^{G(fx, gy, hz)} \varphi(t) dt\right) \leq \psi\left(\int_0^{G(x, y, z)} \varphi(t) dt\right) - \phi\left(\int_0^{G(x, y, z)} \varphi(t) dt\right), \quad \forall x, y, z \in X, \tag{2.6}$$

where,  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is a non-negative mapping which is sub-additive integrable on each compact subset of  $[0, +\infty)$  such that  $\int_0^\epsilon \phi(t) dt > 0$  for each  $\epsilon > 0$ . Then  $f, g$  and  $h$  have a unique common fixed point.

*Proof.* Suppose that  $fv_1 = v_1$ . We prove that  $v_1 = gv_1 = hv_1$ . If not then

$$\psi\left(\int_0^{G(fv_1, gv_1, hv_1)} \varphi(t) dt\right) \leq \psi\left(\int_0^{G(v_1, v_1, v_1)} \varphi(t) dt\right) - \phi\left(\int_0^{G(v_1, v_1, v_1)} \varphi(t) dt\right),$$

which is contradiction. By using the similar arguments to those given above, we obtain a contradiction for  $v_1 \neq gv_1$ , and  $v_1 = hv_1$  or for  $v_1 = gv_1$  and  $v_1 \neq hv_1$ . Hence in all the cases, we conclude that  $fv_1 = gv_1 = hv_1 = v_1$ . Let  $x_0$  be arbitrary in  $X$ , we define a sequence  $x_n$  by the rule,

$$x_{3n+1} = fx_{3n}, \quad x_{3n+2} = gx_{3n+1} \text{ and } x_{3n+3} = hx_{3n+2}, \quad \forall n \in \mathbb{N}.$$

Let

$$G(n) = G(f(x_{3n}), g(x_{3n+1}), h(x_{3n+2})).$$

By following the same lines in the proof of Theorem 2.1 we conclude that  $x_n$  is a  $G$ -Cauchy sequence in  $X$ . Since  $X$  is  $G$ -Complete, there exists  $v_1 \in X$  such that  $x_n \rightarrow v_1$ . We claim that  $fv_1 = v_1$ , if not then,

$$\psi\left(\int_0^{G(fv_1, g(x_{3n+1}), h(x_{3n+2}))} \varphi(t) dt\right) \leq \psi\left(\int_0^{G(v_1, x_{3n+1}, x_{3n+2})} \varphi(t) dt\right) - \phi\left(\int_0^{G(v_1, x_{3n+1}, x_{3n+2})} \varphi(t) dt\right),$$



and

$$\psi\left(\int_0^{G(fv_1, x_{3n+2}, x_{3n+3})} \varphi(t)dt\right) \leq \psi\left(\int_0^{G(v_1, x_{3n+1}, x_{3n+2})} \varphi(t)dt\right) - \phi\left(\int_0^{G(v_1, x_{3n+1}, x_{3n+2})} \varphi(t)dt\right).$$

By taking limit  $n \rightarrow \infty$  and using Lemma 1.19

$$\psi\left(\int_0^{G(f(v_1), v_1, v_1)} \varphi(t)dt\right) = 0.$$

Since,  $\psi$  is altering distance function therefore

$$\int_0^{G(f(v_1), v_1, v_1)} \varphi(t)dt = 0.$$

Hence

$$f(v_1) = v_1. \tag{2.7}$$

Similarly one can show that  $g(v_1) = v_1, h(v_1) = v_1$ .

Finally we show that this fixed point is unique. Assume that there exists another fixed point  $v_2 \in X$  of  $f, g$  and  $h$ . Then we have

$$\begin{aligned} \psi\left(\int_0^{G(v_1, v_2, v_2)} \varphi(t)dt\right) &= \psi\left(\int_0^{G(fv_1, gv_2, hv_2)} \varphi(t)dt\right) \\ &\leq \psi\left(\int_0^{G(v_1, v_2, v_2)} \varphi(t)dt\right) - \phi\left(\int_0^{G(v_1, v_2, v_2)} \varphi(t)dt\right). \end{aligned}$$

The above will not hold unless  $G(v_1, v_2, v_2) = 0$ . Thus  $f, g, h$  have a unique common fixed point in  $X$ .  $\square$

**Example 2.11.** Let  $X = \mathbb{R}^+$  and define  $G : X \times X \times X \rightarrow \mathbb{R}^+$  by

$$G(x, y, z) = |x - y| + |y - z| + |z - x|.$$

Then  $(X, G)$  is complete G-metric space. Let  $f, g : X \rightarrow X, \varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+, \psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  and  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  define by,

$$\varphi(t) = 2t, \quad \psi(t) = \frac{t}{12}, \quad \phi(t) = \frac{t}{24}.$$

$$T(x) = \frac{x}{12}, \quad g(x) = \frac{x}{4}.$$

$$\begin{aligned} \psi\left(\int_0^{G(Tx, Ty, Tz)} \varphi(t)dt\right) &= \psi\left(\left(|Tx - Ty| + |Ty - Tz| + |Tz - Tx|\right)^2\right) \\ &= \psi\left(\frac{\left(|x - y| + |y - z| + |z - x|\right)^2}{144}\right) = \frac{G(x, y, z)^2}{1728}. \end{aligned}$$

Now

$$\begin{aligned} \psi\left(\int_0^{G(fx, fy, fz)} \varphi(t)dt\right) - \phi\left(\int_0^{G(fx, fy, fz)} \varphi(t)dt\right) &= \psi(G(fx, fy, fz)) - \phi(G(fx, fy, fz)) \\ &= \frac{G(x, y, z)^2}{384}. \end{aligned}$$

So

$$\psi\left(\int_0^{G(Tx, Ty, Tz)} \varphi(t)dt\right) \leq \psi\left(\int_0^{G(fx, fy, fz)} \varphi(t)dt\right) - \phi\left(\int_0^{G(fx, fy, fz)} \varphi(t)dt\right).$$

By Theorem 2.1  $f, g$  have a unique common solution.

Remark 2.12.

- Corollaries 2.4 and 2.3 are the result of Mustafa [17].
- Corollary 2.8 is the result of Ayadi [4].
- Corollary 2.7 is the result of Aage and Salunke [1].

### 3. Applications to systems of non-linear integral and differential equations

In this section, we give an existence theorem for the solution of fractional differential equations and non-linear integral equations.

First we consider the following system of integral equations.

$$\begin{cases} x(t) = g(t) + \int_a^b K_1(t, s, V(s))ds, t \in [a, b], \\ y(t) = g(t) + \int_a^b K_2(t, s, U(s))ds, t \in [a, b]. \end{cases} \tag{3.1}$$

In the following theorem, we develop sufficient conditions for the existence of unique solution for the above system of integral equations.

**Theorem 3.1.** *Assume the following hypotheses hold*

(A<sub>1</sub>)  $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , and  $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are continuous;

(A<sub>2</sub>)  $K_1(t, s, U(s)) \leq K_2(t, s, V(s)), t \in [a, b]$ .

Then the system (3.1) of integral equations has a unique solution in  $C[a, b]$ .

*Proof.* Define  $f, g : C([a, b]) \rightarrow C([a, b])$  by

$$\begin{aligned} fx(t) &= g(t) + \int_a^b K_1(t, s, V(s))ds, t \in [a, b], \\ gy(t) &= g(t) + \int_a^b K_2(t, s, U(s))ds, t \in [a, b]. \end{aligned}$$

Consider

$$G(x, y, z) = \max\{x, y\} + \max\{y, z\} + \max\{z, x\}.$$

Now, we have

$$\begin{aligned} G(fx(t), fy(t), fz(t)) &= \max_{t \in [a, b]} \{fx(t), fy(t)\} + \max_{t \in [a, b]} \{fy(t), fz(t)\} + \max_{t \in [a, b]} \{fz(t), fx(t)\} \\ &= \max_{t \in [a, b]} \left\{ \int_a^b K_1(t, s, x(s))ds + g(t), \int_a^b K_1(t, s, y(s))ds + g(t) \right\} \\ &\quad + \max_{t \in [a, b]} \left\{ \int_a^b K_1(t, s, y(s))ds + g(t), \int_a^b K_1(t, s, z(s))ds + g(t) \right\} \\ &\quad + \max_{t \in [a, b]} \left\{ \int_a^b K_1(t, s, z(s))ds + g(t), \int_a^b K_1(t, s, x(s))ds + g(t) \right\} \\ &\leq \max_{t \in [a, b]} \left\{ \int_a^b K_2(t, s, x(s))ds + g(t), \int_a^b K_2(t, s, y(s))ds + g(t) \right\} \end{aligned}$$

$$\begin{aligned}
 & + \max_{t \in [a,b]} \left\{ \int_a^b K_2(t, s, y(s)) ds + g(t), \int_a^b K_2(t, s, z(s)) ds + g(t) \right\} \\
 & + \max_{t \in [a,b]} \left\{ \int_a^b K_2(t, s, z(s)) ds + g(t), \int_a^b K_2(t, s, x(s)) ds + g(t) \right\} \\
 & = \max\{gx(t), gy(t)\} + \max\{gy(t), gz(t)\} + \max\{gy(t), gz(t)\} \\
 & = G(gx(t), gy(t), gz(t)).
 \end{aligned}$$

Thus, we have

$$G(fx(t), fy(t), fz(t)) \leq G(gx(t), gy(t), gz(t)).$$

Thus by Corollary 2.9, the system of equations has a unique solution in  $C[a, b]$ . □

Now, we solve the following system (3.2) of fractional differential equations with the help of Theorem 2.1.

$$\begin{cases}
 {}^c D^\alpha u(t) + \hat{f}(t, v(t)) = 0, \quad {}^c D^\alpha v(t) + \hat{g}(t, u(t)) = 0, \quad 1 < \alpha \leq 2, \quad t \in [0, 1], \\
 u(0) = v(0) = a, \quad u(1) = v(1) = b, \quad \text{where } a, b \text{ are constant,}
 \end{cases} \tag{3.2}$$

where  $\hat{f}, \hat{g} : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ . By using the result [22] we have,

$$I^\alpha [{}^c D^\alpha u(t)] = u(t) + C_0 + C_1 t + C_2 t^2 + \dots + C_{n-1} t^{n-1},$$

where  $n = [\alpha] + 1$  and  $C_i \in \mathbb{R}^+$  and  $I^\alpha$  is the integral operator of fractional order. Then the solution of (3.2) is given by the system of integral equations

$$\begin{cases}
 u(t) = a + t(b - a) + \int_0^1 \mathcal{G}(t, s) \hat{f}(s, v(s)) ds, \\
 v(t) = a + t(b - a) + \int_0^1 \mathcal{G}(t, s) \hat{g}(s, u(s)) ds,
 \end{cases} \tag{3.3}$$

where  $\mathcal{G}(t, s)$  is called Green’s function defined by

$$\mathcal{G}(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t(1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.4}$$

The above system (3.3) is equivalent to the system (3.1) with

$$g(t) = a + t(b - a), V(t) = \int_0^1 \mathcal{G}(t, s) \hat{f}(s, v(s)) ds, U(t) = \int_0^1 \mathcal{G}(t, s) \hat{g}(s, u(s)) ds.$$

Thus by Theorem 2.1, the considered system (3.2) has a unique solution.

Further, we study the unique solution to the following general non-linear system of Fredholm integral equations of second kind given by

$$\begin{cases}
 x(t) = \phi(t) + \int_a^b K_1(t, s, x(s)) ds, \quad t \in [a, b], \\
 y(t) = \phi(t) + \int_a^b K_2(t, s, y(s)) ds, \quad t \in [a, b], \\
 z(t) = \phi(t) + \int_a^b K_3(t, s, z(s)) ds, \quad t \in [a, b].
 \end{cases} \tag{3.5}$$

Let  $X = C[a, b]$  be the set of all continuous functions defined on  $[a, b]$ . Define  $G : X \times X \times X \rightarrow \mathbb{R}^+$  by

$$G(x, y, z) = \|x - y\| + \|y - z\| + \|z - x\|,$$

where  $\|x\| = \sup\{|x(t)| : t \in [a, b]\}$ . Then  $(X, G)$  is a complete  $G$ -metric space on  $X$ . For the derivation of aforesaid condition, we give the following theorem.

**Theorem 3.2.** *Assume that the following assumptions hold*

(A<sub>1</sub>)  $K_i : [a, b] \times [a, b] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , for  $i = 1, 2, 3$  and  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are continuous;

(A<sub>2</sub>) there exists a continuous function  $G : [a, b] \times [a, b] \rightarrow [0, \infty)$  such that,

$$|K_i(t, s, u) - K_j(t, s, v)| \leq G(t, s)|u - v|$$

for each  $t, s \in [a, b]$ ,

(A<sub>3</sub>)  $\sup_{t,s \in [a,b]} \int_0^1 |G(t, s)| \leq r$  for  $r < 1$ .

Then The system of integral equations (3.5) has a unique solution in  $C([a, b])$ .

*Proof.* Define  $f, g, h : C([a, b]) \rightarrow C([a, b])$  by

$$fx(t) = \phi(t) + \int_a^b K_1(t, s, x(s))ds, t \in [a, b].$$

$$gy(t) = \phi(t) + \int_a^b K_2(t, s, y(s))ds, t \in [a, b].$$

$$hz(t) = \phi(t) + \int_a^b K_3(t, s, z(s))ds, t \in [a, b].$$

Now we have

$$\begin{aligned} G(fx(t), gy(t), hz(t)) &= \sup_{t \in [a,b]} |fx(t) - gy(t)| + \sup_{t \in [a,b]} |gy(t) - hz(t)| + \sup_{t \in [a,b]} |hz(t) - fx(t)| \\ &\leq \sup_{t \in [a,b]} \int_a^b |k_1(t, s, x(s)) - k_2(t, s, y(s))|ds + \sup_{t \in [a,b]} \int_a^b |k_2(t, s, y(s)) - k_3(t, s, z(s))|ds \\ &\quad + \sup_{t \in [a,b]} \int_a^b |k_3(t, s, z(s)) - k_1(t, s, x(s))|ds \\ &\leq \sup_{t \in [a,b]} \int_a^b G(t, s)|x(s) - y(s)|ds + \sup_{t \in [a,b]} \int_a^b G(t, s)|y(s) - z(s)|ds \\ &\quad + \sup_{t \in [a,b]} \int_a^b G(t, s)|z(s) - x(s)|ds \\ &\leq \sup_{t \in [a,b]} |x(t) - y(t)| \sup_{t \in [a,b]} \int_a^b G(t, s)ds + \sup_{t \in [a,b]} |y(t) - z(t)| \sup_{t \in [a,b]} \int_a^b G(t, s)ds \\ &\quad + \sup_{t \in [a,b]} |z(t) - x(t)| \sup_{t \in [a,b]} \int_a^b G(t, s)ds \\ &\leq \sup_{t \in [a,b]} |x(t) - y(t)| + \sup_{t \in [a,b]} |y(t) - z(t)| \\ &\quad + \sup_{t \in [a,b]} |z(t) - x(t)| \\ &= \|x(t) - y(t)\| + \|y(t) - z(t)\| + \|z(t) - x(t)\| = G(x(t), y(t), z(t)), \end{aligned}$$

which implies that

$$G(fx(t), gy(t), hz(t)) \leq G(x(t), y(t), z(t)).$$

Define  $\psi(t) = \frac{t}{2}$ ,  $\phi(t) = t$  and  $\varphi(t) = 1$ , then by Theorem 3.1, the system (3.5) has a unique common solution in  $X$ . □

With the help of Theorem 3.1, one can also solve the following coupled system of non-linear fractional ordered differential equations given by

$$\begin{cases} {}^cD^\alpha u(t) + \hat{f}(v(t)) = 0, & 1 < \alpha \leq 2, t \in [0, 1], \\ {}^cD^\alpha v(t) + \hat{g}(w(t)) = 0, & 1 < \alpha \leq 2, t \in [0, 1] \\ {}^cD^\alpha w(t) + \hat{h}(u(t)) = 0, & 1 < \alpha \leq 2, t \in [0, 1], \\ u(0) = v(0) = w(0) = a, & u(1) = v(1) = w(1) = b, \\ \text{where } a, b \text{ are constant,} \end{cases} \tag{3.6}$$

where  $\hat{f}, \hat{g}, \hat{h} : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ . Then the equivalent system of integral equations corresponding to (3.6) is given by

$$\begin{cases} u(t) = \phi(t) + \int_0^1 \mathcal{G}(t, s) \hat{f}(v(s)) ds, & t \in [0, 1], \\ v(t) = \phi(t) + \int_0^1 \mathcal{G}(t, s) \hat{g}(w(s)) ds, & t \in [0, 1], \\ w(t) = \phi(t) + \int_0^1 \mathcal{G}(t, s) \hat{h}(u(s)) ds, & t \in [0, 1], \end{cases} \tag{3.7}$$

where  $\mathcal{G}(t, s)$  is the Green’s function

$$\mathcal{G}(t, s) = \begin{cases} \frac{(t-s)^{\alpha-1} - t(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{-t(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \end{cases} \tag{3.8}$$

and continuous on  $[0, 1] \times [0, 1]$ . Moreover,  $\sup_{t \in [0,1]} \int_0^1 |\mathcal{G}(t, s)| ds \leq 1$ . Further, using  $K(t, s, x(s)) = \mathcal{G}(t, s) \hat{f}(v(s))$  etc. Then the coupled system (3.7) become

$$\begin{cases} x(t) = \phi(t) + \int_0^1 K_1(t, s, x(s)) ds, & t \in [0, 1], \\ y(t) = \phi(t) + \int_0^1 K_2(t, s, x(s)) ds, & t \in [0, 1], \\ z(t) = \phi(t) + \int_0^1 K_2(t, s, x(s)) ds, & t \in [0, 1]. \end{cases} \tag{3.9}$$

Evidently by Theorem 3.2 the system (3.9) has a unique solution, which is the corresponding to the unique solution of the system of non-linear fractional differential equation(3.6).

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