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Hybrid method for the equilibrium problem and a family of generalized nonexpansive mappings in Banach spaces

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Abstract

We introduce a hybrid method for finding a common element of the set of solutions of an equilibrium problem defined on the dual space of a Banach space and the set of common fixed points of a family of generalized nonexpansive mappings and prove strong convergence theorems by using the new hybrid method. Using our main results, we obtain some new strong convergence theorems for finding a solution of an equilibrium problem and a fixed point of a family of generalized nonexpansive mappings in a Banach space.

Keywords: Hybrid method, generalized nonexpansive mapping, NST-condition, equilibrium problem, fixed point problem, Banach space.
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1. Introduction

Let E be a real Banach space, E^* the dual space of E and C a closed subset of E such that JC is a closed and convex subset of E^*, where J is the duality mapping on E. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find

$$\hat{x} \in C \text{ such that } f(J\hat{x}, Jy) \geq 0, \ \forall y \in C.$$

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The set of such solutions $\hat{x}$ is denoted by $EP(f)$. A mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$. A mapping T of C into itself is called *quasi-nonexpansive* if $F(T)$ is nonempty and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. It is easy to see that if T is nonexpansive with $F(T) \neq \emptyset$, then it is quasi-nonexpansive.

Numerous problems in physics, optimization and economics reduce to find a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem in a Hilbert space, see for instance, Blum and Oettli [1], and Combettes and Hirstoaga [2]. On the other hand, Ibaraki and Takahashi [3] introduced a new resolvent of a maximal monotone operator in a Banach space and the concept of a generalized nonexpansive mapping in a Banach space. Ibaraki and Takahashi [3], and Kohsaka and Takahashi [5] also studied some properties for generalized nonexpansive retractions in Banach spaces. Recently, Takahashi and Zembayashi [12] considered the following equilibrium problem with a bifunction defined on the dual space of a Banach space. Moreover, they proved a strong convergence theorem for finding a solution of the equilibrium problem which generalized the result of Combettes and Hirstoaga [2].

Construction of fixed point iteration of nonlinear mappings is an important subject in the theory of nonlinear mappings and has been widely studied by many mathematicians. In 1953, Mann [6] introduced an algorithm which is used to approximate a fixed point of a nonlinear mapping $T : C \to C$. Mann's iterative process is defined as follows: $x_0 \in C$

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$. However, Mann's algorithm have only weak convergence. For example, Reich [9] proved that if $T : C \to C$ is a nonexpansive mapping with a fixed point in a closed and convex subset of a uniformly convex Banach space with a Frechét differentiable norm and $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the Mann's iteration converges weakly to a fixed point of T. Later, Nakajo and Takahashi [8] attempted to modify the Mann's iteration in order to guarantee strong convergence by using the hybrid method in mathematical programming, called normal hybrid method. For a nonexpansive mapping T in a Hilbert space, it is as follows:

$$\begin{cases} x_1 = x \in C, C_0 = C, \\ u_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|z - u_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases} \tag{1.1}$$

for all $n \in \mathbb{N}$ where $\alpha_n \subset [0, a]$ for $a \in [0, 1)$, then sequence $\{x_n\}$ generated by (1.1), converges strongly to $P_{F(T)}x$ which is the metric projection from C onto $F(T)$. Construction the sets C_n and Q_n is difficult to obtain because it has complicated condition. For this reason, Takahashi et al. [11] introduced another hybrid method and proposed the following modification iteration method different from Nakajo and Takahashi 's hybrid method [8]. We call such a method the *shrinking projection method*:

$$\begin{cases} x_1 = x \in C, C_0 = C, \\ u_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|z - u_n\| \leq \|z - x_n\|\}, \\ x_{n+1} = P_{C_n} x \end{cases} \tag{1.2}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1]$. They proved strong convergence of the sequence $\{x_n\}$ generated by (1.2) under an appropriate control condition on the sequence $\{\alpha_n\}$.

In this paper, motivated by Takahashi et al. [11], we introduce a new hybrid method by using the shrinking projection method and Takahashi and Zambayashi [12] for finding a common element of the set of solutions of equilibrium problem and the set of common fixed points of a countable family of generalized nonexpansive mappings in a Banach space and prove strong convergence theorems in a Banach space. Using this results, we obtain some new strong convergence results for finding a solution of an equilibrium problem and a fixed point of a generalized nonexpansive mapping or a family of generalized nonexpansive mappings in a Banach space.

2. Preliminaries

Throughout this paper, we assume that all linear spaces are real. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of all positive integers and real numbers, respectively. Let E be a Banach space and let E^* be the dual space of E. For a sequence $\{x_n\}$ of E and a point $x \in E$, the *weak* convergence of $\{x_n\}$ to x and the *strong* convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \ \forall x \in E.$$

Let $S(E)$ be the unit sphere centered at the origin of E. Then the space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$. It is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in S(E)$. A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be *uniformly convex* if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| < 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \epsilon$. We know the following (see [10]):

(i) if E is smooth, then J is single-valued;

(ii) if E is reflexive, then J is onto;

(iii) if E is strictly convex, then J is one-to-one;

(iv) if E is strictly convex, then J is strictly monotone;

(v) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed convex subset of E. Throughout this paper, define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2, \ \forall y, x \in E. \tag{2.1}$$

Observe that, in a Hilbert space H, (2.1) reduces to $\phi(x, y) = \|x - y\|^2$, for all $x, y \in H$. It is obvious from the definition of the function ϕ that for all $x, y \in E$,

(P1) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$,

(P2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$,

(P3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$.

Let C be a closed convex subset of a Banach space E, and let T be a mapping from C into itself. Recall that a self-mapping $T : C \to C$ is *generalized nonexpansive* if $F(T) \neq \emptyset$ and $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in C$ and $u \in F(T)$. Let R be a mapping from E onto C. Then R is said to be a *retraction* if $R^2 = R$. The mapping R from E onto C is said to be *sunny* if $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t \geq 0$.

A nonempty closed subset C of a smooth Banach space E is said to be a *sunny generalized nonexpansive retract* of E if there exists a sunny generalized nonexpansive retraction R from E onto C. We know the following lemmas for sunny generalized nonexpansive retractions.

Lemma 2.1 ([3])**.** *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E and let R be a retraction from E onto C. Then the following assertions are equivalent:*

(i) *R is sunny generalized nonexpansive;*

(ii) $\langle x - Rx, Jy - JRx\rangle \leq 0, \quad \forall x \in E,\, y \in C.$

Lemma 2.2 ([3])**.** *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.3 ([3])**.** *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C, let $x \in E$ and $z \in C$. Then the following assertions hold:*

(i) *$z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in C$;*

(ii) $\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z)$.

Lemma 2.4 ([5])**.** *Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E. Then the following items are equivalent:*

(i) *C is a sunny generalized nonexpansive retract of E;*

(ii) *JC is closed and convex.*

Lemma 2.5 ([5])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E . Let R be the sunny generalized nonexpansive retraction from E onto C, let $x \in E$ and $z \in C$. Then the following assertions are equivalent:*

(i) *$z = Rx$;*

(ii) $\phi(x, z) = \min_{y\in C} \phi(x, y)$.

Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex. To solve the equilibrium problem, let us assume that a bifunction $f : JC \times JC \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x^*, x^*) = 0$ for all $x^* \in JC$;

(A2) f is monotone, i.e., $f(x^*, y^*) + f(y^*, x^*) \leq 0$ for all $x^*, y^* \in JC$;

(A3) for all $x^*, y^*, z^* \in JC$, $\limsup_{t\downarrow 0} f(tz^* + (1 - t)x^*, y^*) \leq f(x^*, y^*)$;

(A4) for all $x^* \in JC$, $f(x^*, \cdot)$ is convex and lower semicontinuous.

Lemma 2.6 ([1])**.** *Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex, and let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying* (A1)-(A4)*. Then, for $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$f(Jz, Jy) + \frac{1}{r}\langle z - x, Jy - Jz\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.7 ([12])**.** *Let C be a nonempty closed subset of a uniformly smooth, strictly convex and reflexive Banach space E such that JC is closed and convex, let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying* (A1)-(A4)*. For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \{z \in C : f(Jz, Jy) + \frac{1}{r}\langle z - x, Jy - Jz\rangle \geq 0, \ \forall y \in C\}.$$

Then the following statements hold:

(i) *T_r is single-valued;*

(ii) *for all $x, y \in E$, $\langle T_r x - T_r y, JT_r x - JT_r y\rangle \le \langle x - y, JT_r x - JT_r y\rangle$;*

(iii) *$F(T_r) = EP(f)$;*

(iv) *$JEP(f)$ is closed and convex.*

Lemma 2.8 ([12]). *Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex, let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and let $r > 0$. Then, for $x \in E$ and $p \in F(T_r)$,*

$$\phi(x, T_r x) + \phi(T_r x, p) \le \phi(x, p).$$

The following lemmas are also needed for the proof of our main results.

Lemma 2.9 ([4]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.10 ([5]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E and let T be a generalized nonexpansive mapping from C into itself. Then $F(T)$ is closed and $JF(T)$ is closed and convex.*

Lemma 2.11 ([5]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E and let T be a generalized nonexpansive mapping from C into itself. Then $F(T)$ is a sunny generalized nonexpansive retract of E.*

Lemma 2.12 ([4]). *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \le \phi(x, y)$$

for all $x, y \in B_r(0)$, where $B_r(0) = \{z \in E : \|z\| \le r\}$.

Lemma 2.13 ([13]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|tx + (1 - t)y\|^2 \le t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)g(\|x - y\|)$$

for all $x, y \in B_r(0)$ and $t \in [0, 1]$, where $B_r(0) = \{z \in E : \|z\| \le r\}$.

Lemma 2.14 ([4]). *Let E be a smooth and strictly convex Banach space, $z \in E$, and $\{t_i\}_{i=1}^m \subset (0, 1)$ with $\sum_{i=1}^m t_i = 1$. If $\{x_i\}_{i=1}^m$ is a finite sequence in E such that*

$$\phi\Big(\sum_{i=1}^m t_i x_i, z\Big) = \sum_{i=1}^m t_i \phi(x_i, z),$$

then $x_1 = x_2 = ... = x_m$.

Next, we recall some lemmas for NST-condition.

Let E be a real Banach space and C be a closed convex subset of E. Motivated by Nakajo et al. [7], we give the following definition: Let $\{T_n\}$ and $\mathcal{T}$ be two families of generalized nonexpansive mappings of C into E such that $\bigcap_{n=1}^\infty F(T_n) = F(\mathcal{T}) \ne \emptyset$, where $F(T_n)$ is the set of all fixed points of T_n and $F(\mathcal{T})$ is the set of all common fixed points of $\mathcal{T}$. Then, $\{T_n\}$ is said to satisfy the *NST-condition* with $\mathcal{T}$ if for each bounded sequence $\{x_n\} \subset C$,

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0 \Rightarrow \lim_{n\to\infty} \|x_n - T x_n\| = 0, \ \text{for all } T \in \mathcal{T}.$$

In particular, if $\mathcal{T} = \{T\}$, i.e., $\mathcal{T}$ consists of one mapping T, then $\{T_n\}$ is said to satisfy the NST-condition with T. It is obvious that $\{T_n\}$ with $T_n = T$ for all $n \in \mathbb{N}$ satisfies NST-condition with $\mathcal{T} = \{T\}$.

Lemma 2.15. *Let C be a closed subset of a uniformly smooth and uniformly convex Banach space E and let T be a generalized nonexpansive mapping from C into E with $F(T) \neq \emptyset$. Let $\{\beta_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. For $n \in \mathbb{N}$, define the mapping T_n from C into E by*

$$T_n x = \beta_n x + (1-\beta_n)Tx$$

for all $x \in C$. Then, $\{T_n\}$ is a countable family of generalized nonexpansive mappings satisfying the NST-condition with T.

Lemma 2.16. *Let C be a closed subset of a uniformly smooth and uniformly convex Banach space E and let S and T be generalized nonexpansive mappings from C into E with $F(S) \cap F(T) \neq \emptyset$. Let $\{\beta_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. For $n \in \mathbb{N}$, define the mapping T_n from C into E by*

$$T_n x = \beta_n Sx + (1-\beta_n)Tx$$

for all $x \in C$. Then, $\{T_n\}$ is a countable family of generalized nonexpansive mappings satisfying the NST-condition with $\mathcal{T} = \{S,T\}$.

3. Strong convergence theorems

In this section, we introduce and prove a strong convergence theorem of a new hybrid method for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a family of generalized nonexpansive mappings in a Banach space. Recall that an operator T in a Banach space is call *closed*, if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Before proving our main result, we give the following lemma for non-self generalized nonexpansive mappings in a Banach space.

Lemma 3.1. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that JC is closed and convex. Let T be a generalized nonexpansive mapping from C into E such that $F(T) \neq \emptyset$, then $F(T)$ is closed and $JF(T)$ is closed and convex.*

Proof. We first prove that $F(T)$ is closed. Let $\{x_n\} \subset F(T)$ with $x_n \to x$. Since T is generalized nonexpansive, then we have

$$\phi(Tx, x_n) \le \phi(x, x_n)$$

for each $n \in \mathbb{N}$. This implies

$$\phi(Tx, x) = \lim_{n\to\infty} \phi(Tx, x_n) \le \lim_{n\to\infty} \phi(x, x_n) = \phi(x,x) = 0.$$

Therefore, we have $\phi(Tx, x) = 0$ and hence $x \in F(T)$.

We next show that $JF(T)$ is closed. Let $\{x_n^*\} \subset JF(T)$ such that $x_n^* \to x^*$ for some $x^* \in E^*$. Note that since JC is closed and convex, we have $x^* \in JC$. Then, there exist $x \in C$ and $\{x_n\} \subset F(T)$ such that $x^* = Jx$ and $x_n^* = Jx_n$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned}\phi(Tx, x_n) &\le \phi(x, x_n)\\ &= \|x\|^2 - 2\langle x, x_n^*\rangle + \|x_n^*\|^2\\ &\to \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 = 0.\end{aligned}$$

Hence, $\lim_{n\to\infty} \phi(Tx, x_n) = 0$. Since

$$\begin{aligned}0 = \lim_{n\to\infty} \phi(Tx, x_n) &= \lim_{n\to\infty} (\|Tx\|^2 - 2\langle Tx, x_n^*\rangle + \|x_n^*\|^2)\\ &= \|Tx\|^2 - 2\langle Tx, x^*\rangle + \|x^*\|^2 = \phi(Tx, x),\end{aligned}$$

we have $\phi(Tx, x) = 0$ and hence $x = Tx$. This implies $x^* = Jx \in JF(T)$.

We finally show that $JF(T)$ is convex. Let $x^*, y^* \in JF(T)$ and let $\alpha \in (0, 1)$ and $\beta = 1 - \alpha$. Then we have $x, y \in F(T)$ such that $x^* = Jx$ and $y^* = Jy$. Thus, we have

$$\begin{aligned}
&\phi(TJ^{-1}(\alpha Jx + \beta Jy), J^{-1}(\alpha Jx + \beta Jy))\\
&\qquad =\|TJ^{-1}(\alpha Jx + \beta Jy)\|^2 - 2\langle TJ^{-1}(\alpha Jx + \beta Jy), \alpha Jx + \beta Jy\rangle + \|J^{-1}(\alpha Jx + \beta Jy)\|^2\\
&\qquad\quad + \alpha\|x\|^2 + \beta\|y\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2)\\
&\qquad =\alpha\phi(TJ^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(TJ^{-1}(\alpha Jx + \beta Jy), y) + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2).
\end{aligned}$$

Since $x, y \in F(T)$ and T is generalized nonexpansive, we have

$$\begin{aligned}
&\alpha\phi(TJ^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(TJ^{-1}(\alpha Jx + \beta Jy), y) + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2)\\
&\qquad \le\alpha\phi(J^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(J^{-1}(\alpha Jx + \beta Jy), y) + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2)\\
&\qquad =\alpha\{\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), Jx\rangle + \|x\|^2\}\\
&\qquad\quad + \beta\{\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), Jy\rangle + \|y\|^2\}\\
&\qquad\quad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2)\\
&\qquad =2\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), \alpha Jx + \beta Jy\rangle\\
&\qquad =2\|\alpha Jx + \beta Jy\|^2 - 2\|\alpha Jx + \beta Jy\|^2 = 0.
\end{aligned}$$

Then we have $TJ^{-1}(\alpha Jx + \beta Jy) = J^{-1}(\alpha Jx + \beta Jy)$ and hence $J^{-1}(\alpha Jx + \beta Jy) \in JF(T)$. This implies that $\alpha Jx + \beta Jy \in JF(T)$. Therefore, $JF(T)$ is convex and the proof is complete. □

Using Lemmas 2.4 and 3.1, we obtain the following lemma.

Lemma 3.2. *Let E be a smooth, strictly convex and reflexive Banach space and C be a closed subset of E such that JC is closed and convex. Let T be a generalized nonexpansive mapping from C into E such that $F(T) \neq \emptyset$, then $F(T)$ is a sunny generalized nonexpansive retract of E.*

Theorem 3.3. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and $\{T_n\}$ be a countable family of generalized nonexpansive mappings from C into E, and let $\mathcal{T}$ be a family of closed generalized nonexpansive mappings from C into E such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$ and $F(\mathcal{T}) \cap EP(f) \neq \emptyset$. Suppose that $\{T_n\}$ satisfies the NST-condition with $\mathcal{T}$. Let $\{x_n\}$ be the sequence generated by $x_0 = x \in C, C_0 = C$ and*

$$\begin{cases}
y_n = \alpha_n x_n + (1 - \alpha_n)T_n x_n,\\
u_n \in C, \text{ such that } f(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - y_n, Jy - Ju_n\rangle \ge 0, \ \ \forall y \in C,\\
C_{n+1} = \{z \in C_n : \phi(u_n, z) \le \phi(x_n, z)\},\\
x_{n+1} = R_{C_{n+1}}x
\end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $R_{F(\mathcal{T})\cap EP(f)}x$, where $R_{F(\mathcal{T})\cap EP(f)}$ is the sunny generalized nonexpansive retraction from E onto $F(\mathcal{T}) \cap EP(f)$.

Proof. Since the proof of Theorem 3.3 is very long, so we divide it into 5 steps.

Step1 : We begin by proving that $\{x_n\}$ is well-defined. Putting $u_n = T_{r_n} y_n$ for all $n \in \mathbb{N} \cup \{0\}$, we have from Lemma 2.8 that T_{r_n} is generalized nonexpansive. We first show that $F(\mathcal{T}) \cap EP(f)$ is a sunny generalized nonexpansive retract of E and JC_n is closed and convex. From Lemmas 2.7 and 3.1, we have $JEP(f)$ and $JF(T)$ are closed and convex, respectively. Since E is uniformly convex, J is injective and hence

$$J(F(T) \cap JEP(f)) = JF(T) \cap JEP(f),$$

which is also closed and convex. Using Lemma 2.4, we have $F(T)\cap EP(f)$ is a sunny generalized nonexpansive retract of E. It is obvious that JC_0 is closed and convex. Since $\phi(u_n, z) \le \phi(x_n, z)$ is equivalent to

$$0 \le \|x_n\|^2 - \|u_n\|^2 - 2\langle x_n - u_n, Jz\rangle,$$

which is affine in Jz, hence JC_n is closed and convex. Next, we show by the induction that $F(\mathcal{T})\cap EP(f) \subset C_n$ for all $n \in \mathbb{N}\cup\{0\}$. From $C_0 = C$, we have $F(\mathcal{T}) \cap EP(f) \subset C_0$. Suppose that $F(\mathcal{T}) \cap EP(f) \subset C_k$ for some $k \in \mathbb{N}\cup\{0\}$. Since T_{r_k} and T_n are generalized nonexpansive, we have

$$\begin{aligned}\phi(u_k, u) &= \phi(T_{r_k} y_k, u) \le \phi(y_k, u)\\ &= \phi(\alpha_k x_k + (1-\alpha_k)T_k x_k, u)\\ &= \|\alpha_k x_k + (1-\alpha_k)T_k x_k\|^2 - 2\langle \alpha_k x_k + (1-\alpha_k)T_k x_k, Ju\rangle + \|u\|^2\\ &\le \alpha_k\|x_k\|^2 + (1-\alpha_k)\|T_k x_k\|^2 - 2\alpha_n\langle x_k, Ju\rangle - 2(1-\alpha_k)\langle T_k x_k, Ju\rangle + \|u\|^2\\ &= \alpha_k\phi(x_k, u) + (1-\alpha_k)\phi(T_k x_k, u)\\ &\le \alpha_k\phi(x_k, u) + (1-\alpha_k)\phi(x_k, u) = \phi(x_k, u).\end{aligned} \tag{3.1}$$

Hence, we have $u \in C_{k+1}$. This implies that $F(\mathcal{T})\cap EP(f) \subset C_n$ for all $n \in \mathbb{N}\cup\{0\}$. So, $\{x_n\}$ is well-defined.

Step2 : We will show that $\lim_{n\to\infty}\|x_n - u_n\| = 0$. It follows from Lemma 2.3 (ii) and $x_n = R_{C_n}x$ that

$$\phi(x, x_n) = \phi(x, R_{C_n}x) \le \phi(x, u) - \phi(R_{C_n}x, u) \le \phi(x, u)$$

for all $u \in F(\mathcal{T}) \cap EP(f) \subset C_n$. Then, $\{\phi(x, x_n)\}$ is bounded. Moreover, by definition of ϕ, we have that $\{x_n\}$ is bounded. From $C_{n+1} \subset C_n$ and $x_n = R_{C_n}x$, we have

$$\phi(x, x_n) \le \phi(x, x_{n+1}),\ n \ge 0.$$

So, the limit of $\{\phi(x, x_n)\}$ exists. From $x_n = R_{C_n}x$, and for any positive integer k, we have

$$\phi(x_n, x_{n+k}) = \phi(R_{C_n}x, x_{n+k}) \le \phi(x, x_{n+k}) - \phi(x, R_{C_n}x) = \phi(x, x_{n+k}) - \phi(x, x_n).$$

This implies that $\lim_{n\to\infty}\phi(x_n, x_{n+k}) = 0$. Using Lemma 2.12, we have that, for $m, n \in \mathbb{N}$ with $m > n$,

$$g(\|x_n - x_m\|) \le \phi(x_n, x_m) \le \phi(x, x_m) - \phi(x, x_n),$$

where $g : [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing and convex function with $g(0) = 0$. Then the property of the function g yields that $\{x_n\}$ is a Cauchy sequence in C, so there exists $w \in C$ such that $x_n \to w$. In view of $x_{n+1} = R_{C_{n+1}}x \in C_{n+1}$ and definition of C_{n+1}, we also have

$$\phi(u_n, x_{n+1}) \le \phi(x_n, x_{n+1}).$$

It follows that $\lim_{n\to\infty}\phi(u_n, x_{n+1}) = \lim_{n\to\infty}\phi(x_n, x_{n+1}) = 0$. Since E is uniformly convex and smooth, we have from Lemma 2.9 that

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = \lim_{n\to\infty}\|u_n - x_{n+1}\| = 0.$$

So, we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.2}$$

Step3 : We will prove that $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. Put $r = \max\{\sup_n\|x_n\|, \sup_n\|Tx_n\|\}$. Since E is a uniformly convex Banach space, there exists a continuous, strictly increasing and convex function g with $g(0) = 0$ such that

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|)$$

for all $x, y \in B_r(0)$ and $t \in [0,1]$. So, we have that for $u \in \Omega$,

$$\begin{aligned}\phi(u_n, u) &= \phi(T_{r_n} y_n, u) \le \phi(y_n, u)\\ &= \phi(\alpha_n x_n + (1-\alpha_n) T_n x_n, u)\\ &= \|\alpha_n x_n + (1-\alpha_n) T_n x_n\|^2 - 2\langle \alpha_n x_n + (1-\alpha_n) T_n x_n, Ju\rangle + \|u\|^2\\ &\le \alpha_n \|x_n\|^2 + (1-\alpha_n)\|T_n x_n\|^2 - 2\alpha_n \langle x_n, Ju\rangle - 2(1-\alpha_n)\langle T_n x_n, Ju\rangle + \|u\|^2\\ &\quad - \alpha_n(1-\alpha_n) g(\|x_n - T_n x_n\|)\\ &= \alpha_n \phi(x_n, u) + (1-\alpha_n)\phi(T_n x_n, u) - \alpha_n(1-\alpha_n) g(\|x_n - T_n x_n\|)\\ &\le \alpha_n \phi(x_n, u) + (1-\alpha_k)\phi(x_n, u) - \alpha_n(1-\alpha_n) g(\|x_n - T_n x_n\|)\\ &= \phi(x_n, u) - \alpha_n(1-\alpha_n) g(\|x_n - T_n x_n\|).\end{aligned}$$

Therefore, we have

$$\alpha_n(1-\alpha_n) g(\|x_n - T_n x_n\|) \le \phi(x_n, u) - \phi(u_n, u), \ \ \forall n \in \mathbb{N} \cup \{0\}.$$

Since

$$\begin{aligned}\phi(x_n, u) - \phi(u_n, u) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle x_n - u_n, Ju\rangle\\ &\le |\|x_n\|^2 - \|u_n\|^2| + 2|\langle x_n - u_n, Ju\rangle|\\ &\le |\|x_n\| - \|u_n\||(\|x_n\| + \|u_n\|) + 2\|x_n - u_n\|\|Ju\|\\ &\le \|x_n - u_n\||(\|x_n\| + \|u_n\|) + 2\|x_n - u_n\|\|Ju\|,\end{aligned}$$

it follows that

$$\lim_{n\to\infty} \big(\phi(x_n, u) - \phi(u_n, u)\big) = 0.$$

From $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have $\lim_{n\to\infty} g(\|x_n - T_n x_n\|) = 0$. By properties of the function g, we have $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$.

Step4 : We will show that $w \in F(T) \cap EP(f)$. Since $\{T_n\}$ satisfies the NST-condition with $\mathcal{T}$, we have that

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0, \text{ for all } T \in \mathcal{T}.$$

Since $x_n \to w$ and T is closed, it follows that w is a fixed point of T, that is, $w \in F(\mathcal{T})$ and by (3.2), we have that $u_n \to w$. On the other hand, from $u_n = T_{r_n} y_n$, Lemma 2.8 and (3.1) we have that

$$\begin{aligned}\phi(y_n, u_n) &= \phi(y_n, T_{r_n} y_n)\\ &\le \phi(y_n, u) - \phi(T_{r_n} y_n, u)\\ &\le \phi(x_n, u) - \phi(T_{r_n} y_n, u)\\ &= \phi(x_n, u) - \phi(u_n, u).\end{aligned}$$

Since $\lim_{n\to\infty} \big(\phi(x_n, u) - \phi(u_n, u)\big) = 0$, we have that $\lim_{n\to\infty} \phi(y_n, u_n) = 0$. Since E is uniformly convex and smooth, we have from Lemma 2.9 that $\lim\limits_{n\to\infty} \|y_n - u_n\| = 0$. From $r_n \ge a$, we have

$$\lim_{n\to\infty} \frac{\|y_n - u_n\|}{r_n} = 0. \tag{3.3}$$

By $u_n = T_{r_n} y_n$, we have

$$f(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - y_n, Jy - Ju_n\rangle \ge 0, \ \forall y \in C.$$

By (A2), we have that

$$\frac{1}{r_n}\langle u_n - y_n, Jy - Ju_n\rangle \ge -f(Ju_n, Jy) \ge f(Jy, Ju_n), \ \forall y \in C. \tag{3.4}$$

Since $f(x, \cdot)$ is convex and lower semicontinuous and $u_n \to w$, it follows from (3.3) and (3.4) that

$$f(Jy, Jw) \leq 0, \ \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t^* = tJy + (1-t)Jw$. Since JC is convex, we have $y_t^* \in JC$ and hence $f(y_t^*, Jw) \leq 0$. So, from (A1) we have

$$0 = f(y_t^*, y_t^*) \leq tf(y_t^*, Jy) + (1-t)f(y_t^*, Jw) \leq tf(y_t^*, Jy).$$

Hence

$$f(y_t^*, Jy) \geq 0, \ \forall y \in C.$$

Letting $t \downarrow 0$, from (A3) we have

$$f(Jw, Jy) \geq 0, \ \forall y \in C.$$

Therefore, we have $Jw \in JEP(f)$ that is $w \in EP(f)$.

Step5 : We will show that x_n converges strongly to $R_{F(\mathcal{T})\cap EP(f)}x$ by proving $w = R_{F(\mathcal{T})\cap EP(f)}x$. From Lemma 2.3 (ii), we have

$$\phi(x, R_{F(\mathcal{T})\cap EP(f)}x) + \phi(R_{F(\mathcal{T})\cap EP(f)}x, w) \leq \phi(x, w).$$

Since $x_{n+1} = R_{C_{n+1}}x$ and $w \in F(\mathcal{T}) \cap EP(f) \subset C_n$, we get from Lemma 2.3 (ii) that

$$\phi(x, x_{n+1}) + \phi(x_{n+1}, R_{F(\mathcal{T})\cap EP(f)}x) \leq \phi(x, R_{F(\mathcal{T})\cap EP(f)}x). \tag{3.5}$$

Since $x_n \to w$, it follows by definition of ϕ that $\phi(x, x_{n+1}) \to \phi(x, w)$. This implies by (3.5) that $\phi(x, w) \leq \phi(x, R_{F(\mathcal{T}\cap EP(f))}x)$. But since $\phi(x, w) \geq \phi(x, R_{F(\mathcal{T})\cap EP(f)}x)$, we obtain $\phi(x, w) = \phi(x, R_{F(\mathcal{T})\cap EP(f)}x)$. Therefore, it follows from the uniqueness of $R_{F(\mathcal{T})\cap EP(f)}x$ that $w = R_{F(\mathcal{T})\cap EP(f)}x$. This completes the proof. □

Corollary 3.4. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying* (A1)-(A4)*. Let $\{x_n\}$ be the sequence generated by $x_0 = x \in C, C_0 = C$ and*

$$\begin{cases} u_n \in C, \text{ such that } f(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - x_n, Jy - Ju_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $R_{EP(f)}x$, where $R_{EP(f)}$ is the sunny generalized nonexpansive retraction from E onto $EP(f)$.

Proof. Putting $T_n = I$ for all $n \in \mathbb{N} \cup \{0\}$ in Theorem 3.3, we obtain the desired result. □

Corollary 3.5. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $\{T_n\}$ be a countable family of generalized nonexpansive mappings from C into E and, let $\mathcal{T}$ be a family of closed generalized nonexpansive mappings from C into E such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$. Suppose that $\{T_n\}$ satisfies the NST-condition with $\mathcal{T}$. Let $\{x_n\}$ be the sequence generated by $x_0 = x \in C, C_0 = C$ and*

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $R_{F(\mathcal{T})}x$, where $R_{F(\mathcal{T})}$ is the sunny generalized nonexpansive retraction from E onto $F(\mathcal{T})$.

Proof. Putting $f(Jx, Jy) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$ in Theorem 3.3, we obtain the desired result. □

4. Deduced results

In this section, using Theorem 3.3, we obtain some new convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of one and two of generalized nonexpansive mappings in a Banach space.

Theorem 4.1. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and T be a closed generalized nonexpansive mapping from C into E such that $F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0 = x \in C, C_0 = C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ u_n \in C, \text{ such that } f(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - y_n, Jy - Ju_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $R_{F(T)\cap EP(f)}x$, where $R_{F(T)\cap EP(f)}$ is the sunny generalized nonexpansive retraction from E onto $F(T) \cap EP(f)$.

Proof. Put $T_n = T$ for all $n \in \mathbb{N}$. It is obvious that $\{T_n\}$ satisfies the NST-condition with T, so we obtain the desired result by using Theorem 3.3. □

Theorem 4.2. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and T be a closed generalized nonexpansive mapping from C into E such that $F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0 = x \in C, C_0 = C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n), \\ u_n \in C, \text{ such that } f(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - y_n, Jy - Ju_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x \end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\{r_n\} \subset [a,\infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $R_{F(T)\cap EP(f)}x$, where $R_{F(T)\cap EP(f)}$ is the sunny generalized nonexpansive retraction from E onto $F(T) \cap EP(f)$.

Proof. Define $T_n x = \beta_n x + (1-\beta_n)Tx$ for all $n \in \mathbb{N}$ and $x \in C$. By Lemma 2.15, we know that $\{T_n\}$ satisfies the NST-condition with T, so we obtain the desired result by using Theorem 3.3. □

Theorem 4.3. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and S, T be closed generalized nonexpansive mappings from C into E such that $\Omega = F(S) \cap F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0 = x \in C, C_0 = C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)(\beta_n Sx_n + (1-\beta_n)Tx_n), \\ u_n \in C, \text{ such that } f(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - y_n, Jy - Ju_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x \end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying

$\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ *and* $\{r_n\} \subset [a,\infty)$ *for some* $a > 0$. *Then,* $\{x_n\}$ *converges strongly to* $R_\Omega x$*, where* R_Ω *is the sunny generalized nonexpansive retraction from* E *onto* Ω.

Proof. Define $T_n x = \beta_n Sx + (1-\beta_n)Tx$ for all $n \in \mathbb{N}$ and $x \in C$. By Lemma 2.15, we know that $\{T_n\}$ satisfies the NST-condition with T. So, we obtain the desired result by using Theorem 3.3. □

5. Applications

In this section, we give a strong convergence theorem for a countable family of nonexpansive mappings in a Hilbert space. In a Hilbert space, we know that $\phi(x,y) = \|x-y\|^2$ for all $x, y \in H$, $J = I$, where I is an identity mapping and every nonexpansive mapping is closed generalized nonexpansive. The following two lemmas are directly obtained by Lemmas 2.15 and 2.16, respectively.

Lemma 5.1 ([11, Lemma 2.1]). *Let* C *be a closed and convex subset of a Hilbert space* H *and let* T *be a nonexpansive mapping from* C *into itself with* $F(T) \neq \emptyset$. *Let* $\{\beta_n\} \subset [0,1]$ *satisfies* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. *For* $n \in \mathbb{N}$*, define the mapping* T_n *of* C *into itself by*

$$T_n x = \beta_n x + (1-\beta_n)Tx$$

for all $x \in C$. *Then,* $\{T_n\}$ *is a countable family of nonexpansive mappings satisfying the NST-condition with* T.

Lemma 5.2 ([11, Lemma 2.3]). *Let* C *be a closed and convex subset of a Hilbert space* H *and let* S *and* T *be nonexpansive mappings from* C *into itself with* $F(S) \cap F(T) \neq \emptyset$. *Let* $\{\beta_n\} \subset [0,1]$ *satisfies* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. *For* $n \in \mathbb{N}$*, define the mapping* T_n *of* C *into itself by*

$$T_n x = \beta_n Sx + (1-\beta_n)Tx$$

for all $x \in C$. *Then,* $\{T_n\}$ *is a countable family of nonexpansive mappings satisfying the NST-condition with* $\{S, T\}$.

Theorem 5.3. *Let* H *be a Hilbert space and let* C *be a nonempty closed and convex subset of* H. *Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)-(A4). *Let* $\{T_n\}$ *and* $\mathcal{T}$ *be families of nonexpansive mappings of* C *into itself such that* $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$ *and* $F(\mathcal{T}) \cap EP(f) \neq \emptyset$. *Suppose that* $\{T_n\}$ *satisfies the NST-condition with* $\mathcal{T}$. *Let* $\{x_n\}$ *be the sequence generated by* $x_0 = x \in C, C_0 = C$ *and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ u_n \in C, \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle u_n - y_n, y - u_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \|z - u_n\| \leq \|z - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x \end{cases}$$

for all $n \in \mathbb{N}$*, where* $\{\alpha_n\} \subset [0,1]$ *satisfies* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\{r_n\} \subset [a,\infty)$ *for some* $a > 0$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F(\mathcal{T})\cap EP(f)}x$*, where* $P_{F(\mathcal{T})\cap EP(f)}$ *is the metric projection from* C *onto* $F(\mathcal{T}) \cap EP(f)$.

Proof. In a Hilbert space, we know that $\phi(x,y) = \|x-y\|^2$ for all $x, y \in H$, $J = I$, where I is the identity mapping. By using Theorem 3.3, we obtain the desired conclusion. □

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