Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4990–4999
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

On some recent fixed point results for (ψ, φ)-contractive mappings in ordered partial b-metric spaces

Huaping Huang[a,*], Diana Dolićanin-Đekić[b], Guantie Deng[a]

[a] *School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, China.*
[b] *Faculty of Technical Sciences, University of Prištna-Kosovska Mitrovica, Serbia.*

Communicated by N. Hussain

Abstract

In this paper we unite, complement, improve, and generalize the recent fixed point results in ordered partial b-metric spaces, established by Mustafa et al. [Z. Mustafa, J. R. Roshan, V. Parvaneh, Z. Kadelburg, J. Inequal. Appl., **2013** (2013), 26 pages], with much shorter proofs. An example is given to show the superiority of the results obtained.

Keywords: Fixed point, b-metric space, partial metric space, ordered partial b-metric space, partially ordered set.
2010 MSC: 47H10, 54H25.

---

1. Introduction and preliminaries

Partial metric spaces [3, 12, 16] and b-metric spaces [5, 6] are two well-known generalizations of the usual metric spaces. Also, the Banach contraction principle is a fundamental result in the fixed point theory, which has been used and extended in many different directions (see [5, 15, 20, 25, 27]).

The following two definitions are consistent with [5] and [16].

Definition 1.1 ([16]). A partial metric on a nonempty set X is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$, if

*Corresponding author
Email addresses: mathhhp@163.com (Huaping Huang), diana.dolicanin@pr.ac.rs (Diana Dolićanin-Đekić), denggt@bnu.edu.cn (Guantie Deng)

Received 2015-12-20

(p1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;

(p2) $p(x,x) \leq p(x,y)$;

(p3) $p(x,y) = p(y,x)$;

(p4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

For a partial metric p on X, the function $p^s : X \times X \to [0,\infty)$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a (usual) metric on X. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

For more details on partial metric spaces, see [9, 16–19, 24, 26].

Definition 1.2 ([5]). Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $b : X \times X \to [0,\infty)$ is called a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(b1) $b(x,y) = 0$ if and only if $x = y$;

(b2) $b(x,y) = b(y,x)$;

(b3) $b(x,z) \leq s[b(x,y) + b(y,z)]$.

In this case, the pair (X,b) is called a b-metric space.

For more details on b-metric spaces we refer the reader to [2, 4, 6, 10, 11, 13, 17] and references therein.

As a generalization and unification of partial metric and b-metric spaces, Shukla [28] introduced the concept of partial b-metric space as follows:

Definition 1.3 ([28]). A partial b-metric on a nonempty set X is a mapping $p_b : X \times X \to [0,\infty)$ such that for a constant $s \geq 1$ and all $x, y, z \in X$, if

(pb1) $x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$;

(pb2) $p_b(x,x) \leq p_b(x,y)$;

(pb3) $p_b(x,y) = p_b(y,x)$;

(pb4) $p_b(x,z) \leq s[p_b(x,y) + p_b(y,z)] - p_b(y,y)$.

A partial b-metric space is a pair (X,p_b) such that X is a nonempty set and p_b is a partial b-metric on X. The number $s \geq 1$ is called the coefficient of (X,p_b).

In a partial b-metric space (X,p_b), if $x, y \in X$ and $p_b(x,y) = 0$, then $x = y$, but the converse may not be true. It is clear that every partial metric space is a partial b-metric space with the coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of these facts does not necessarily hold. For some examples, see [9, 18, 28].

In [18] the authors said that (X,p_b) is a partial b-metric space if (pb4) is substituted for the following: for all $x, y, z \in X$,

$$p_b(x,z) \leq s[p_b(x,y) + p_b(y,z) - p_b(y,y)] + \frac{1-s}{2}(p_b(x,x) + p_b(y,y)).$$

Further, for some other notions such as p_b-convergence, p_b-completeness, and p_b-Cauchy sequence in the setting of partial b-metric spaces, the reader can refer to [9, 18, 28].

Definition 1.4 ([18])**.** A triple $(X, \preceq, p_b)$ is called an ordered partial b-metric space if $(X, \preceq)$ is a partially ordered set and p_b is a partial b-metric on X.

Definition 1.5 ([10])**.** An ordered partial b-metric space $(X, \preceq, p_b)$ is called regular if one of the following conditions holds:

(r1) if for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to x$, as $n \to \infty$, one has $x_n \preceq x$ for all $n \in \mathbb{N}$;

(r2) if for any nonincreasing sequence $\{y_n\}$ in X such that $y_n \to y$, as $n \to \infty$, one has $y_n \succeq y$ for all $n \in \mathbb{N}$.

Let $(X, \preceq)$ be a partially ordered set and let f, g be two self-maps on X. We shall use the following terminology (see [10]):

(1) two elements $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds;

(2) a subset K of X is said to be well-ordered if every two elements of K are comparable;

(3) f is called nondecreasing with respect to $\preceq$ if $x \preceq y$ implies $fx \preceq fy$;

(4) the pair (f, g) is said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$;

(5) f is said to be g-weakly isotone increasing if for all $x \in X$ it satisfies $fx \preceq gfx \preceq fgfx$.

Otherwise, fixed point results in ordered partial metric spaces were firstly presented by Ran and Reurings [25], and then by Nieto and López [20], [21]. Subsequently, many authors obtained several interesting results in ordered metric spaces, ordered b-metric spaces and ordered partial metric spaces (see [1, 2, 11, 19, 22]).

Altering distance functions were introduced by Khan et al. in [14] as follows.

Definition 1.6. A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following items are satisfied:

(a) ψ is continuous and nondecreasing;

(b) $\psi(t) = 0$ if and only if $t = 0$.

So far, many authors have studied fixed point theorems which are based on altering distance functions (see, e.g., [6, 7, 11, 23]).

In [18] the authors introduced the following denotations and notions, and proved the corresponding fixed point theorems.

Let $(X, \preceq, p_b, s > 1)$ be an ordered partial b-metric space, and let $f, g : X \to X$ be mappings. Set

$$M_s^{f,g}(x, y) = \max\left\{p_b(x, y), p_b(x, fx), p_b(y, gy), \frac{p_b(x, gy) + p_b(fx, y)}{2s}\right\} \tag{1.1}$$

and

$$M_s^{f}(x, y) = \max\left\{p_b(x, y), p_b(x, fx), p_b(y, fy), \frac{p_b(x, fy) + p_b(fx, y)}{2s}\right\}. \tag{1.2}$$

Definition 1.7. Let $(X, \preceq, p_b, s > 1)$ be an ordered partial b-metric space, and let ψ and φ be altering distance functions. The pair (f, g) of mappings $f, g : X \to X$ is called a generalized $(\psi, \varphi)_{s,2}$-contraction pair if

$$\psi\left(s^2 p_b(fx, gy)\right) \leq \psi\left(M_s^{f,g}(x, y)\right) - \varphi\left(M_s^{f,g}(x, y)\right)$$

for all comparable $x, y \in X$.

Theorem 1.8. *Let $(X, \preceq, p_b, s > 1)$ be a p_b-complete ordered partial b-metric space, and let $f, g : X \to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose that (f, g) is a generalized $(\psi, \varphi)_{s,2}$-contraction pair for some altering distance functions ψ and φ. If f and g are continuous (resp. $(X, \preceq, p_b, s > 1)$ is regular), then f and g have a common fixed point.*

Definition 1.9. Let $(X, \preceq, p_b, s > 1)$ be an ordered partial b-metric space, and let ψ and φ be altering distance functions. A mapping $f : X \to X$ is called a generalized $(\psi, \varphi)_{s,1}$-weakly contractive mapping if

$$\psi\left(sp_b\left(fx, fy\right)\right) \leq \psi\left(M_s^f\left(x, y\right)\right) - \varphi\left(M_s^f\left(x, y\right)\right)$$

for all comparable $x, y \in X$.

Theorem 1.10. *Let $(X, \preceq, p_b, s > 1)$ be a p_b-complete ordered partial b-metric space. Let $f : X \to X$ be a nondecreasing, with respect to $\preceq$, continuous (resp. $(X, \preceq, p_b, s > 1)$ is regular) mapping. Suppose that f is a generalized $(\psi, \varphi)_{s,1}$-weakly contractive mapping. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.*

It shows, specifically, the following crucial lemma is often used in proving of all main results in [18].

Lemma 1.11. *Let $(X, \preceq, p_b, s > 1)$ be a partial b-metric space and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x and y, respectively. Then we have*

$$\begin{aligned} \frac{1}{s^2}p_b\left(x, y\right) - \frac{1}{s}p_b\left(x, x\right) - p_b\left(y, y\right) &\leq \liminf_{n\to\infty} p_b\left(x_n, y_n\right) \\ &\leq \limsup_{n\to\infty} p_b\left(x_n, y_n\right) \\ &\leq sp_b\left(x, x\right) + s^2p_b\left(y, y\right) + s^2p_b\left(x, y\right). \end{aligned}$$

In particular, if $p_b\left(x, y\right) = 0$, then we have $\lim_{n\to\infty} p_b\left(x_n, y_n\right) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s}p_b\left(x, z\right) - p_b\left(x, x\right) \leq \liminf_{n\to\infty} p_b\left(x_n, z\right) \leq \limsup_{n\to\infty} p_b\left(x_n, z\right) \leq sp_b\left(x_n, z\right) + sp_b\left(x, x\right).$$

In particular, if $p_b\left(x, x\right) = 0$, then we have

$$\frac{1}{s}p_b\left(x, z\right) \leq \liminf_{n\to\infty} p_b\left(x_n, z\right) \leq \limsup_{n\to\infty} p_b\left(x_n, z\right) \leq sp_b\left(x, z\right).$$

2. Main results

In what follows, we shall introduce two concepts which greatly generalize Definition 1.7 and Definition 1.9. Further, we shall present very simple proofs of some common fixed point theorems in the new framework for not only without considering the assumptions of Theorem 1.8 and Theorem 1.10, but also without utilizing Lemma 1.11 in the proofs.

Definition 2.1. Let $(X, \preceq, p_b, s > 1)$ be an ordered partial b-metric space, and let ψ and φ be altering distance functions. The pair (f, g) of mappings $f, g : X \to X$ is called a generalized $(\psi, \varphi)_{s,\varepsilon}$-contraction pair if

$$\psi\left(s^{\varepsilon}p_b\left(fx, gy\right)\right) \leq \psi\left(M_s^{f,g}\left(x, y\right)\right) - \varphi\left(M_s^{f,g}\left(x, y\right)\right) \tag{2.1}$$

for all comparable $x, y \in X$, where $\varepsilon > 1$ is a constant and $M_s^{f,g}(x, y)$ is defined by (1.1).

Remark 2.2. Definition 1.7 is the special case of Definition 2.1. Indeed, take $\varepsilon = 2$ in Definition 2.1, then generalized $(\psi, \varphi)_{s,\varepsilon}$-contraction pair is reduced to generalized $(\psi,\ \varphi)_{s,2}$-contraction pair. Accordingly, Definition 2.1 is more useful and meaningful in applications.

Theorem 2.3. *Let $(X, \preceq, p_b, s > 1)$ be a p_b-complete ordered partial b-metric space, and let $f, g : X \to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose that (f, g) is a generalized $(\psi, \varphi)_{s,\varepsilon}$-contraction pair for some altering distance functions ψ and φ and $\varepsilon > 1$. If f and g are continuous (resp. $(X, \preceq, p_b, s > 1)$ is regular), then f and g have a common fixed point.*

Proof. It is clear that (2.1) implies

$$s^{\varepsilon} p_b(fx, gy) \le M_s^{f,g}(x, y) \tag{2.2}$$

for all comparable $x, y \in X$.

Now, it follows immediately from (2.2) that $z \in X$ is a fixed point of f if and only if z is a fixed point of g. Take $x_0 \in X$ and construct a sequence $\{x_n\}$ in X such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all nonnegative integers n. Since f and g are weakly increasing with respect to $\preceq$, we have that

$$fx_0 = x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots . \tag{2.3}$$

If $x_{2n} = x_{2n+1}$ for some n or $x_{2n+1} = x_{2n+2}$ for some n, then obviously f and g have at least one common fixed point. Therefore, we may assume without loss of generality, that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$. Now, we complete the proof via three steps:

Step I. We shall prove that

$$p_b(x_{n+1}, x_{n+2}) \le \lambda p_b(x_n, x_{n+1}) \tag{2.4}$$

for all $n \ge 1$, where $\lambda \in [0, \frac{1}{s})$. Indeed, by (2.3), x_{2n} and x_{2n+1} are comparable, then from (2.2) it establishes that

$$\begin{aligned} s^{\varepsilon} p_b(x_{2n+1}, x_{2n+2}) &= s^{\varepsilon} p_b(fx_{2n}, gx_{2n+1}) \\ &\le M_s^{f,g}(x_{2n}, x_{2n+1}) \\ &= \max\left\{ p_b(x_{2n}, x_{2n+1}), p_b(x_{2n+1}, x_{2n+2}), \frac{p_b(x_{2n}, x_{2n+2}) + p_b(x_{2n+1}, x_{2n+1})}{2s} \right\} \\ &\le \max\left\{ p_b(x_{2n}, x_{2n+1}), p_b(x_{2n+1}, x_{2n+2}), \frac{p_b(x_{2n}, x_{2n+1}) + p_b(x_{2n+1}, x_{2n+2})}{2} \right\} \\ &\le \max\{ p_b(x_{2n}, x_{2n+1}), p_b(x_{2n+1}, x_{2n+2}) \} . \end{aligned} \tag{2.5}$$

If $p_b(x_{2n}, x_{2n+1}) \le p_b(x_{2n+1}, x_{2n+2})$, then (2.5) becomes

$$s^{\varepsilon} p_b(x_{2n+1}, x_{2n+2}) \le p_b(x_{2n+1}, x_{2n+2}),$$

which leads to a contradiction (because $s^{\varepsilon} > 1$). Accordingly, we deduce that

$$s^{\varepsilon} p_b(x_{2n+1}, x_{2n+2}) \le p_b(x_{2n}, x_{2n+1}) . \tag{2.6}$$

Again by (2.3), x_{2n} and x_{2n-1} are comparable, then from (2.2) it establishes that

$$\begin{aligned} s^{\varepsilon} p_b(x_{2n}, x_{2n+1}) &= s^{\varepsilon} p_b(gx_{2n-1}, fx_{2n}) \\ &= s^{\varepsilon} p_b(fx_{2n}, gx_{2n-1}) \\ &\le M_s^{f,g}(x_{2n}, x_{2n-1}) \\ &= \max\left\{ p_b(x_{2n}, x_{2n-1}), p_b(x_{2n}, x_{2n+1}), \frac{p_b(x_{2n}, x_{2n}) + p_b(x_{2n+1}, x_{2n-1})}{2s} \right\} \\ &\le \max\left\{ p_b(x_{2n}, x_{2n-1}), p_b(x_{2n}, x_{2n+1}), \frac{p_b(x_{2n+1}, x_{2n}) + p_b(x_{2n}, x_{2n-1})}{2} \right\} \\ &\le \max\{ p_b(x_{2n}, x_{2n-1}), p_b(x_{2n}, x_{2n+1}) \} . \end{aligned} \tag{2.7}$$

If $p_b(x_{2n}, x_{2n-1}) \leq p_b(x_{2n}, x_{2n+1})$, then (2.7) implies that

$$s^{\varepsilon} p_b(x_{2n}, x_{2n+1}) \leq p_b(x_{2n}, x_{2n+1}).$$

This contradiction is valid (because $s^{\varepsilon} > 1$). Consequently, we demonstrate that

$$s^{\varepsilon} p_b(x_{2n}, x_{2n+1}) \leq p_b(x_{2n-1}, x_{2n}). \tag{2.8}$$

Hence by (2.6) and (2.8), we get (2.4), where $\lambda = \frac{1}{s^{\varepsilon}} \in [0, \frac{1}{s})$.

Step II. We shall prove that $\{x_n\}$ is a p_b-Cauchy sequence. In order to end this, for $m, n \in \mathbb{N}$ and $m < n$, applying the triangle-type inequality (pb4), we arrive at

$$\begin{aligned}
p_b(x_m, x_n) &\leq s\left[p_b(x_m, x_{m+1}) + p_b(x_{m+1}, x_n)\right] \\
&\leq s p_b(x_m, x_{m+1}) + s^2\left[p_b(x_{m+1}, x_{m+2}) + p_b(x_{m+2}, x_n)\right] \\
&\vdots \\
&\leq s p_b(x_m, x_{m+1}) + s^2 p_b(x_{m+1}, x_{m+2}) + \cdots \\
&\quad + s^{n-m-1}\left[p_b(x_{n-2}, x_{n-1}) + p_b(x_{n-1}, x_n)\right] \\
&\leq s p_b(x_m, x_{m+1}) + s^2 p_b(x_{m+1}, x_{m+2}) \\
&\quad + \cdots + s^{n-m-1} p_b(x_{n-2}, x_{n-1}) + s^{n-m} p_b(x_{n-1}, x_n).
\end{aligned}$$

Note that by (2.4) and $s\lambda < 1$, it is easy to see that

$$\begin{aligned}
p_b(x_m, x_n) &\leq \left(s\lambda^m + s^2\lambda^{m+1} + \ldots + s^{n-m}\lambda^{n-1}\right) p_b(x_0, x_1) \\
&= s\lambda^m \left[1 + (s\lambda) + \ldots + (s\lambda)^{n-m-1}\right] p_b(x_0, x_1) \\
&\leq \frac{s\lambda^m}{1 - s\lambda} p_b(x_0, x_1) \to 0, \ (m \to \infty).
\end{aligned}$$

It follows that $\{x_n\}$ is a p_b-Cauchy sequence. Since (X, p_b) is p_b-complete, then from [17, Lemma 1], it implies that $\{x_n\}$ converges to some $z \in X$. Again by [17, Lemma 1] it may be verified that

$$\lim_{n\to\infty} p_b(x_n, z) = \lim_{n,m\to\infty} p_b(x_n, x_m) = 0 = p_b(z, z).$$

Step III. Now, we prove the existence of a common fixed point for f, g.

(i) Let f and g be continuous. Then, by using (pb4), we acquire that

$$\frac{1}{s} p_b(z, fz) \leq p_b(z, fx_{2n}) + p_b(fx_{2n}, fz) \tag{2.9}$$

and

$$\frac{1}{s} p_b(z, gz) \leq p_b(z, gx_{2n+1}) + p_b(gx_{2n+1}, gz). \tag{2.10}$$

Letting $n \to \infty$ in (2.9) and (2.10), and using the continuity of f and g, we claim that

$$\frac{1}{s} p_b(z, fz) \leq p_b(fz, fz), \qquad \frac{1}{s} p_b(z, gz) \leq p_b(gz, gz). \tag{2.11}$$

Now, we derive from (2.11) and (pb2) that

$$\frac{1}{s} \max\{p_b(z, fz), p_b(z, gz)\} \leq p_b(fz, gz). \tag{2.12}$$

In view of $p_b(z,z) \leq p_b(z,fz)$ or $p_b(z,z) \leq p_b(z,gz)$ and by (2.2), it establishes that

$$\begin{aligned} s^{\varepsilon} p_b(fz,gz) &\leq \max\left\{p_b(z,z), p_b(z,fz), p_b(z,gz), \frac{p_b(z,gz)+p_b(fz,z)}{2s}\right\} \\ &\leq \max\left\{p_b(z,fz), p_b(z,gz), \max\left\{\frac{p_b(z,fz)}{s}, \frac{p_b(z,gz)}{s}\right\}\right\} \\ &= \max\left\{p_b(z,fz), p_b(z,gz)\right\}. \end{aligned} \tag{2.13}$$

Further, combining (2.12) and (2.13), we speculate that

$$\frac{1}{s}\max\left\{p_b(z,fz), p_b(z,gz)\right\} \leq \frac{1}{s^{\varepsilon}}\max\left\{p_b(z,fz), p_b(z,gz)\right\}. \tag{2.14}$$

If $p_b(z,fz) > 0$ or $p_b(z,gz) > 0$, then from (2.14) it leads to a contradiction. Hence, we have proved that f and g have at least one common fixed point.

(ii) Let $(X, \preceq, p_b)$ be a regular ordered partial b-metric space. Using the given assumption on $(X, \preceq, p_b)$, we have that $x_n \preceq z$ for all $n \in \mathbb{N}$. Finally, we show that $fz = gz = z$. Actually, by (2.2), it ensures us that

$$s^{\varepsilon} p_b(x_{2n+1}, gz) \leq \max\left\{p_b(x_{2n}, z), p_b(x_{2n}, x_{2n+1}), p_b(z,gz), \frac{p_b(x_{2n}, gz)+p_b(x_{2n+1}, z)}{2s}\right\}. \tag{2.15}$$

Noting that $p_b(x_{2n+1}, gz) \to p_b(z,gz)$, $p_b(x_{2n}, z) \to p_b(z,z)$, $p_b(x_{2n}, x_{2n+1}) \to p_b(z,z)$, $p_b(x_{2n}, gz) \to p_b(z,gz)$ and $p_b(x_{2n+1}, z) \to p_b(z,z)$ as $n \to \infty$, and taking the limit from both sides of (2.15), we obtain that

$$\begin{aligned} s^{\varepsilon} p_b(z,gz) &\leq \max\left\{p_b(z,z), p_b(z,gz), \frac{p_b(z,gz)+p_b(z,z)}{2s}\right\} \\ &\leq \max\left\{p_b(z,gz), \frac{p_b(z,gz)}{s}\right\} \\ &= p_b(z,gz), \end{aligned}$$

which is a contradiction if $p_b(z,gz) > 0$. That is to say, $z = gz$. Similarly, we can show $z = fz$. Therefore, z is a common fixed point of f and g. □

Remark 2.4. Since any generalized $(\psi,\varphi)_{s,2}$-contraction pair must be a generalized $(\psi,\varphi)_{s,\varepsilon}$-contraction pair, thus Theorem 2.3 greatly improves and expands Theorems 3, 4 as well as Corollaries 3 and 4 of [18].

Remark 2.5. The proof of Theorem 2.3 does not rely on Lemma 1.11 as compared to the proofs of the main results of [17]. Moreover, our proof is much shorter than [17]. As a result, our statement is more acceptable and applicable in applications.

The following example illustrates our conclusions to be genuine generalizations.

Example 2.6. Let $X = \{0,1,2\}$ be equipped with the following partial order:

$$\preceq := \{(0,0), (1,1), (2,2), (0,1)\}.$$

Define a partial b-metric $p_b : X \times X \to [0,\infty)$ by

$$p_b(x,y) = \begin{cases} 0, & \text{if } x = y, \\ (x+y)^2, & \text{if } x \neq y. \end{cases}$$

It is easy to see that (X, p_b) is a p_b-complete partial b-metric space with $s = \frac{9}{5}$. Define self-maps $f = g$ with $f0 = g0 = 0$ and $f1 = f2 = g1 = g2 = 2$. Simple calculations show that f and g are weakly increasing mappings with respect to $\preceq$ and that f and g are continuous. In order to check that $(f,g) = (f,f)$ is a generalized $(\psi,\varphi)_{s,\varepsilon}$-contraction pair, we observe that only the case $x=0, y=1$ is nontrivial. For this case we arrive at

$$s^{\varepsilon}p_b\left(f0,g1\right)=\left(\frac{9}{5}\right)^{\varepsilon}p_b\left(0,2\right)=\left(\frac{9}{5}\right)^{\varepsilon}\cdot 4$$

and

$$\begin{aligned} M_s^{f,g}\left(0,1\right)=M_s^{f,f}\left(0,1\right) &= \max\left\{p_b\left(0,1\right),p_b\left(0,f0\right),p_b\left(1,f1\right),\frac{p_b\left(0,f1\right)+p_b\left(1,f0\right)}{2\cdot\frac{9}{5}}\right\}\\ &= \max\left\{1,0,9,\frac{25}{18}\right\}=9. \end{aligned}$$

Now that $\left(\frac{9}{5}\right)^2\cdot 4>9$, then the pair $(f,g)=(f,f)$ is not a generalized $(\psi,\varphi)_{\frac{9}{5},2}$-contraction for any altering functions ψ and φ. However, there exists $\varepsilon\in(1,2)$ such that $\left(\frac{9}{5}\right)^{\varepsilon}\cdot 4\le 9$, that is, the pair $(f,g)=(f,f)$ satisfies the condition (2.2). Furthermore, there exist altering functions ψ and φ such that $(f,g)=(f,f)$ is a $(\psi,\varphi)_{\frac{9}{5},\varepsilon}$-contractive. Indeed, putting $\psi(t)=t$, we can find $\varphi(t)=kt, k\in(0,1)$ such that

$$\left(\frac{9}{5}\right)^{\varepsilon}\cdot 4\le 9-\varphi\left(9\right).$$

Hence, Theorem 2.3 is a real generalization compared with Theorems 3 and 4, and Corollaries 3 and 4 of [18].

Finally, we state the generalizations of Definition 1.9 and Theorem 1.10 as follows.

Definition 2.7. Let $(X,\preceq,p_b,s>1)$ be an ordered partial b-metric space, and let ψ and φ be altering distance functions. A mapping $f:X\to X$ is called a generalized $(\psi,\varphi)_{s,\varepsilon}$-weakly contractive mapping if

$$\psi\left(s^{\varepsilon}p_b\left(fx,fy\right)\right)\le\psi\left(M_s^f\left(x,y\right)\right)-\varphi\left(M_s^f\left(x,y\right)\right) \tag{2.16}$$

for all comparable $x,y\in X$, where $\varepsilon>1$ is a constant and $M_s^f(x,y)$ is defined by (1.2).

Theorem 2.8. *Let $(X,\preceq,p_b,s>1)$ be a p_b-complete ordered partial b-metric space. Let $f:X\to X$ be a nondecreasing, with respect to $\preceq$, continuous mapping (resp. $(X,\preceq,p_b,s>1)$ is regular). Suppose that f is a generalized $(\psi,\varphi)_{s,\varepsilon}$-weakly contractive mapping. If there exists $x_0\in X$ such that $x_0\preceq fx_0$, then f has a fixed point.*

Proof. First of all, the condition (2.16) implies

$$s^{\varepsilon}p_b\left(fx,fy\right)\le M_s^f\left(x,y\right)$$

for all comparable $x,y\in X$. The rest of the proof is further similar to the proof of Theorem 2.3 so long as putting $g=f$ and therefore we omit it. □

Remark 2.9. Although Theorem 1 of [18] is more general than Theorem 2.3, the proof of Theorem 2.3 has nothing to do with Lemma 1.11, whereas, the proof of Theorem 1 of [18] is strongly dependent of this lemma. Consequently, our conclusion is more meaningful and valuable.

Remark 2.10. According to [9], some fixed point generalizations to partial metric spaces can be obtained from the corresponding results in metric spaces. However, our results to partial b-metric spaces cannot be reduced to the case of b-metric spaces. This is because the b-metric and partial b-metric do not satisfy the usual continuity. That is, $x_n\to x$ and $y_n\to y$ can not lead to $b(x_n,y_n)\to b(x,y)$. Whereas, [9] got all the assertions based on the continuity of metric and partial metric in the case when $p(x_n,x)$ tends to $p(x,x)=0$.

Remark 2.11. To the author's knowledge, there has no metric version result so far for the counterpart of our main results in the framework of partial b-metric spaces. Therefore, even if by using the method of [8] and [9], our results cannot be derived from the existing results of ordinary metric spaces because the metric cases have not appeared so far. Furthermore, our generalizations are indeed real generalizations because of the arbitrary character of the constant ε from (2.1) and (2.16). In addition, the proofs of our results are much simpler than the previous results in the literature.

Acknowledgment

The research was partially supported by the National Natural Science Foundation of China (11271045).

References

[1] M. Abbas, V. Parvaneh, A. Razani, *Periodic points of T-Ciric generalized contraction mappings in ordered metric spaces*, Georgian Math. J., **19** (2012), 597–610. 1

[2] A. Aghajani, M. Abbas, J. R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces*, Math. Slovaca, **64** (2014), 941–960. 1, 1

[3] J. Ahmad, A. Azam, M. Arshad, *Fixed points of multivalued mappings in partial metric spaces*, Fixed Point Theory Appl., **2013** (2013), 9 pages. 1

[4] A. Azam, N. Mehmood, J. Ahmad, S. Radenović, *Multivalued fixed point theorems in cone b-metric spaces*, J. Inequal. Appl., **2013** (2013), 9 pages. 1

[5] I. A. Bakhtin, *The contraction mapping principle in quasimetric spaces*, Funct. Anal., Gos. Ped. Inst., Unianowsk, **30** (1989), 26–37. 1, 1.2

[6] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis, **1** (1993), 5–11. 1, 1, 1

[7] D. Đorić, *Common fixed point for generalized (ψ, ϕ)-weak contractions*, Appl. Math. Lett., **22** (2009), 1896–1900. 1

[8] X. Ge, S. Lin, *A note on partial b-metric spaces*, Mediterr. J. Math., **13** (2016), 1273–1276. 2.11

[9] R. H. Haghi, Sh. Rezapour, N. Shahzad, *Be careful on partial metric fixed point results*, Topology Appl., **160** (2013), 450–454. 1, 1, 2.10, 2.11

[10] H. Huang, J. Vujaković, S. Radenović, *A note on common fixed point theorems for isotone increasing mappings in ordered b-metric spaces*, J. Nonlinear Sci. Appl., **8** (2015), 808–815. 1, 1.5, 1

[11] N. Hussain, V. Parvaneh, J. R. Roshan, Z. Kadelburg, *Fixed points of cyclic weakly (ψ, φ, L, A, B)-contractive mappings in ordered b-metric spaces with applications*, Fixed Point Theory Appl., **2013** (2013), 18 pages. 1, 1, 1

[12] N. Hussain, J. R. Roshan, V. Parvaneh, A. Latif, *A unification of G-metric, partial metric, and b-metric spaces*, Abstr. Appl. Anal., **2014** (2014), 14 pages. 1

[13] N. Hussain, R. Saadati, R. P. Agarwal, *On the topology and wt-distance on metric type spaces*, Fixed Point Theory Appl., **2014** (2014), 14 pages. 1

[14] M. S. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Austral. Math. Soc., **30** (1984), 1–9. 1

[15] M. A. Kutbi, J. Ahmad, N. Hussain, M. Arshad, *Common fixed point results for mappings with rational expressions*, **2013** (2013), 11 pages. 1

[16] S. G. Matthews, *Partial metric topology*, Papers on general topology and applications, Ann. New York Acad. Sci., **728** (1994), 183–197. 1, 1.1, 1

[17] A. Mukheimer, *α-ψ-ϕ-contractive mappings in ordered partial b-metric spaces*, J. Nonlinear Sci. Appl., **7** (2014), 168–179. 1, 2, 2.5

[18] Z. Mustafa, J. R. Roshan, V. Parvaneh, Z. Kadelburg, *Some common fixed point results in ordered partial b-metric spaces*, J. Inequal. Appl., **2013** (2013), 26 pages. 1, 1.4, 1, 1, 2.4, 2.6, 2.9

[19] H. K. Nashine, Z. Kadelburg, S. Radenović, *Common fixed point theorems for weakly isotone increasing mappings in ordered partial metric spaces*, Math. Comput. Modelling, **57** (2013), 2355–2365. 1, 1

[20] J. J. Nieto, R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22** (2005), 223–239. 1, 1

[21] J. J. Nieto, R. Rodríguez-López, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sin. (Engl. Ser.), **23** (2007), 2205–2212. 1

[22] V. Parvaneh, J. R. Roshan, S. Radenović, *Existence of tripled coincidence points in ordered b-metric spaces and an application to a system of integral equations*, Fixed Point Theory Appl., **2013** (2013), 19 pages. 1

[23] O. Popescu, *Fixed points for (ψ, ϕ)-weak contractions*, Appl. Math. Lett., **24** (2011), 1–4. 1

[24] S. Radenović, *Coincidence point results for nonlinear contraction in ordered partial metric spaces*, J. Indian Math. Soc. (N.S.), **81** (2014), 319–333. 1

[25] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435–1443. 1, 1

[26] J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei, W. Shatanawi, *Common fixed points of almost generalized $(\psi, \varphi)_s$-contractive mappings in ordered b-metric spaces*, Fixed Point Theory Appl., **2013** (2013), 123 pages. 1

[27] P. Salimi, N. Hussain, S. Shukla, Sh. Fathollahi, S. Radenović, *Fixed point results for cyclic $\alpha - \psi\varphi$-contractions with application to integral equations*, J. Comput. Appl. Math., **290** (2015), 445–458. 1

[28] S. Shukla, *Partial b-metric spaces and fixed point theorems*, Mediterr. J. Math., **11** (2014), 703–711. 1, 1.3, 1