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On some classical soft intersection properties

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Abstract

In this paper, we investigate on the finite soft intersection property of a family of soft sets that is indexed by another soft set, so that such family is represented by a soft set-valued map. We show that the finite soft intersection property is characterized by some appropriate conditions on such maps. ©2016 all rights reserved.

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1. Introduction

A soft set, as introduced by Molodtsov [13], is an important and effective device in rigorous studies of uncertainty and ambiguity. The theory of soft set has been improved and developed from the very successful concept of fuzzy sets [19].

Always, a fuzzy set is characterized by a real function whose values fall within the closed interval $[0, 1]$. This function explains the degree of belonging of a certain point to the imposed set. An advantage of a fuzzy set to a soft set is that it is quantitative so that the algebraic calculation is valid. It is, however, narrower applicable compared to a soft set. This can be seen explicitly as Molodtsov has mentioned in his paper [13] that a fuzzy set is indeed a special case of a soft set.

Numerous improvements and investigations have been introduced to confirm the basis of soft set theory, for examples, on basic notions and properties of soft sets [11, 18], on the soft set operations and relations

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[6, 14, 20], on the interconnections to other mathematical approaches to uncertainty [9, 17], on the generalized soft sets [2, 4, 5, 7], and on the algebraic structures with soft sets [1, 3]. One of the most important concepts we would like to emphasize on, and would mention separately is the very successful introduction of the soft topology, which was announced independently in [8] and [15]. They were later improved to the present version in [10, 12]. These soft topological results enable us to adopt soft topological notions in parallel to the study of classical topological spaces.

Let us turn to another important and powerful ingredient in topology – the finite intersection property (F.I.P.). A collection of sets is said to have the F.I.P. whenever each of its finite subcollection has a nonempty intersection. It is known that, together with some additional hypothesis and under the right setting, the F.I.P. could be very useful in showing the existence of an object of interests.

In soft set theory, an empty soft set (or a null soft set) is, imprecisely, a soft set that soft contains only soft empty elements. In the same manner as of ordinary topological spaces, we may develop the finite soft intersection property (F.S.I.P.) for a collection of soft sets, i.e., a collection of soft sets whose the soft intersection of any of its finite subcollection soft contains at least one nonempty soft element, i.e., the soft intersection is nonempty.

Practically, when one is working with a large group of individuals and with a wide standard of interpretations, the soft intersection and finite soft intersection could be very useful to give satisfactory to the group. We shall now illustrate a simple and general situation, as an example.

Example 1.1. Suppose that $N := \{1, 2, \cdots, n\}$ is the set of individuals, and they are sharing their money to buy exactly one product listed in the set C of commodities. Assume that each individual $i \in N$ has criteria, collected in the set K_i, on his preference over commodities. We thus can summarize the preferences relative to the criteria of each individual $i \in N$ with a map $P_i : K_i \to 2^C$. Here, we can consider C as the universe and $K := \bigcup_{i\in N} K_i$ as the parameter set, so that (P_i, K_i) is a soft set. The soft intersection $\tilde{\bigcap}_{i\in N} P_i$ gives some valuable information on the non-conflict choice of commodities.

Motivated by the above importance of soft intersection, we consider F.S.I.P. for a family indexed by another soft set. Here, soft set-valued map characterization is taken into account. We show that such characterized family has F.S.I.P. if and only if it is a KKM map. Moreover, we also provide further extensions where the soft intersection over the whole index is not a null soft set.

2. Preliminaries

In this section, we recollect, as concise as possible, some background materials which will be used and mentioned in the sequel.

2.1. Soft sets and basic soft operations

Let $\mathbb{U}$ be an initial universe and $\mathbb{E}$ be a set of parameters. A pair (F, A) is called a *soft set over* $\mathbb{U}$ if $A \subset \mathbb{E}$ is nonempty and $F : A \to 2^{\mathbb{U}}$ is a set-valued map. Unless otherwise specified, always assume that soft sets appearing in this present paper are defined over $\mathbb{U}$ and $\mathbb{E}$.

Without loss of generality, we can consider a soft set (F, A) as a soft set $(F^*, \mathbb{E})$ such that $F^*(\varepsilon) = F(\varepsilon)$ for all $\varepsilon \in A$ and $F^*(\varepsilon) = \emptyset$ for all $\varepsilon \notin A$. Thus, it should not be confused if we neglect the importance of the parameter set A and write F in place of (F, A).

In particular, the *absolute soft set* (resp., *null soft set*), written $\tilde{\mathbb{U}}$ (resp., $\tilde{\emptyset}$), is the soft set $(U, \mathbb{E})$ such that $U(\varepsilon) = \mathbb{U}$ (resp., $U(\varepsilon) = \emptyset$) for all $\varepsilon \in \mathbb{E}$.

Given two soft sets (F, A) and (G, B), the *soft union* of (F, A) and (G, B), denoted by $F \tilde{\cup} G$, is the soft set (H, C) with $C = A \cup B$ and

$$H(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A \setminus B, \\ G(\varepsilon) & \text{if } \varepsilon \in B \setminus A, \\ F(\varepsilon) \cup G(\varepsilon) & \text{if } \varepsilon \in A \cap B. \end{cases}$$

Similarly, the *soft intersection* of (F, A) and (G, B), written as $F \tilde{\cap} G$, is the soft set (H, C) given by $C = A \cap B$ and $H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ for each $\varepsilon \in C$. Moreover, if $A \subset B$ and $F(\varepsilon) \subset G(\varepsilon)$ for all $\varepsilon \in A$, we say that (F, A) is a *soft subset* of (G, B), which is denoted by $F \tilde{\subset} G$. If $F \tilde{\subset} G$ and $G \tilde{\subset} F$, then we say that they are *soft equal*.

The *soft difference* of two soft sets $(F, \mathbb{E})$ and $(G, \mathbb{E})$, written as $F \tilde{\setminus} G$, is the soft set $(H, \mathbb{E})$ given by $H(\varepsilon) = F(\varepsilon) \setminus G(\varepsilon)$ for each $\varepsilon \in \mathbb{E}$. In particular, for a soft set F, the soft difference $\tilde{\mathbb{U}} \tilde{\setminus} F$, denoted by $F^{\tilde{\complement}}$, is called the *soft relative complement* (or shortly, *soft complement*) of F.

2.2. Soft elements

Very recently, Wardowski [16] introduced a promising concept of soft elements. This concept produces some natural properties which made the understanding of soft sets simpler. A pair (ε, I), where $\varepsilon \in \mathbb{E}$ and I is a set containing at most one element, is said to be a *soft element* of a soft set (F, A), expressed with $(\varepsilon, I) \tilde{\in} (F, A)$, if $\varepsilon \in A$ and $I \subset F(\varepsilon)$. Certainly, we also say that (F, A) *soft contains* the soft element (ε, I). In case $I = \emptyset$, we say that (ε, I) is an *empty soft element*. The following basic properties are essential.

Proposition 2.1 ([16])**.** *Suppose that F, F_1, F_2 are three soft sets over the same universe with suitable parameters. Assume that α is a soft element. Then, the following statements are true:*

(i) *every soft set soft contains an empty soft element;*

(ii) $\alpha \tilde{\in} F$ *if and only if* $\{\alpha\} \tilde{\subset} F$*;*

(iii) $F = \tilde{\bigcup}_{\alpha \tilde{\in} F} \{\alpha\}$*;*

(iv) $F_1 \tilde{\subset} F_2$ *if and only if soft elements of* F_1 *are soft contained in* F_2*;*

(v) $\alpha \tilde{\in} F_1 \tilde{\cup} F_2$ *if and only if either* $\alpha \tilde{\in} F_1$ *or* $\alpha \tilde{\in} F_2$*;*

(vi) $\alpha \tilde{\in} F_1 \tilde{\cap} F_2$ *if and only if both* $\alpha \tilde{\in} F_1$ *and* $\alpha \tilde{\in} F_2$*;*

(vii) $\alpha \tilde{\in} F_1 \tilde{\setminus} F_2$ *if and only if* $\alpha \tilde{\in} F_1$ *but* $\alpha \tilde{\notin} F_2$.

2.3. Soft Cartesian products and soft maps

There are various concepts of soft maps in the literature of soft set theory. However, we will stick to the one in [16]. Recall as well that a soft product between two soft sets (F, A) and (G, B), denoted by $F \tilde{\times} G$, is the soft set (H, C), where $C = A \times B$ and $H(a, b) = F(a) \times G(b)$, for each $(a, b) \in C$.

A soft map T [16] from (F, A) into (G, B), notated by $T : F \tilde{\to} G$, is a soft subset of $F \tilde{\times} G$ such that to each nonempty soft element $\alpha \tilde{\in} F$, there corresponds a unique nonempty soft element $\beta \tilde{\in} G$ such that $\{\alpha\} \tilde{\times} \{\beta\} \tilde{\in} T$. In this case, we shall write $T(\alpha)$ to represent β. To be simple, we always assume that $T(\alpha)$ if and only if α is an empty soft element.

For the aspect of this paper, we consider the soft set-valued maps instead. A soft set-valued map T from (F, A) into (G, B), written by $T : F \tilde{\rightrightarrows} G$, is a soft subset of $F \tilde{\times} G$ such that to each nonempty soft element $\alpha \tilde{\in} F$, there corresponds a unique soft subset $\tilde{\emptyset} \neq \Lambda \tilde{\subset} G$ such that $\{\alpha\} \tilde{\times} \Lambda \tilde{\subset} T$. Again, we write $T(\alpha)$ to represent Λ. The image of a soft set $X \tilde{\subset} F$ under T is defined to be the soft set $T(X) = \tilde{\bigcup}_{x \tilde{\in} X} T(x)$.

2.4. Soft topological spaces

Now, we have arrived at the pioneering concept of a soft topology. Some definitions and related properties are collected and presented in the following.

Definition 2.2 ([8, 15])**.** A *soft topology* on a soft set $\tilde{\mathbb{U}}$ is a collection τ of soft subsets (with parameters in $\mathbb{E}$) over $\mathbb{U}$ satisfying

(i) both $\tilde{\emptyset}$ and $\tilde{\mathbb{U}}$ belong to τ,

(ii) soft union of members of τ belongs to τ,

(iii) soft intersection of finitely many members of τ belongs to τ.

We call $\mathbb{U}$ and $\mathbb{E}$ together with a soft topology τ a *soft topological space*, denoted by $(\tilde{\mathbb{U}}, \tau, \mathbb{E})$. Members of τ are said to be *soft open*, and a soft complement of an open soft set is said to be *soft closed*.

In particular, we say that τ is *soft Hausdorff* if for any two distinct points $x, y \tilde{\in} X$, we can find two disjoint soft open sets U, V such that $U \tilde{\supset} x$ and $V \tilde{\supset} y$. In this case, we call (X, τ, E) a *soft Hausdorff space*.

Let $X \tilde{\subset} \tilde{\mathbb{U}}$, we may adopt a topology $\tau|_X$ on X from $\tilde{\mathbb{U}}$, called the *subspace topology* defined by

$$\tau|_X = \{U \tilde{\cap} X, \, U \in \tau\}.$$

If $U \in \tau|_X$, we say that U is soft open in X.

One may see that the ordinary topological spaces are included in the class of soft topological spaces. Take a nonempty set X, $Y \subset X$, and $\mathbf{1} = \{p\}$ an appropriate singleton parameter set of $\mathbb{E}$. Denoted by $\tilde{Y}_{\mathbf{1}} = (\tilde{Y}, \mathbf{1})$, the soft set $\tilde{Y}(p) = Y$. In particular, $U \subset V$ if and only if $\tilde{U}_{\mathbf{1}} \tilde{\subset} \tilde{V}_{\mathbf{1}}$. By letting τ be a topology on X, we may define a soft topology $\tau_{\mathbf{1}}$ on $\tilde{X}_{\mathbf{1}}$ by

$$\tau_{\mathbf{1}} = \{\tilde{U}_{\mathbf{1}} \tilde{\subset} \tilde{X}_{\mathbf{1}}, \, U \in \tau\}.$$

Clearly, $\tilde{U}_{\mathbf{1}} \in \tau_{\mathbf{1}}$ if and only if $U \in \tau$. Also, $\tilde{D}_{\mathbf{1}} \tilde{\subset} \tilde{X}_{\mathbf{1}}$ is soft compact if and only if D is compact.

A soft map (single-valued) T from a soft topological space X into another soft topological space Y is said to be *soft continuous* if $T^{-1}(V) := \{\alpha \tilde{\in} X, \, T(\alpha) \tilde{\in} V\}$ is soft open (or soft closed, resp.) in X for each soft open (or soft closed, resp.) subset $V \tilde{\subset} Y$. If $K \tilde{\subset} X$ is compact, then its image $T(K)$ is also compact (in Y).

3. Soft KKM maps

We give in this section a version of KKM maps in the setting of soft topological space. Always assume throughout this section that (X, E) is a nonempty soft set and (Y, τ, F) is a soft topological space.

Definition 3.1. A soft set-valued map $T : X \tilde{\rightrightarrows} Y$ is said to be a *soft KKM map* if for any finite soft subset $\{x_0, x_1, \cdots, x_n\} \tilde{\subset} X$, there exists a soft continuous map $\varphi : (\tilde{\Delta}_n)_{\mathbf{1}} \tilde{\to} Y$ such that $\varphi(z)$ is a nonempty element if z is a nonempty element of $(\tilde{\Delta}_n)_{\mathbf{1}}$, and

$$\varphi((\tilde{\Delta}_k)_{\mathbf{1}}) \tilde{\subset} \tilde{\bigcup}_{j=0}^{k} T(x_{i_j})$$

for every $\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\} \subset \{e_0, e_1, \cdots, e_n\}$, where $\Delta_k = \mathrm{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}$. The corresponding map φ here will be referred to as a *controller* of $\{x_0, x_1, \cdots, x_n\}$.

We next associate the F.S.I.P. to the KKM map by showing that if $\{T(x), \, x \tilde{\in} X\}$ has the F.S.I.P., then T is a KKM map. The next theorem actually shows that the F.S.I.P. cannot be extended beyond this soft KKM map.

Theorem 3.2. *Suppose that $T : X \tilde{\rightrightarrows} Y$ is a soft set-valued map such that the family $\{T(x), \, x \tilde{\in} X\}$ has the F.S.I.P. Then, T is a soft KKM map.*

Proof. Let $\{x_0, x_1, \cdots, x_n\} \tilde{\subset} X$ be an arbitrary finite soft subset. Thus, by the hypothesis, the soft intersection $\tilde{\bigcap}_{i=0}^{n} T(x_i)$ contains a nonempty soft element β. Now, we consider the map $\varphi : (\tilde{\Delta}_n)_{\mathbf{1}} \tilde{\to} Y$ such that $\varphi(\alpha)$ is an empty soft element whenever α is an empty element and $\varphi(\alpha) = \beta$ whenever $\alpha \tilde{\in} (\tilde{\Delta}_n)_{\mathbf{1}}$ is a nonempty. The map φ constructed by this way is soft continuous. Moreover, we may obtain the following inclusions:

$$\varphi((\tilde{\Delta}_k)_{\mathbf{1}}) = \{\beta\} \tilde{\subset} \tilde{\bigcap}_{i=0}^{n} T(x_i) \tilde{\subset} \tilde{\bigcup}_{j=0}^{k} T(x_{i_j})$$

for every $\{i_0, i_1, \cdots, i_k\} \subset \{1, 2, \cdots, n\}$ with $\Delta_k = \mathrm{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}$. Hence, T is soft KKM. $\square$

The converse holds as a consequence of the following stronger result, provided that T produces soft open values.

Theorem 3.3. *Suppose that $T : X \tilde{\rightrightarrows} Y$ is a soft KKM map with soft open values. Then, the soft intersection $\varphi((\tilde{\Delta}_n)_{\mathbf{1}}) \tilde{\cap} (\tilde{\bigcap}_{i=1}^n T(x_i))$ is nonempty for each finite soft subset $\{x_0, x_1, \cdots, x_n\} \tilde{\subset} X$, where φ is the corresponding controller.*

Proof. Suppose to the contrary, that there is a finite soft subset $\{x_0, x_1, \cdots, x_n\} \tilde{\subset} X$ such that $\varphi((\tilde{\Delta}_n)_{\mathbf{1}}) \tilde{\cap} (\tilde{\bigcap}_{i=1}^n T(x_i))$ contains no nonempty elements, where φ is the corresponding controller. Thus, we may see that

$$\begin{aligned}\varphi((\tilde{\Delta}_n)_{\mathbf{1}}) &= \varphi((\tilde{\Delta}_n)_{\mathbf{1}}) \tilde{\setminus} \left(\tilde{\bigcap}_{i=1}^n \left(\varphi((\tilde{\Delta}_n)_{\mathbf{1}}) \tilde{\cap} T(x_i) \right) \right) \\ &= \tilde{\bigcup}_{i=0}^n \left[\varphi((\tilde{\Delta}_n)_{\mathbf{1}}) \tilde{\setminus} \left(\varphi((\tilde{\Delta}_n)_{\mathbf{1}}) \tilde{\cap} T(x_i) \right) \right].\end{aligned}$$

For each nonempty element $\tilde{z} \tilde{\in} (\tilde{\Delta}_n)_{\mathbf{1}}$, we define

$$I(\tilde{z}) = \{i \in \{0, 1, \cdots, n\}, \varphi(\tilde{z}) \tilde{\notin} T(x_i)\}.$$

We may see that $I(\tilde{z})$ is nonempty for each nonempty element $\tilde{z} \tilde{\in} (\tilde{\Delta}_n)_{\mathbf{1}}$, otherwise $\varphi(\tilde{z}) \tilde{\in} \tilde{\bigcap}_{i=0}^n T(x_i)$, which contradicts our supposition. Let $U \subset \Delta_n$ be a set such that

$$\tilde{U}_{\mathbf{1}} = (\tilde{\Delta}_n)_{\mathbf{1}} \tilde{\setminus} \varphi^{-1} \left(\tilde{\bigcup}_{i \notin I(\tilde{z})} \left[\varphi((\tilde{\Delta}_n)_{\mathbf{1}}) \tilde{\setminus} \left(\varphi((\tilde{\Delta}_n)_{\mathbf{1}}) \tilde{\cap} T(x_i) \right) \right] \right).$$

We may see that $\tilde{U}_{\mathbf{1}}$ is soft open, so that U is open. Let $\zeta \in U$, notice that we have $\varphi((p, \{\zeta\})) \in \tilde{\bigcap}_{i \notin I(\tilde{z})} T(x_i)$. Thus, $I((p, \{\zeta\})) \subset I(\tilde{z})$ for every $\zeta \in U$. Moreover, for some $\bar{z} \in U$, it must be the case that $\tilde{z} = (p, \{\bar{z}\})$. This implies that U is an open neighborhood of $\bar{z}$. By defining $\Gamma : \Delta_n \rightrightarrows \Delta_n$ with

$$\Gamma(z) = \mathrm{co}\{e_i, \, i \in I(p, \{z\})\} \quad \forall z \in \Delta_n,$$

we can simply declare that Γ is u.s.c., by applying the above property of the set U. Furthermore, Γ has the compact convex values. Thus, by applying the Kakutani's fixed point theorem, we may see that $\zeta_0 \in \Gamma(\zeta_0)$, for some $\zeta_0 \in \Delta_n$. Equivalently, we can say that $(p, \{\zeta_0\}) \tilde{\in} \Gamma(\zeta_0)_{\mathbf{1}}$. Since T is a soft KKM map, we may obtain the following inclusion:

$$\varphi((p, \{\zeta_0\})) \tilde{\in} \varphi(\Gamma(\zeta_0)_{\mathbf{1}}) \tilde{\subset} \tilde{\bigcup}_{i \in I((p, \{\zeta_0\}))} T(x_i),$$

which contradicts the definition of $I(\cdot)$. Therefore, the soft intersection

$$\varphi((\tilde{\Delta}_n)_{\mathbf{1}}) \tilde{\cap} (\tilde{\bigcap}_{i=1}^n T(x_i))$$

is nonempty, as desired. □

Thus, we have the converse of Theorem 3.2 holds true if the soft map T has soft open values, as is summarized in the next corollary.

Corollary 3.4. *Suppose that $T : X \tilde{\rightrightarrows} Y$ is a soft KKM map with soft open values. Then, the soft intersection $\tilde{\bigcap}_{i=1}^n T(x_i)$ is nonempty for each finite soft subset $\{x_0, x_1, \cdots, x_n\} \tilde{\subset} X$.*

Next, we impose a condition on the soft map T guaranteeing that $\{T(x), \, x \tilde{\in} X\}$ has a nonempty soft intersection. To do this, we need an auxiliary result.

Lemma 3.5. *Suppose that (K, τ, E) is a soft topological space. Then, the following statements are equivalent:*

(C1) *K is soft compact.*

(C2) *Every collection of soft closed soft subsets of K with the F.S.I.P. has a nonempty soft intersection.*

Proof. (Sufficiency) Suppose that K is soft compact and $\{F_\alpha\}_{\alpha\in\Lambda}$ be a collection of soft closed soft subsets of K with the F.S.I.P.. Let us assume to the contrary that the soft intersection $\tilde{\bigcap}_{\alpha\in\Lambda}F_\alpha$ is empty. By this, we obtain that

$$K = K\,\tilde{\setminus}\,\tilde{\bigcap}_{\alpha\in\Lambda}F_\alpha = \tilde{\bigcup}_{\alpha\in\Lambda}(K\,\tilde{\setminus}\,F_\alpha).$$

This means $\Gamma := \{K\,\tilde{\setminus}\,F_\alpha\}_{\alpha\in\Lambda}$ is a soft open cover of K. By the soft compactness of K, we have

$$K = \tilde{\bigcup}_{i=0}^{N}(K\,\tilde{\setminus}\,F_{\alpha_i}) = K\,\tilde{\setminus}\,\tilde{\bigcap}_{i=0}^{N}F_{\alpha_i}$$

for some finite subsets $\{\alpha_1,\alpha_2,\cdots,\alpha_N\}\subset\Lambda$. Thus, $\tilde{\bigcap}_{i=0}^{N}F_{\alpha_i}$ must contain only empty soft element, this contradicts the F.S.I.P.. Therefore, $\{F_\alpha\}_{\alpha\in\Lambda}$ has a nonempty soft intersection.

(Necessity) Suppose that K is not soft compact. Then, we can find a soft open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ such that each of its finite collection does not cover K. Therefore, we can see that $K\,\tilde{\setminus}\,\tilde{\bigcup}_{\alpha\in\Lambda}U_\alpha$ is empty, while $K\,\tilde{\setminus}\,\tilde{\bigcup}_{i\in I}U_i$ is nonempty for each nonempty finite subset $I\subset\Lambda$. By defining

$$F_\alpha := K\,\tilde{\setminus}\,U_\alpha, \quad \forall\alpha\in\Lambda,$$

we conclude that $\{F_\alpha\}_{\alpha\in\Lambda}$ is a family of soft closed soft subset of K in which 3.5 fails to hold. □

With Theorem 3.3 (or Corollary 3.4) and Lemma 3.5, we may now conclude the behavior of the whole soft intersection.

Corollary 3.6. *Let $T : X \tilde{\rightrightarrows} Y$ be a soft KKM map with soft open values. Assume that at some $x_0\,\tilde{\in}\,X$, the soft set $T(x_0)$ is relatively soft compact, i.e., $T(x_0)$ is soft contained in some soft compact set $K\,\tilde{\subset}\,Y$. Then, the soft intersection $\tilde{\bigcap}_{x\,\tilde{\in}\,X}\,\tilde{\mathrm{cl}}_Y\,T(x)$ is nonempty, where $\tilde{\mathrm{cl}}_Y(\cdot)$ is defined for each $A\,\tilde{\subset}\,Y$ by $\tilde{\mathrm{cl}}_Y\,A := \tilde{\bigcap}\{C\,\tilde{\subset}\,Y,\ C$ is soft closed and $C\,\tilde{\supset}\,A\}$.*

Proof. By Corollary 3.4, we know that $\{T(x),\ x\,\tilde{\in}\,X\}$ has the F.S.I.P., and hence the family $\{\tilde{\mathrm{cl}}_Y\,T(x),\ x\,\tilde{\in}\,X\}$ also does. Moreover, we may see that for each $x\,\tilde{\in}\,X$, the soft set $K\,\tilde{\cap}\,\tilde{\mathrm{cl}}_Y\,T(x)$ is nonempty and soft closed in K. Since $\{K\,\tilde{\cap}\,\tilde{\mathrm{cl}}_Y\,T(x), x\,\tilde{\in}\,X\}$ has the F.S.I.P., Lemma 3.5 is then applied to obtain that the soft intersection $\tilde{\bigcap}_{x\,\tilde{\in}\,X}\,\tilde{\mathrm{cl}}_Y\,T(x)$ is nonempty. □

Next, we would like to deduce the soft intersection property for open valued maps, with an additional assumption.

Definition 3.7. A soft map $T : X \tilde{\rightrightarrows} Y$ is said to transfer soft closed value at $x\in X$ if for each $y\,\tilde{\in}\,Y\,\tilde{\setminus}\,T(x)$, there exists a point $x'\,\tilde{\in}\,X$ such that $V\,\tilde{\cap}\,T(x')$ is soft empty for some soft open subset $V\,\tilde{\subset}\,Y$ with $V\,\tilde{\ni}\,y$. If T transfers soft closed value at every $x\,\tilde{\in}\,X$, we simply say that T transfers soft closed values.

Lemma 3.8. *Suppose that $T : X \tilde{\rightrightarrows} Y$. Then, $\tilde{\bigcap}_{x\,\tilde{\in}\,X}T(x) = \tilde{\bigcap}_{x\,\tilde{\in}\,X}\,\tilde{\mathrm{cl}}_Y\,T(x)$ if and only if T transfers soft closed values.*

Proof. (Sufficiency) Suppose that $y\,\tilde{\notin}\,\tilde{\bigcap}_{x\,\tilde{\in}\,X}T(x)$. Then, we have $y\,\tilde{\notin}\,T(z)$ at some $z\,\tilde{\in}\,X$. Since T transfers soft closed values, we can find some $z'\tilde{\in}X$ such that $y\,\tilde{\notin}\,\tilde{\mathrm{cl}}_Y\,T(x')$. Consequently, we have $y\,\tilde{\notin}\,\tilde{\bigcap}_{x\,\tilde{\in}\,X}\,\tilde{\mathrm{cl}}_Y\,T(x)$, and hence

$$\tilde{\bigcap}_{x\,\tilde{\in}\,X}\,\tilde{\mathrm{cl}}_Y\,T(x)\,\tilde{\subset}\,\tilde{\bigcap}_{x\,\tilde{\in}\,X}T(x).$$

Since the reverse is clear, we have thus proved the sufficiency.

(Necessity) Let $x\,\tilde{\in}\,X$ and $y\,\tilde{\in}\,Y$ be two soft elements with $y\,\tilde{\notin}\,T(x)$, so that $y\,\tilde{\notin}\,\tilde{\bigcap}_{z\,\tilde{\in}\,X}T(z) = \tilde{\bigcap}_{z\,\tilde{\in}\,X}\,\tilde{\mathrm{cl}}_Y\,T(z)$. Therefore, there exists a soft element $x'\,\tilde{\in}\,X$ satisfying $y\,\tilde{\in}\,[Y\,\tilde{\setminus}\,\tilde{\mathrm{cl}}_Y\,T(x')]$. As $Y\,\tilde{\setminus}\,\tilde{\mathrm{cl}}_Y\,T(x')$ is soft open, we can find a soft open set $Y\,\tilde{\supset}\,V\,\tilde{\ni}\,y$ in which $V\,\tilde{\cap}\,\tilde{\mathrm{cl}}_Y\,T(x')$ is soft empty. Hence, $V\,\tilde{\cap}\,T(x')$ is also soft empty, showing that T transfers soft closed values. □

Corollary 3.9. *Suppose that $T : X \tilde{\rightrightarrows} Y$ is a soft KKM map with soft open values, and $T(x_0)$ is relatively soft compact at some $x_0 \tilde{\in} X$. If T transfers soft closed values, then the soft intersection $\bigcap_{x \tilde{\in} X} T(x)$ is nonempty.*

Proof. By Corollary 3.6 and Lemma 3.8 we have immediately that $\tilde{\bigcap}_{x \tilde{\in} X} T(x) = \tilde{\bigcap}_{x \tilde{\in} X} \tilde{\mathrm{cl}}_Y\, T(x)$ is nonempty. □

Conclusion

As we have already mentioned about the importance of soft intersection, we investigated in this paper some soft intersection property. We adopted the notion of soft KKM maps and showed that this notion is actually equivalent to the finite soft intersection property for a family of soft open sets. We also studied the duality results for a family of soft closed sets. Some extensions to arbitrary soft intersections are also examined.

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