Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 5189–5195
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Almost monotone contractions on weighted graphs

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Communicated by P. Kumam

Abstract

Almost contraction mappings were introduced as an extension to the contraction mappings for which the conclusion of the Banach contraction principle (BCP in short) holds. In this paper, the concept of monotone almost contractions defined on a weighted graph is introduced. Then a fixed point theorem for such mappings is given.

Keywords: Almost contraction, directed graph, fixed point, monotone mapping, multivalued mapping.
2010 MSC: 47H09, 46B20, 47H10, 47E10.

1. Introduction

The BCP [2] has laid the foundation of metric fixed point theory for contraction mappings on a complete metric space. Fixed point theory of certain important single-valued mappings is very interesting in its own right due to their results having constructive proofs and applications in industrial fields such as image processing engineering, physics, computer science, economics and telecommunication. Multivalued mappings are a useful tool in convex optimization, differential inclusions, control theory and economics. Following the BCP, Nadler [14] gave the definition of multivalued contractions and established multivalued contraction version of the BCP.

Following the publication of Ran and Reurings paper [16] in which the authors investigated the extension of the BCP to metric spaces endowed with a partial order, many mathematicians got interested into the investigation of the metric fixed point problem for monotone mappings defined in a partially ordered metric space. The ideas behind the main fixed point theorem of [16] are found in the original paper [6]. The main

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Received 2016-06-12

fixed point result of [16] was discovered while investigating the solutions to some special matrix equations. Nieto and Rodríguez-López [15] extended the main fixed point theorem of [16] and used it to solve some differential equations. In [10], Jachymski was the first to use graphs instead of partial orders (see also [1]). Before we close these historical facts, let us point out that in fact the first attempt to generalize the BCP to partially ordered metric spaces was carried by Turinici in [20]. In this work, we extend the concept of monotone almost contraction single-valued and multivalued mappings defined in a weighted graph.

The interested reader into fixed point theory may consult the books [8, 12].

2. Preliminaries and basic results

Following the publication of the BCP [2], many mathematicians tried to weaken its main assumptions. Most of the attention was focused on the Lipschitzian condition. One of the first attempt was done by Kannan [11] followed by Chatterjea [5], Rus [17, 18], Tasković [19] and Zamfirescu [23]. Berinde [3] was able to give a condition that captures most of the new concepts which he called weak contraction and later on almost contraction [4].

Definition 2.1 ([4])**.** Let (M, d) be a metric space. A map $T : M \to M$ is said to be an almost contraction if there exists $k < 1$ and $\theta \geq 0$ such that

$$d(T(x), T(y)) \leq k\ d(x, y) + \theta\ d(y, T(x)) \tag{AC}$$

for any $x, y \in M$.

It is obvious that by symmetry, the condition (AC) is equivalent to

$$d(T(x), T(y)) \leq k\ d(x, y) + \theta\ d(x, T(y))$$

for any $x, y \in M$. The following definition of an almost contraction is introduced because it captures most of the ideas behind the proofs of the existence of fixed points of such mappings.

Definition 2.2. Let (M, d) be a metric space. A map $T : M \to M$ is said to be a generalized almost contraction if there exist $k < 1$ and a function $\theta : [0, +\infty) \to [0, +\infty)$ which satisfies $\lim_{t\to 0+} \theta(t) = \theta(0) = 0$, such that

$$d(T(x), T(y)) \leq k\ d(x, y) + \min\Big\{\theta\ \Big(d(x, T(y))\Big), \theta\ \Big(d(y, T(x))\Big)\Big\}$$

for any $x, y \in M$.

Remark 2.3. In the paper [21], the authors gave an example when a convex averaging of a nonexpansive mapping and the identity happens to be an almost contraction. In particular, they considered a property, which we call (XU). Indeed, let (M, d) be a metric space. A map $T : M \to M$ is said to satisfy the property (XU) in a nonempty subset K of M if there exists $C \geq 1$ such that

$$d(x, y) \leq d(x, T(y)) \quad \text{implies} \quad d(x, T(y)) \leq C\ d(x, y)$$

for any $x \neq y$ in K. In fact, the authors of [21] took K to be an open subset of M. In this case, let us show that the only map which satisfies the property (XU) is the identity map of K. Indeed, notice that since $C \geq 1$, then we have

$$d(y, T(x)) \leq C\ d(x, y)$$

for any $x \neq y$ in K. Hence we must have

$$d(x, T(x)) \leq d(x, y) + d(y, T(x)) \leq (1 + C)\ d(x, y)$$

for any $x \neq y$ in K. Since K is open, we may choose $y \neq x$ and as close as one wishes to x. Therefore, $d(x, T(x))$ is as small as one wishes it to be. In other words, $T(x) = x$ for any $x \in K$.

In the next section, we introduce the concept of monotone almost contractions defined on weighted graphs.

3. Monotone almost contractions on weighted graphs

A weighted digraph is a digraph that has a numeric label associated to each edge. Such numeric labels may be interpreted as a distance or a connection costs for example. As an example of a weighted digraph, one would take a graph which represents the roads between cities. So if we are interested to find the shortest way to travel cross-country, then it is not appropriate for all edges to be equal. For more details on this example, we refer to the page 340 by Goodrich and Tamassia [9].

The basic concepts related to a graph may be found in any textbook on graph theory, see for example [22]. Recall that a graph G is determined by a nonempty set $V(G)$ known as the vertices and another subset $E(G) \subset V(G) \times V(G)$ known as the set of edges. If a direction is imposed on each edge, we call the graph a directed graph or digraph. Digraphs may have two edges with the same endpoints, provided they have opposite directions. Given a digraph G, one may generate a graph $\widetilde{G}$ where we ignore the directions and replace the resulting multiple edges by single edges [22]. Throughout the paper, we assume that digraphs are reflexive i.e., each vertex has a loop. Moreover, we will say that a digraph G is transitive whenever

$$(f, g) \in E(G) \ \text{ and } \ (g, h) \in E(G) \Rightarrow (f, h) \in E(G)$$

for any $f, g, h \in V(G)$. Throughout this work, we consider the weighted digraphs where the weight of each edge is given by a distance function between vertices. Moreover, we will use the concept of increasing or decreasing sequences in the sense of a digraph. Therefore, the following definition is needed.

Definition 3.1. Let G be a digraph. A sequence $\{x_n\} \in V(G)$ is said to be

(a) G-increasing if $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$;

(b) G-decreasing if $(x_{n+1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$;

(c) G-monotone if $\{x_n\}$ is either G-increasing or G-decreasing.

Let G be a digraph. Next we define the concept of G-completeness.

Definition 3.2. Let (G, d) be a weighted digraph. G is said to be G-complete if any Cauchy G-monotone sequence $\{x_n\}$ is convergent to a point in $V(G)$.

Example 3.3. Consider the weighted digraph (G, d) where

$$V(G) = \{(x, y) \in \mathbb{R}^2;\ x \in [0, 1) \text{ and } y \in [0, 1]\},$$

and d is the usual Euclidean distance. Define the edges by

$$\Big((x, y), (a, b)\Big) \in E(G) \iff x = a \text{ and } y \leq b.$$

It is clear that if $\Big((x_n, y_n)\Big)$ is a G-monotone sequence, then there exists $s_0 \in [0, 1)$ such that $x_n = s$ for all $n \in \mathbb{N}$, and $\{y_n\}$ is a monotone sequence in $[0, 1]$. Clearly this will imply that $\{(x_n, y_n)\}$ is convergent to a point in $V(G)$. Moreover, it is also clear that $V(G)$ is not complete in the usual metric definition.

This example shows that the G-completeness is finer than the usual completeness. This is interesting since most of the results published for monotone mappings defined on a metric space endowed with a graph assumes the usual completeness [10, 15, 16].

Before we give the definition of a G-monotone almost contraction mapping, let us introduce the family $\mathcal{C}_{\prime}(\mathbb{R}_+)$ defined by $\theta \in \mathcal{C}_{\prime}(\mathbb{R}_+)$ if and only if $\theta : [0, +\infty) \to [0, +\infty)$ which satisfies $\lim\limits_{t \to 0+} \theta(t) = \theta(0) = 0$.

Definition 3.4. Let (G, d) be a weighted digraph. A map $T : V(G) \to V(G)$ is said to be a monotone generalized almost contraction if

(i) T is G-monotone, i.e., $(x, y) \in E(G)$ implies $(T(x), T(y)) \in E(G)$;

(ii) there exist $k < 1$ and a function $\theta \in \mathcal{C}_{\prime}(\mathbb{R}_+)$ such that

$$d(T(x), T(y)) \leq k \ d(x, y) + \min\left\{\theta\ \Big(d(x, T(y))\Big), \theta\ \Big(d(y, T(x))\Big)\right\}$$

for any $x, y \in V(G)$ such that $(x, y) \in E(G)$.

The set of fixed points of T is given by $Fix(T) = \{x \in V(G);\ T(x) = x\}$.

Before we state an analogue to the main fixed point theorem of [3] to the case of monotone mappings, we will need to assume a property initially introduced in [15] in partially ordered sets and in [10] in metric spaces endowed with a graph.

Property (JNRL)**.** G is said to satisfy the property $(JNRL)$ if for any G-monotone increasing sequence (resp. increasing sequence) $\{x_n\}$ which converges to some $x \in V(G)$, we have $(x_n, x) \in E(G)$ (resp. $(x, x_n) \in E(G)$) for any $n \in \mathbb{N}$.

Theorem 3.5. *Let (G, d) be a weighted digraph. Assume G is G-complete and satisfies the property $(JNRL)$. Let $T : V(G) \to V(G)$ be a monotone generalized almost contraction. Then T has a fixed point provided there exists $x_0 \in V(G)$ such that $(x_0, T(x_0)) \in E(\tilde{G})$.*

Proof. Without loss of generality, we assume that $(x_0, T(x_0)) \in E(G)$. Since T is G-monotone, we have $(T^n(x_0), T^{n+1}(x_0)) \in E(G)$ for any $n \in \mathbb{N}$. Therefore $\{T^n(x_0)\}$ is a G-monotone sequence. Moreover, since T is a monotone generalized almost contraction, there exist $k < 1$ and a function $\theta \in \mathcal{C}_{\prime}(\mathbb{R}_+)$ such that

$$d(T(x), T(y)) \leq k \ d(x, y) + \min\left\{\theta\ \Big(d(x, T(y))\Big), \theta\ \Big(d(y, T(x))\Big)\right\}$$

for any $x, y \in V(G)$ such that $(x, y) \in E(G)$. Hence

$$d(T^{n+2}(x_0), T^{n+1}(x_0)) \leq k \ d(T^{n+1}(x_0), T^n(x_0))$$

for any $n \in \mathbb{N}$. Obviously this implies

$$d(T^{n+1}(x_0), T^n(x_0)) \leq k^n \ d(T(x_0), x_0)$$

for any $n \in \mathbb{N}$. Since $k \in [0, 1)$, we conclude that $\{T^n(x_0)\}$ is a Cauchy G-monotone sequence. Since G is G-complete, $\{T^n(x_0)\}$ converges to some point $z \in V(G)$. We claim that z is a fixed point of T. Indeed, note that if T is continuous, then this conclusion is obvious. Otherwise, we use the property $(JNRL)$ satisfied by G. Indeed, this property implies $(T^n(x_0), z) \in E(G)$ for any $n \in \mathbb{N}$. Hence

$$d(T^{n+1}(x_0), T(z)) \leq k \ d(T^n(x_0), z) + \min\left\{\theta(d(T^n(x_0), T(z))), \theta(d(T^{n+1}(x_0), z))\right\}$$

for any $n \in \mathbb{N}$. Using the properties of the function θ, we conclude that

$$\lim_{n \to +\infty} \ \theta(d(T^{n+1}(x_0), z)) = 0,$$

which implies $\lim\limits_{n \to +\infty} \ d(T^{n+1}(x_0), T(z)) = 0$, i.e., $\{T^{n+1}(x_0)\}$ converges to $T(z)$. Uniqueness of the limit implies $T(z) = z$. $\square$

In the next section, we discuss the multi-valued version of Theorem 3.5.

4. Monotone multivalued almost contractions on weighted graphs

Nadler [14] obtained the multivalued version of the BCP. Extensions and generalizations of Nadler's fixed point theorem were obtained by many mathematicians [7, 13].

Let (X, d) be a metric space. The Hausdorff-Pompeiu distance H defined on $\mathcal{CB}(\mathcal{X})$, the set of nonempty bounded and closed subsets of X, is given by

$$H(A, B) = \max\left\{\sup_{b\in B}\inf_{a\in A} d(b, a),\ \sup_{a\in A}\inf_{b\in B} d(a, b)\right\}$$

for any $A, B \in \mathcal{CB}(\mathcal{X})$. The following technical result is useful to explain our definition later on.

Lemma 4.1 ([14])**.** *Let (X, d) be a metric space. For any $A, B \in \mathcal{CB}(\mathcal{X})$ and $\varepsilon > 0$, and for any $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \varepsilon.$$

Using Lemma 4.1, we are able to give a simpler formulation of what is a monotone multivalued almost contraction which avoids the use of Hausdorff-Pompeiu distance.

Definition 4.2. Let (G, d) be a weighted digraph. A map $T : V(G) \to \mathcal{C}(\mathcal{V}(\mathcal{G}))$ is said to be monotone if for any $x, y \in V(G)$ such that $(x, y) \in E(G)$, then for any $\alpha \in T(x)$, there exists $\beta \in T(y)$ such that $(\alpha, \beta) \in E(G)$. T is said to be a monotone generalized almost contraction if T is monotone and there exist $k < 1$ and a function $\theta \in \mathcal{C}_\prime(\mathbb{R}_+)$ such that for any $x, y \in V(G)$ with $(x, y) \in E(G)$ and any $\alpha \in T(x)$, there exists $\beta \in T(y)$ such that $(\alpha, \beta) \in E(G)$ and

$$d(\alpha, \beta) \leq k\ d(x, y) + \min\left\{\theta\left(dist(x, T(y))\right), \theta\left(dist(y, T(x))\right)\right\},$$

where $dist(x, A) = \inf\{d(x, a);\ a \in A\}$. A point $x \in V(G)$ is said to be a fixed point of T whenever $x \in T(x)$.

Next, we give the multi-valued version of Theorem 3.5.

Theorem 4.3. *Let (G, d) be a weighted digraph. Assume G is G-complete and satisfies the property $(JNRL)$. Let $T : V(G) \to \mathcal{C}(\mathcal{V}(\mathcal{G}))$ be a monotone generalized almost contraction. If the set*

$$E_T = \{x \in V(G);\ \textit{there exists } y \in T(x) \textit{ such that } (x, y) \in E(\tilde{G})\}$$

is not empty, then T has a fixed point.

Proof. Note that if T has a fixed point $z \in V(G)$, then $z \in E_T$. Assume E_T is not empty and let $x_0 \in E_T$. Without loss of generality, we assume that there exists $x_1 \in T(x_0)$ such that $(x_0, x_1) \in E(G)$. Since T is a monotone generalized almost contraction, there exist $k < 1$ and a function $\theta \in \mathcal{C}_\prime(\mathbb{R}_+)$ such that for any $x, y \in V(G)$ with $(x, y) \in E(G)$ and any $a \in T(x)$ there exists $b \in T(y)$ such that $(a, b) \in E(G)$ and

$$d(a, b) \leq k\ d(x, y) + \min\left\{\theta\left(dist(x, T(y))\right), \theta\left(dist(y, T(x))\right)\right\}.$$

In this case, there exists $x_2 \in T(x_1)$ such that

$$d(x_1, x_2) \leq k\ d(x_0, x_1) + \min\left\{\theta\left(dist(x_1, T(x_0))\right), \theta\left(dist(x_0, T(x_1))\right)\right\}.$$

Since $x_1 \in T(x_0)$, we get $dist(x_1, T(x_0)) = 0$, which implies

$$d(x_1, x_2) \leq k\ d(x_0, x_1).$$

By the induction, we construct a sequence $\{x_n\}$ in $V(G)$ such that

(1) $(x_n, x_{n+1}) \in E(G)$;

(2) $x_{n+1} \in T(x_n)$;

(3) $d(x_{n+1}, x_{n+2}) \leq k\ d(x_n, x_{n+1})$

for any $n \in \mathbb{N}$. Condition (2) implies $d(x_{n+1}, x_n) \leq k^n\ d(x_0, x_1)$ for any $n \in \mathbb{N}$. Since $k \in [0, 1)$, we conclude that $\{x_n\}$ is a Cauchy G-monotone sequence (because of (1)). Since G is G-complete, $\{x_n\}$ converges to some point $z \in V(G)$. We claim that z is a fixed point of T. Indeed, using the property $(JNRL)$ satisfied by G, we know that $(x_n, z) \in V(G)$ holds for any $n \in \mathbb{N}$. Hence for any $n \in \mathbb{N}$, there exists $z_n \in T(z)$ such that

$$d(x_{n+1}, z_n) \leq k\ d(x_n, z) + \min\Big\{\theta(dist(x_n, T(z))), \theta(dist(z, T(x_n))\Big\}$$

for any $n \in \mathbb{N}$. Since $x_{n+1} \in T(x_n)$, we get $dist(z, T(x_n)) \leq d(z, x_{n+1})$ for any $n \in \mathbb{N}$. Hence,

$$\lim_{n\to+\infty} dist(z, T(x_n)) = 0.$$

The main property satisfied by θ implies

$$\lim_{n\to+\infty} \min\Big\{\theta(dist(x_n, T(z))), \theta(dist(z, T(x_n))\Big\} \leq \lim_{n\to+\infty} \theta(d(z, T(x_n)) = 0.$$

Therefore, we have $\lim_{n\to+\infty} dist(x_{n+1}, z_n) = 0$. From this, we conclude that $\{z_n\}$ also converges to z. Since $T(z)$ is closed, we get $z \in T(z)$, i.e., z is a fixed point of T. □

Acknowledgement

The second and third authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this Research group No. (RG-1435-079).

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