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On the fixed point theory in bicomplete quasi-metric spaces

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Abstract

We show that some important fixed point theorems on complete metric spaces as Browder's fixed point theorem and Matkowski's fixed point theorem can be easily generalized to the framework of bicomplete quasi-metric spaces. From these generalizations we deduce quasi-metric versions of well-known fixed point theorems due to Krasnoselskiĭ and Stetsenko; Khan, Swalesh and Sessa; and Dutta and Choudhury, respectively. In fact, our approach shows that many fixed point theorems for φ-contractions on bicomplete quasi-metric spaces, and hence on complete G-metric spaces, are actually consequences of the corresponding fixed point theorems for complete metric spaces.

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1. Introduction and preliminaries

The study of the fixed point theory in quasi-metric spaces has received an increasing attention in the last years (see e.g. [1–4, 8, 10, 15, 20, 21, 27]) due, in great part, to the usefulness of these spaces and other related structures, as the so-called partial metric spaces, to the theory of computation, the complexity analysis of algorithm (see e.g. [5, 25, 26, 28]), as well as to the fixed point theory for G-metric spaces [1, 14].

The purpose of this paper is to show that some important fixed point theorems on complete metric spaces as Browder's fixed point theorem and Matkowski's fixed point theorem can be easily generalized

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to the framework of bicomplete quasi-metric spaces. Then, and with the help of some useful equivalences proved by Jachymski [13, Lemma 1], we deduce quasi-metric versions of well-known fixed point theorems due to Krasnoselskiĭ and Stetsenko [17], Khan et al. [16], and Dutta and Choudhury [11]. In fact, our approach shows that many fixed point theorems for φ-contractions on bicomplete quasi-metric spaces, and hence on complete G-metric spaces, are actually consequences of the corresponding fixed point theorems for complete metric spaces. We also consider the problem of extending the famous Boyd and Wong fixed point theorem [6] to this framework.

Next we recall some concepts and properties of the theory of quasi-metric spaces. (By $\mathbb{R}^+$ we shall denote the set of all non-negative real numbers.)

Following the modern terminology (see [9]) by a quasi-metric on the set X we mean a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(i) $x = y \Leftrightarrow d(x, y) = d(y, x) = 0$;

(ii) $d(x, z) \leq d(x, y) + d(y, z)$.

A quasi-metric space is a pair (X, d) such that X is a set and d is a quasi-metric on X.

Given a quasi-metric d on a set X the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x, y \in X$, is a metric on X.

Each quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x, r) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If τ_d is a T_1 topology on X, we say that (X, d) is a T_1 quasi-metric space.

There exist many different notions of quasi-metric completeness in the literature (see e.g. [9, 19, 23]). For our purposes here we will consider the following one: A quasi-metric space (X, d) is said to be bicomplete if the metric space (X, d^s) is complete.

It is interesting to point out that bicompleteness is a very useful notion of quasi-metric completeness in solving the problem of quasi-metric completion. Furthermore, a class of bicomplete quasi-metric spaces (the so-called Smyth complete quasi-metric spaces) provides a suitable tool in constructing mathematical models in theoretical computer science and complexity analysis of algorithms (see e.g. [24, 26, 28]).

2. The results

Given a quasi-metric space (X, d), a mapping $T : X \to X$ and functions $\varphi, \eta, \psi : \mathbb{R}^+ \to \mathbb{R}^+$, consider the following conditions:

(1) φ is non-decreasing, $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, and

$$d(Tx, Ty) \leq \varphi(d(x, y))$$

for all $x, y \in X$.

(2) φ is non-decreasing, right continuous, $\varphi(t) < t$ for all $t > 0$, and

$$d(Tx, Ty) \leq \varphi(d(x, y))$$

for all $x, y \in X$.

(3) η is continuous, $0 < \eta(t) < t$ for all $t > 0$, and

$$d(Tx, Ty) \leq d(x, y) - \eta(d(x, y))$$

for all $x, y \in X$.

(4) ψ is non-decreasing, continuous, $\psi^{-1}(0) = \{0\}$, and

$$\psi(d(Tx, Ty)) \leq \alpha\psi(d(x, y))$$

for all $x, y \in X$ and some $\alpha \in [0, 1)$.

(5) η and ψ are non-decreasing, continuous, $\eta^{-1}(0) = \psi^{-1}(0) = \{0\}$, and

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \eta(d(x, y))$$

for all $x, y \in X$.

As in the metric case (see e.g. [13]), given a quasi-metric space (X, d), a mapping $T : X \to X$ and a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(t) < t$ for all $t > 0$, we say that T is a φ-contraction if $d(Tx, Ty) \leq \varphi(d(x, y))$, for all $x, y \in X$.

It is clear that every self-mapping T on a quasi-metric space (X, d) and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a function for which condition (1) or condition (2) is satisfied, then T is a φ-contraction.

It is well-known that if (X, d) is a complete metric space, and some of the above conditions (1)-(5) is satisfied, then T has a unique fixed point. In fact, Matkowski [22] and Browder [7], respectively proved the following.

Theorem 2.1 ([22])**.** *Let (X, d) be a complete metric space, T be a self-mapping of X and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function for which condition* (1) *above is satisfied. Then T has a unique fixed point.*

Theorem 2.2 ([7])**.** *Let (X, d) be a complete metric space, T be a self-mapping of X and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function for which condition* (2) *above is satisfied. Then T has a unique fixed point.*

If T is a self-map of a complete metric space (X, d), Krasnoselskiĭ and Stetsenko [17] (see also [18]) proved the existence of a unique fixed point for T whenever condition (3) is satisfied, whereas Khan et al. [16], and Dutta and Choudhury [11], respectively, proved that T has a unique fixed point when condition (4), respectively (5), is satisfied.

Next we easily generalize Theorems 2.1 and 2.2 to bicomplete quasi-metric spaces.

Theorem 2.3. *Let (X, d) be a bicomplete quasi-metric space, $T : X \to X$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$, for all $t > 0$, and*

$$d(Tx, Ty) \leq \varphi(d(x, y))$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. Since (X, d) is bicomplete, (X, d^s) is a complete metric space. Let $x, y \in X$, and suppose, without loss of generality that $d^s(Tx, Ty) = d(Tx, Ty)$. Since $d(x, y) \leq d^s(x, y)$ and φ is non-decreasing, we deduce

$$d^s(Tx, Ty) = d(Tx, Ty) \leq \varphi(d(x, y)) \leq \varphi(d^s(x, y)).$$

Consequently, we can apply Matkowki's fixed point theorem (Theorem 2.1) to (X, d^s), and thus T has a unique fixed point. □

Theorem 2.4. *Let (X, d) be a bicomplete quasi-metric space, $T : X \to X$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing and right continuous function such that $\varphi(t) < t$ for all $t > 0$, and*

$$d(Tx, Ty) \leq \varphi(d(x, y))$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. It is well-known, and easy to check, that if φ is non-decreasing, right continuous, and satisfies $\varphi(t) < t$ for all $t > 0$, then $\lim_{n\to\infty} \varphi^n(t) = 0$. Hence T has a fixed point by Theorem 2.3. □

Remark 2.5. Observe that similar to the proof of Theorems 2.3 and 2.4 can be also directly deduced from Browder's theorem (Theorem 2.2).

Our next theorem allows us to deduce quasi-metric generalizations of the fixed point theorems in [11, 16, 18] mentioned above. To this end, the following result due to Jachymski [13] will be crucial.

Lemma 2.6 ([13, Lemma 1: (viii), (xii)]). *Let D be a nonempty subset of $\mathbb{R}^+ \times \mathbb{R}^+$. The following statements are equivalent:*

(i) *there exists a non-decreasing and continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(t) < t$ for all $t > 0$, and $D \subseteq E_\varphi$, where $E_\varphi = \{(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+ : v \leq \varphi(u)\}$;*

(ii) *there exist a non-decreasing and continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi^{-1}(0) = \{0\}$ and $\lim_{t\to\infty} \phi(t) = \infty$, and a lower semicontinuous function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\eta^{-1}(0) = \{0\}$, such that $D \subseteq E_{\phi,\eta}$, where $E_{\phi,\eta} = \{(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+ : \phi(v) \leq \phi(u) - \eta(u)\}$.*

Theorem 2.7. *Let (X, d) be a bicomplete quasi-metric space, $T : X \to X$ and $\eta, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ be functions such that ψ is non-decreasing and continuous, η is lower semicontinuous, $\eta^{-1}(0) = \psi^{-1}(0) = \{0\}$, and*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \eta(d(x, y))$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. We first observe that for each $x, y \in X$ one has $d(Tx, Ty) \leq d(x, y)$.

Define a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\phi(t) = t + \psi(t)$ for all $t \in \mathbb{R}^+$. Then ϕ is non-decreasing and continuous on $\mathbb{R}^+$ and satisfies $\phi^{-1}(0) = \{0\}$ and $\lim_{t\to\infty} \phi(t) = \infty$.

Now define

$$D = \{(d(x, y), d(Tx, Ty)) : x, y \in X\},$$

and

$$E_{\phi,\eta} = \{(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+ : \phi(v) \leq \phi(u) - \eta(u)\}.$$

We show that $D \subseteq E_{\phi,\eta}$. Indeed, given $x, y \in X$ we have

$$\begin{aligned}\phi(d(Tx, Ty) &= d(Tx, Ty) + \psi(d(Tx, Ty)) \leq d(x, y) + \psi(d(x, y)) - \eta(d(x, y))\\ &= \phi(d(x, y)) - \eta(d(x, y)).\end{aligned}$$

Therefore, $D \subseteq E_{\phi,\eta}$. By Lemma 2.6, there exists a continuous and non-decreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(t) < t$ for all $t > 0$, and $D \subseteq E_\varphi$, where $E_\varphi = \{(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+ : v \leq \varphi(u)\}$. Hence

$$d(Tx, Ty) \leq \varphi(d(x, y))$$

for all $x, y \in X$. By Theorem 2.4 we conclude that T has a unique fixed point. □

Corollary 2.8. *Let (X, d) be a bicomplete quasi-metric space, $T : X \to X$ and $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ be a lower semicontinuous function such that $\eta^{-1}(0) = \{0\}$, and*

$$d(Tx, Ty) \leq d(x, y) - \eta(d(x, y))$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. Apply Theorem 2.7 with $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\psi(t) = t$ for all $t \in \mathbb{R}^+$. □

Corollary 2.9. *Let (X, d) a bicomplete quasi-metric space, $T : X \to X$, $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing and continuous function with $\psi^{-1}(0) = \{0\}$, and $\alpha \in [0, 1)$ be a constant such that*

$$\psi(d(Tx, Ty)) \leq \alpha\psi(d(x, y))$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. Apply Theorem 2.7 with $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\eta(t) = (1-c)t$ for all $t \in \mathbb{R}^+$. □

Corollary 2.10. *Let (X, d) be a bicomplete quasi-metric space, $T : X \to X$ and $\alpha \in [0, 1)$ be a constant such that*

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point.

Remark 2.11. Jleli and Samet [14, Theorem 3.2] proved Corollary 2.8 for the case that (X, d) is a bicomplete T_1 quasi-metric space and $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with $\eta^{-1}(0) = \{0\}$. Corollary 2.10 is the well-known bicomplete quasi-metric version of the Banach contraction principle. Obviously, it is also an immediate consequence of Theorem 2.3 or Theorem 2.4.

The following two examples illustrate the preceding results.

Example 2.12. Denote by Σ a non-empty alphabet (i.e., a non-empty set) and by Σ^F the set of all finite words (or strings) on Σ. We assume that the empty word ϕ is an element of Σ^F. Denote by $\sqsubseteq$ the prefix order on Σ^F and by $\ell(x)$ the length of each $x \in \Sigma^F$. In particular $\phi \sqsubseteq x$ for all $x \in \Sigma^F$, and $\ell(\phi) = 0$.

Now let d be the quasi-metric on Σ^F defined as $d(x, y) = 0$ if $x \sqsubseteq y$, and $d(x, y) = \ell(x)$ otherwise. Since for each $x, y \in \Sigma^F$ we have $d^s(x, y) = \max\{\ell(x), \ell(y)\}$ it immediately follows that every Cauchy sequence in the metric space (X, d^s) is eventually constant, and thus (X, d) is obviously a bicomplete quasi-metric space.

Define $T : \Sigma^F \to \Sigma^F$ as follows: $T\phi = \phi$, and for each $x \in \Sigma^F \backslash \{\phi\}$, Tx is the element of Σ^F obtained by deleting the last letter of x, i.e., if $x := x_1x_2...x_n$, with $x_k \in \Sigma$ for all $k = 1, ..., n$, then $Tx = x_1x_2...x_{n-1}$. In particular $Tx = \phi$ whenever $\ell(x) = 1$. Observe also that $\ell(Tx) = \ell(x) - 1$ whenever $x \in \Sigma^F \backslash \{\phi\}$. Now consider the function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\eta(0) = 0$ and $\eta(t) = 1$ for all $t > 0$. Clearly, η is lower semicontinuous on $\mathbb{R}^+$. Finally, let $x, y \in \Sigma^F$. If Tx is a prefix of Ty we have $d(Tx, Ty) = 0$. Otherwise, it follows that x is not a prefix of y, so $d(Tx, Ty) = \ell(Tx) = \ell(x) - 1 = d(x, y) - \eta(\ell(x)) = d(x, y) - \eta(d(x, y))$.

We have shown that all conditions of Corollary 2.8 are satisfied, so T has a unique fixed point. In fact ϕ is that unique fixed point.

Example 2.13. Let $X = [0, 1/3]$ and let d be the quasi-metric on X defined as $d(x, y) = y - x$, if $x \leq y$, and $d(x, y) = x$ otherwise. Clearly, (X, d) is a bicomplete T_1 quasi-metric space (note that if $(x_n)_n$ is a non-eventually constant Cauchy sequence in (X, d^s), then $x_n \to 0$ with respect to the usual topology and thus $d^s(0, x_n) \to 0$, as $n \to \infty$).

Define $T : X \to X$ as $Tx = x^2$ for all $x \in X$, $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\psi(t) = \sqrt{t}$ for all $t \in \mathbb{R}^+$, and let $\alpha = \sqrt{2/3}$. If $x \leq y$, we obtain

$$\begin{aligned}\psi(d(Tx, Ty)) &= \psi(y^2 - x^2) = \sqrt{y^2 - x^2} = \sqrt{y + x}\sqrt{y - x} \\ &\leq \sqrt{2/3}\sqrt{y - x} = \alpha\psi(d(x, y)).\end{aligned}$$

If $x > y$, we obtain

$$\psi(d(Tx, Ty)) = \psi(x^2) = x \leq \sqrt{1/3}\sqrt{x} < \alpha\psi(x) = \alpha\psi(d(x, y)).$$

We have shown that all conditions of Corollary 2.9 are satisfied, so T has a unique fixed point. In fact 0 is that unique fixed point.

We conclude the paper with some remarks on the question of extending the famous Boyd and Wong fixed point theorem [6] to bicomplete quasi-metric spaces. This theorem, that provides a substantial improvement of Browder's fixed point theorem and is independent from Matkowski's fixed point theorem (see [12]) establishes that if T is a φ-contraction on a complete metric space (X, d) such that the function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is right upper semicontinuous, then T has a unique fixed point.

In contrast to the Boyd and Wong theorem, we give an easy example of a φ-contraction T on a bicomplete quasi-metric such that φ is right upper semicontinuous but T has no fixed points.

Example 2.14. Let $X = \{0, 1\}$ and let d be the quasi-metric on X defined as $d(0, 0) = d(1, 1) = d(0, 1) = 0$, and $d(1, 0) = 1$. Since d^s is the discrete metric on X it follows that (X, d) is a bicomplete quasi-metric space. Now define $T : X \to X$ as $T0 = 1$ and $T1 = 0$. Finally, we show that T is a φ-contraction, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is the (right) upper semicontinuous function given by $\varphi(0) = 1$ and $\varphi(t) = t/2$ for all $t > 0$.

Indeed, we have $\varphi(t) < t$ for all $t > 0$, and $d(T1, T0) = d(0, 1) = 0$, and $d(T0, T1) = d(1, 0) = 1 = \varphi(0) = \varphi(d(0, 1))$.

The above example suggests that a possible extension of the Boyd and Wong fixed point theorem to bicomplete quasi-metric spaces requires some additional condition. This question will be discussed elsewhere.

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