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Strong convergence theorem for common solutions to quasi variational inclusion and fixed point problems

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Abstract

In this paper, we consider a problem that consists of finding a common solution to quasi variational inclusion and fixed point problems. We first present a simple proof to the strong convergence theorem established by Zhang et al. recently. Next, we propose a new algorithm to solve such a problem. Under some mild conditions, we establish the strong convergence of iterative sequence of the proposed algorithm.

Keywords: Variational inclusion, fixed point problem, inverse strongly monotone operator, nonexpansive mapping, multi-valued maximal monotone mapping.
2010 MSC: 47H05, 47H09, 47J05, 49J25.

1. Introduction

In this paper, we consider a quasi variational inequality problem that requires to find a point $u \in \mathcal{H}$ so that

$$\theta \in A(u) + M(u), \tag{1.1}$$

where $\mathcal{H}$ is a real Hilbert space, $A : \mathcal{H} \to \mathcal{H}$ is a single-valued mapping and $M : \mathcal{H} \to 2^{\mathcal{H}}$ is a multi-valued mapping. The solution set of problem (1.1) is denoted by $VI(\mathcal{H}, A, M)$. We recall one of special cases of

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such problem. If $M = \partial\delta_C$ with C a nonempty closed convex subset of $\mathcal{H}$, and $\delta_C : \mathcal{H} \to [0, +\infty]$ is the indicator function of C, that is,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

In this case, problem (1.1) is reduced to finding a point u so that

$$\langle A(u), v - u\rangle \geq 0, \ \ \forall v \in C, \tag{1.2}$$

which is called Hartman-Stampacchia variational inequality problem. A fixed point problem requires to find a point u so that

$$Su = u, \tag{1.3}$$

where $S : \mathcal{H} \to \mathcal{H}$ is a nonlinear mapping. The set of fixed points of S is denoted by $F(S)$.

Takahashi and Toyoda [6] considered the problem for finding a common solution to Hartman-Stampacchia variational inequality problem (1.2) and fixed point problem (1.3), that is, find a point u such that

$$u \in F(S) \text{ and } \langle A(u), v - u\rangle \geq 0, \ \ \forall v \in C. \tag{1.4}$$

Since then much efforts have gone into constructing algorithms to solve such a problem; see e.g., [3, 5, 6, 10] and references therein. Recently, Zhang et al. [11] considered a problem to find a common solution of problems (1.1) and (1.3), that is, find a point u such that

$$u \in F(S) \cap VI(\mathcal{H}, A, M). \tag{1.5}$$

It is obvious that problem (1.5) is an extension of the problem (1.4) considered by Takahashi and Toyoda [6]. In [11], Zhang et al. constructed an algorithm, which generates a sequence (x_n) by

$$\begin{cases} x_0 = x \in \mathcal{H}, \\ y_n = J_{M,\lambda}(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) S y_n, \end{cases} \tag{1.6}$$

where $J_{M,\lambda}$ is the resolvent related to M with λ a positive constant. Under some certain assumptions, they proved the sequence (x_n) generated by (1.6) converges in norm to a solution of problem (1.5).

The aim of this paper is to introduce some new iterative algorithms to solve problem (1.5). We first prove the strong convergence of algorithm (1.6) by employing a new simple proof. Some other iterative schemes to approximate the solution of problem (1.5) are proposed and also the strong convergence properties for these new algorithms are proved.

2. Preliminaries

Throughout this paper, I denotes the identity operator on $\mathcal{H}$, "$\to$" strong convergence, "$\rightharpoonup$" weak convergence, and $\omega_w(x_n)$ the set of weak cluster points of the sequence (x_n). Let P_C denote the projection from $\mathcal{H}$ onto a nonempty closed convex subset C of $\mathcal{H}$, that is,

$$P_C x = \arg\min_{y \in C} \|x - y\|, \quad x \in \mathcal{H}.$$

It is well-known that $P_C x$ is characterized by the inequality:

$$\langle x - P_C x, c - P_C x\rangle \leq 0, \quad c \in C. \tag{2.1}$$

Let T be a mapping defined on $\mathcal{H}$. Recall that T is contractive if there is a $\kappa \in (0, 1)$ so that $\|Tx - Ty\| \leq \kappa\|x - y\|$ for any $x, y \in \mathcal{H}$; and nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in \mathcal{H}$. The fixed point problem (1.3) for the nonexpansive mapping has been widely investigated and studied. There are two iterative schemes with strong convergence for approximating a fixed point of a nonexpansive mapping. One is the Halpern iteration, which generates an iterative sequence by

$$x_{n+1} = \alpha_n x + (1-\alpha_n)Sx_n,\ x \in \mathcal{H}. \tag{2.2}$$

This iteration is originally constructed by Halpern [2] and further studied by Wittmann [7] and Xu [8]. It is well-known that if $F(S) \neq \emptyset$, then the sequence (x_n) generated by (2.2) converges strongly to $P_{F(S)}x$, whenever (α_n) is a sequence in $(0,1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=0}^{\infty} \alpha_n = \infty$;
(C2) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n|/\alpha_n = 0$.

Another is viscosity approximation method, which generates an iterative sequence by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Sx_n,\ x \in \mathcal{H}, \tag{2.3}$$

where f is a contraction. This iteration is originally proposed by Moudafi [4] and further studied by Xu [9]. It is well-known that if $F(S) \neq \emptyset$, then the sequence (x_n) generated by (2.3) converges in norm to $P_{F(S)}f$.

A mapping T is called ν-averaged if there exist a constant $\nu \in (0,1)$ and a nonexpansive mapping S such that $T = (1-\nu)I + \nu S$; and ν-inverse strongly monotone (ν-ism) if there is a constant $\nu > 0$ such that $\langle Tx - Ty, x - y\rangle \geq \nu\|Tx - Ty\|^2$ for any $x, y \in \mathcal{H}$. The following lemma collects some useful properties of averaged and inverse-strongly mappings.

Lemma 2.1 ([1])**.** *The following assertions hold.*

(i) *T is averaged if and only if $I - T$ is ν-ism for some $\nu > 1/2$;*
(ii) *The composition of two averaged mappings is also averaged;*
(iii) *If T is ν-ism with $\nu > 0$ and if $\lambda > 0$, then λT is (ν/λ)-ism;*
(iv) *If T is 1-ism with $\nu > 0$, then it is averaged;*
(v) *If T is ν-averaged with $\nu \in (0,1)$, there holds the inequality:*

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \frac{1-\nu}{\nu}\|Tx - x\|^2,$$

where $x \in \mathcal{H}$ and $z \in F(T)$.

Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued maximal monotone mapping. Then the mapping $J_{M,\lambda}$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u),\ u \in \mathcal{H},$$

is called the resolvent operator associated with M, where λ is any given positive constant. The mapping $J_{M,\lambda}$ has the following properties:

Lemma 2.2. *Let A be α-ism and let $\lambda \in (0, 2\alpha)$. Then the following assertions hold.*

(i) *$J_{M,\lambda}$ is single-valued and 1-ism;*
(ii) *$VI(\mathcal{H}, A, M) = F(J_{M,\lambda}(I - \lambda A))$;*
(iii) *$J_{M,\lambda}(I - \lambda A)$ is averaged.*

Proof. Assertions (i) and (ii) are proved in [11]. Since A is α-ism, λA is α/λ-ism (Lemma 2.1 (iii)). It is easy to check that $\alpha/\lambda > 1/2$, and hence by Lemma 2.1 (i), $I - \lambda A$ is averaged. Since $J_{M,\lambda}$ is 1-ism, then it is also averaged (Lemma 2.1 (iv)) and therefore by Lemma 2.1 (ii), the composition $J_{M,\lambda}(I - \lambda A)$ is averaged, too. □

The following lemmas will be used in the subsequent section.

Lemma 2.3 (demiclosedness principle)**.** *Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If (x_n) is a sequence in $\mathcal{H}$ so that $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $x \in F(T)$.*

Lemma 2.4 ([8])**.** *Let (a_n) be a nonnegative real sequence satisfying*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \mu_n,$$

where the sequences $(\alpha_n) \subset (0, 1)$ and (μ_n) satisfy the conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;
(ii) *either* $\sum_{n=0}^{\infty} |\alpha_n \mu_n| < \infty$ *or* $\overline{\lim}_{n\to\infty} \mu_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section we first give a simple proof of [11, Theorem 2.1]. The key of the proof is the following lemma.

Lemma 3.1. *Let A be a nonexpansive operator and B a ν-averaged operator. If $F(A) \cap F(B) \neq \emptyset$, then $F(A) \cap F(B) = F(AB)$.*

Proof. It is obvious that $F(A)\cap F(B) \subseteq F(AB)$. To see the converse, let $x \in F(AB)$. Since $F(A)\cap F(B) \neq \emptyset$, we can pick $u \in F(A) \cap F(B)$. Hence

$$\begin{aligned}
\|x - u\|^2 + \frac{1-\nu}{\nu}\|Bx - x\|^2 &= \|A(Bx) - u\|^2 + \frac{1-\nu}{\nu}\|Bx - x\|^2 \\
&= \|A(Bx) - Au\|^2 + \frac{1-\nu}{\nu}\|Bx - x\|^2 \\
&\leq \|Bx - u\|^2 + \frac{1-\nu}{\nu}\|Bx - x\|^2 \\
&\leq \|x - u\|^2,
\end{aligned}$$

where the last inequality follows from Lemma 2.1 (v). This implies that $\|Bx - x\| \leq 0$, or equivalently, $Bx = x$ and further

$$x = A(Bx) = Ax.$$

Altogether we get the result as desired. □

Remark 3.2. In [1], Byrne proved that if A and B are averaged and if $F(A)\cap F(B) \neq \emptyset$, then the intersection $F(A) \cap F(B)$ and $F(AB)$ are coincident. So our result is an extension of this assertion.

Theorem 3.3. *Let $A : \mathcal{H} \to \mathcal{H}$ be α-ism with $\alpha > 0$, $M : \mathcal{H} \to 2^{\mathcal{H}}$ a maximal monotone mapping, and $S : \mathcal{H} \to \mathcal{H}$ a nonexpansive mapping. If (α_n) is chosen in $(0, 1)$ so that the conditions* (C1) *and* (C2) *are satisfied, then the sequence (x_n) generated by* (1.6) *converges strongly to $x^* = P_{F(S)\cap VI(\mathcal{H},A,M)}x$, whenever $F(S) \cap VI(\mathcal{H}, A, M) \neq \emptyset$.*

Proof. Set $T = SJ_{M,\lambda}(I - \lambda A)$. Since S and $J_{M,\lambda}(I - \lambda A)$ are both nonexpansive, the operator T is nonexpansive, too. Thus algorithm (1.6) has the following form:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n,$$

which is a standard iterative scheme of Halpern iteration. Since $J_{M,\lambda}(I - \lambda A)$ is averaged, it follows from Lemmas 3.1 and 2.2 that

$$F(T) = F(S) \cap VI(\mathcal{H}, A, M) \neq \emptyset.$$

The sequence (x_n) therefore converges in norm to $P_{F(S)\cap VI(\mathcal{H},A,M)}x$. □

Remark 3.4. Here we choose $\lambda \in (0, 2\alpha)$, while it is assumed that $\lambda \in (0, 2\alpha]$ in [11]. We show that λ can not be equal to 2α. In fact, it is proved in [11, page 577, line 11]

$$(1-\alpha_n)\lambda(2\alpha-\lambda)\|Ax_n - Au\|^2 \to 0, \text{ as } n \to \infty,$$

from which they obtained $\|Ax_n - Au\| \to 0$ as $n \to \infty$. So, if $\lambda = 2\alpha$, one can not deduce $\|Ax_n - Au\| \to 0$ as $n \to \infty$.

Theorem 3.5. *Let $A : \mathcal{H} \to \mathcal{H}$ be α-ism with $\alpha > 0$, $M : \mathcal{H} \to 2^{\mathcal{H}}$ a maximal monotone mapping and $S : \mathcal{H} \to \mathcal{H}$ a nonexpansive mapping. Choose $\lambda \in (0, 2\alpha)$ and define a sequence (x_n) by the iterative procedure:*

$$\begin{cases} x_0 = x \in \mathcal{H}, \\ y_n = J_{M,\lambda}(x_n - \lambda A x_n), \\ x_{n+1} = S(\alpha_n x + (1-\alpha_n) y_n). \end{cases} \tag{3.1}$$

If (α_n) is chosen in $(0,1)$ so that the conditions (C1) *and* (C2) *are satisfied, then the sequence (x_n) generated by* (3.1) *converges strongly to $x^* = P_{F(S)\cap VI(\mathcal{H},A,M)}x$, whenever $F(S) \cap VI(\mathcal{H}, A, M) \neq \emptyset$.*

Proof. Take $u \in F(S) \cap VI(\mathcal{H}, A, M)$. We divide our proof into several steps.

STEP 1. The sequence (x_n) is bounded.

Since $J_{M,\lambda}(I - \lambda A)$ is nonexpansive, we have

$$\|y_n - u\| = \|J_{M,\lambda}(I-\lambda A)x_n - u\| \leq \|x - u\|,$$

which implies that

$$\begin{aligned} \|x_{n+1} - u\| &\leq \|\alpha_n x + (1-\alpha_n)y_n - u\| \\ &\leq \alpha_n\|x-u\| + (1-\alpha_n)\|x_n - u\| \\ &\leq \max\{\|x-u\|, \|x_n - u\|\} \\ &\ \vdots \\ &\leq \max\{\|x_0 - u\|, \|x-u\|\} = \|x-u\|. \end{aligned}$$

This shows (x_n) is bounded and so is (y_n).

STEP 2. $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

It follows from (3.1) that

$$\|y_n - y_{n-1}\| = \|J_{M,\lambda}(I-\lambda A)x_n - J_{M,\lambda}(I-\lambda A)x_{n-1}\| \leq \|x_n - x_{n-1}\|,$$

and also that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|[\alpha_n x + (1-\alpha_n)y_n] - [\alpha_{n-1}x + (1-\alpha_{n-1})y_{n-1}]\| \\ &= \|(\alpha_n - \alpha_{n-1})(x - y_{n-1}) + (1-\alpha_n)(y_n - y_{n-1})\| \\ &\leq |\alpha_n - \alpha_{n-1}|\|x - y_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\ &\leq M|\alpha_n - \alpha_{n-1}| + (1-\alpha_n)\|x_n - x_{n-1}\|, \end{aligned} \tag{3.2}$$

where $M = (\|x\| + \sup_{n\geq 0}\|y_n\|)$. By virtue of conditions (C1) and (C2), we can apply Lemma 2.4 to (3.2) to obtain $x_{n+1} - x_n \to 0$. Consequently, we also have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.3}$$

In fact, since $J_{M,\lambda}(I-\lambda A)$ is averaged, we may assume that it is κ-averaged for some $\kappa \in (0,1)$. Then it follows from Lemma 2.1 (v) that

$$\begin{aligned}\|x_{n+1}-u\|^2 &\le \|\alpha_n x+(1-\alpha_n)y_n-u\|^2\\ &\le \alpha_n\|x-u\|^2+(1-\alpha_n)\|y_n-u\|^2\\ &\le \alpha_n\|x-u\|^2+\|J_{M,\lambda}(I-\lambda A)x_n-u\|^2\\ &\le \alpha_n\|x-u\|^2+\|x_n-u\|^2-\frac{1-\kappa}{\kappa}\|J_{M,\lambda}(I-\lambda A)x_n-x_n\|^2\\ &= \alpha_n\|x-u\|^2+\|x_n-u\|^2-\frac{1-\kappa}{\kappa}\|y_n-x_n\|^2.\end{aligned}$$

Letting $L=2\sup_{n\ge 0}\|x_n\|$, we get

$$\begin{aligned}\frac{1-\kappa}{\kappa}\|y_n-x_n\|^2 &\le \|x_n-u\|^2-\|x_{n+1}-u\|^2+\alpha_n\|x-u\|^2\\ &\le L\|x_n-x_{n+1}\|+\alpha_n,\end{aligned} \tag{3.4}$$

and therefore (3.3) follows from (3.4) by tending $n\to\infty$.

STEP 3. If $z\in\omega_w(x_n)$, then $z\in F(S)\cap VI(\mathcal{H},A,M)$.

To see this, we set $z_n=\alpha_n x+(1-\alpha_n)y_n$. Then we conclude that

$$\|x_n-z_n\|\le\|x_n-y_n\|+\|y_n-z_n\|=\|x_n-y_n\|+\alpha_n\|x-y_n\|\to 0,\ \text{as } n\to\infty,$$

which further gives that

$$\|Sz_n-z_n\|\le\|x_{n+1}-x_n\|+\|x_n-z_n\|\to 0,\ \text{as } n\to\infty.$$

Take a subsequence (x_{n_k}) of (x_n) such that $x_{n_k}\rightharpoonup z$; hence $x_{n_k}\rightharpoonup z$. By Lemma 2.3, we have $z\in F(S)$. Since

$$\|J_{M,\lambda}(I-\lambda A)x_n-x_n\|=\|y_n-x_n\|\to 0,\ \text{as } n\to\infty,$$

we get, by using Lemma 2.3 again, $z\in F(J_{M,\lambda}V_\beta)=VI(\mathcal{H},A,M)$.

STEP 4. $x_n\to x^*:=P_{F(S)\cap VI(\mathcal{H},A,M)}x$.

It follows from the definition of x^* that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \|\alpha_n x+(1-\alpha_n)y_n-x^*\|^2\\ &=(1-\alpha_n)^2\|y_n-x^*\|^2+\alpha_n^2\|x-x^*\|^2+2\alpha_n(1-\alpha_n)\langle y_n-x^*,x-x^*\rangle\\ &\le(1-\alpha_n)\|x_n-x^*\|^2+\alpha_n^2\|x-x^*\|^2+2\alpha_n(1-\alpha_n)\langle y_n-x^*,x-x^*\rangle\\ &=(1-\alpha_n)\|x_n-x^*\|^2+\alpha_n^2\|x-x^*\|^2+2\alpha_n(1-\alpha_n)\langle x_n-x^*,x-x^*\rangle.\end{aligned}$$

In view of Lemma 2.4, if we show that

$$\overline{\lim_{n\to\infty}}\langle x_n-x^*,x-x^*\rangle\le 0,$$

then the proof is finished. To this end, let (x_{n_k}) be a subsequence of (x_n) converging weakly to z and

$$\overline{\lim_{n\to\infty}}\langle x_n-x^*,x-x^*\rangle=\lim_{k\to\infty}\langle x_{n_k}-x^*,x-x^*\rangle.$$

By STEP 3, $z\in F(S)\cap VI(\mathcal{H},A,M)$. This together with (2.1) and $x^*:=P_{F(S)\cap VI(\mathcal{H},A,M)}x$ implies that

$$\overline{\lim_{n\to\infty}}\langle x_n-x^*,x-x^*\rangle=\langle z-x^*,x-x^*\rangle\le 0,$$

which is the result as desired. □

Analogously, by using the viscosity approximation method, one can easily get some other algorithms for approximating a solution to problem (1.5).

Theorem 3.6. *Let $A : \mathcal{H} \to \mathcal{H}$ be α-ism with $\alpha > 0$, $M : \mathcal{H} \to 2^{\mathcal{H}}$ a maximal monotone mapping, $f : \mathcal{H} \to \mathcal{H}$ a contractive mapping, and $S : \mathcal{H} \to \mathcal{H}$ a nonexpansive mapping. Choose $\lambda \in (0, 2\alpha)$ and define a sequence (x_n) by the iterative procedure:*

$$\begin{cases} x_0 = x \in \mathcal{H}, \\ y_n = J_{M,\lambda}(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S y_n. \end{cases} \tag{3.5}$$

If (α_n) is chosen in $(0, 1)$ so that conditions (C1) and (C2) are satisfied, then the sequence (x_n) generated by (3.5) converges strongly to $x^* = P_{F(S)\cap VI(\mathcal{H},A,M)} f$, whenever $F(S) \cap VI(\mathcal{H}, A, M) \neq \emptyset$.

Theorem 3.7. *Let $A : \mathcal{H} \to \mathcal{H}$ be α-ism with $\alpha > 0$, $M : \mathcal{H} \to 2^{\mathcal{H}}$ a maximal monotone mapping, $f : \mathcal{H} \to \mathcal{H}$ a contractive mapping, and $S : \mathcal{H} \to \mathcal{H}$ a nonexpansive mapping. Choose $\lambda \in (0, 2\alpha)$ and define a sequence (x_n) by the iterative procedure:*

$$\begin{cases} x_0 = x \in \mathcal{H}, \\ y_n = J_{M,\lambda}(x_n - \lambda A x_n), \\ x_{n+1} = S[\alpha_n f(x_n) + (1 - \alpha_n) y_n]. \end{cases} \tag{3.6}$$

If (α_n) is chosen in $(0, 1)$ so that the conditions (C1) and (C2) are satisfied, then the sequence (x_n) generated by (3.6) converges strongly to $x^* = P_{F(S)\cap VI(\mathcal{H},A,M)} f$, whenever $F(S) \cap VI(\mathcal{H}, A, M) \neq \emptyset$.

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