Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 5259–5270
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Multivalent guiding functions in the bifurcation problem of differential inclusions

Sergey Kornev[a,*], Yeong-Cheng Liou[b]

[a]*Faculty of Physics and Mathematics, Voronezh State Pedagogical University, Lenina 86, 394043 Voronezh, Russia.*

[b]*Department of Healthcare Administration and Medical Informatics; and Research Center of Nonlinear Analysis and Optimization and Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung 807, Taiwan.*

Communicated by Y. H. Yao

Abstract

In this paper we use the multivalent guiding functions method to study the bifurcation problem for differential inclusions with convex-valued right-hand part satisfying the upper Carathéodory and the sublinear growth conditions.

Keywords: Differential inclusion, bifurcation of periodic solution, multivalent guiding function, topological degree.
2010 MSC: 34C25, 34A60.

1. Introduction and preliminaries

The base of the method of guiding functions was laid by Krasnosel'skii and Perov (see [17–19]). The method of multivalent guiding functions became one of the most important directions of its development in the case of differential equations (see [28]).

It is well-known that the application of topological degree methods to the study of various problems of the theory of differential inclusions is very effective (see [2–6]).

A number of works was devoted to the extension of the guiding functions method to the case of differential inclusions and this approach demonstrated its effectiveness to the study of periodic problems. The classical method of guiding potential was used by Borisovich et al. [2] and Górniewicz [5]. The method of integral guiding functions and some of its versions were developed in [8, 9, 13, 16, 27] and the method of multivalent

*Corresponding author
Email addresses: kornev_vrn@rambler.ru (Sergey Kornev), simplex_liou@hotmail.com (Yeong-Cheng Liou)

Received 2016-08-16

guiding functions was extended to differential inclusions in [7, 10, 12]. For some other applications of the guiding functions method see, for example, [11, 14, 15, 25].

Notice that now the bifurcation phenomena in dynamical systems governed by the various classes of differential inclusions were studied by not only the classical method of guiding functions [20], but also by the method of integral guiding functions [21–24, 26, 27].

In the present paper, developing the abstract approach proposed in [20], the method of multivalent guiding functions is used to investigate the bifurcation problem for some classes of differential inclusions. More precisely, we consider the bifurcation problem for nonlinear systems governed by differential inclusions with convex-valued right-hand parts satisfying T-periodicity condition in the first argument, the upper Carathéodory and the sublinear growth conditions.

In what follows we will use some known notions and notations from the theory of multivalued maps (multimaps) (see [2–6]). We recall some of them as follows.

Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. By the symbols $P(Y)$ and $K(Y)$ we denote the collections of all nonempty and, respectively, nonempty and compact subsets of the space Y. If Y is a normed space, $Kv(Y)$ denotes the collection of all nonempty convex compact subsets of Y.

Definition 1.1. A multimap $F : X \to P(Y)$ is called upper semicontinuous (u.s.c.) at the point $x \in X$ if for each open set $V \subset Y$ such that $F(x) \subset Y$ there exists $\delta > 0$ such that $d_X(x, x') < \delta$ implies $F(x') \subset V$. A multimap $F : X \to P(Y)$ is called u.s.c. if it is u.s.c. at each point $x \in X$.

Definition 1.2. A multimap $F : X \to P(Y)$ is called closed if its graph

$$\Gamma_F = \{(x, y) \mid \ (x, y) \in X \times Y, \quad y \in F(x)\}$$

is a closed subset of the space $X \times Y$.

Definition 1.3. A multimap $F : X \to P(Y)$ is called compact if its range $F(X)$ is relatively compact in Y.

Remark 1.4. If multimap $F : X \to P(Y)$ is closed and compact, then it is u.s.c..

A multimap will be called multifunction if it is defined on a subset of $\mathbb{R}$. Let I be a closed subset of $\mathbb{R}$ endowed with the Lebesgue measure.

Definition 1.5. A multifunction $F : I \to K(Y)$ is called measurable if, for each open subset $V \subset Y$, its pre-image

$$F^{-1}(V) = \{t \in I : F(t) \subset V\}$$

is the measurable subset of I.

Remark 1.6. Each measurable multifunction $F : I \to K(Y)$ has a measurable selection, i.e., there exists such measurable function $f : I \to Y$, that $f(t) \in F(t)$ for almost every (a.e.) $t \in I$.

Let Δ be a compact subset of $\mathbb{R}$.

Definition 1.7. A multimap $F : I \times \mathbb{R}^n \times \Delta \to Kv(\mathbb{R}^n)$ is called the upper Carathéodory multimap if

(i) for each $x \in \mathbb{R}^n$, $\mu \in \Delta$, multifunction $F(\cdot, x, \mu) : I \to Kv(\mathbb{R}^n)$ is measurable;

(ii) for μ-a.e. $t \in I$ multimap $F(t, \cdot, \cdot) : \mathbb{R}^n \times \Delta \to Kv(\mathbb{R}^n)$ is u.s.c..

Definition 1.8. A multimap $F : I \times \mathbb{R}^n \times \Delta \to Kv(\mathbb{R}^n)$ satisfies the sublinear growth, if there is a positive Lebesgue integrable function $\alpha(\cdot)$ such that for all $x \in \mathbb{R}^n$, $\mu \in \Delta$, at a.e. $t \in I$

$$\|F(t, x, \mu)\| := \max_{y \in F(t,x,\mu)} \|y\| \le \alpha(t)(1 + \|x\|).$$

In the sequel, we use some aspects of the bifurcation theory in the following situation (see [5, 20]).

Let $A \subset U \subset \mathbb{R}^n$, where A is compact and U is open in $\mathbb{R}^n$. We identify the n-sphere $S^n = \{x \in \mathbb{R}^n | \|x\| = 1\}$ with $\mathbb{R}^n \cup \{\infty\}$. By Π_k, $k \geq 0$, we denote the k-th stable homotopy group of spheres, i.e.,

$$\Pi_k := \varinjlim_{n \geq 0} \pi_{n+k}(S^n).$$

Let $\mathcal{A}$ be a sheaf of Abelian groups over Y, $f : X \to Y$ be a closed surjection, the symbol $\mathcal{A}_y$ denotes the fibre of a sheaf $\mathcal{A}$ over $y \in Y$. By $\mathcal{A}^*$ we denote the inverse image of a sheaf $\mathcal{A}$ under a map f. For an integer $k \geq 1$, define

$$s^0(f; \mathcal{A}) := \{y \in Y \mid H^0(f^{-1}(y); \mathcal{A}^*) \neq \mathcal{A}_y\},$$
$$s^k(f; \mathcal{A}) := \{y \in Y \mid H^k(f^{-1}(y); \mathcal{A}^*) \neq 0\},$$

where $H^*(\cdot; \mathcal{A})$ denotes the Čhech cohomology groups with coefficients in the sheaf $\mathcal{A}$ and, for integers $N \geq 1$, let us define the Vietoris indices of f by

$$i^N(f; \mathcal{A}) := \inf\{n \geq 0 \mid \max_{0 \leq k \leq N-1} \{rd_Y(s^k(f; \mathcal{A})) + k\} + 1 < n\},$$

where for $A \subset Y$, $rd_Y(A) := \sup\{\dim C \mid C \text{ is closed in } Y,\ C \subset A\}$ and dim denotes the topological dimension of a set (see [1]).

We set $i(f; \mathcal{A}) = \sup_{N \geq 0} i^N(f; \mathcal{A})$. If a sheaf $\mathcal{A}$ is constant and equals $\mathbb{Z}$, then $i(f; \mathbb{Z})$ is denoted by $i(f)$.

Definition 1.9. Let $\nu : Z \to X$. We say that ν belongs to the class $\mathcal{V}$ (ν is a $\mathcal{V}$-map) if

(i) ν is the perfect surjection, i.e., the surjection with compact fibres;

(ii) $i(\nu) < \infty$.

We say that $\nu : Z \to X$ is a $\widetilde{\mathcal{V}}$-map if ν is a $\mathcal{V}$-map and

(iii) $\dim \nu := \sup_{x \in X} \dim \nu^{-1}(x) < \infty$.

Let $(X, X'), (Y, Y')$ be pairs of spaces and $m \geq 0$. By $D_m(X, X'; Y, Y')$ (resp. $\widetilde{D}_m(X, X'; Y, Y')$) we denote the class of all cotriads

$$(X, X') \overset{\nu}{\leftarrow} (Z, Z') \overset{\chi}{\rightarrow} (Y, Y'),$$

where ν is a $\mathcal{V}_m$-map (resp. $\widetilde{\mathcal{V}}_m$-map) and χ is a continuous map. Additionally, we put

$$D(X, X'; Y, Y') := \bigcup_{m \geq 0} D_m(X, X'; Y, Y'),$$

($\widetilde{D} = \bigcup_{m \geq 0} \widetilde{D}_m$); hence $\widetilde{D} \subset D$.

Definition 1.10. We say that cotriads

$$(X, X') \overset{\nu_i}{\leftarrow} (Z_i, Z_i') \overset{\chi_i}{\rightarrow} (Y, Y'), \quad i = 1, 2$$

from $D(X, X'; Y, Y')$ (resp. $\widetilde{D}$) are equivalent (written $(\nu_1, \chi_1) \approx (\nu_2, \chi_2)$) if there exists a cotriad

$$(X, X') \overset{\nu}{\leftarrow} (Z, Z') \overset{\chi}{\rightarrow} (Y, Y')$$

(resp. with finite-dimensional map ν) and $\mathcal{V}_0$-maps $f_i : (Z, Z') \to (Z_i, Z_i')$ such that $\nu_i \circ f_i = \nu$ and $\chi_i \circ f_i = \chi$, $i = 1, 2$.

Definition 1.11. Elements of the quotient

$$M(X, X'; Y, Y') = D(X, X'; Y, Y') / \approx$$

or

$$\widetilde{M}(X, X'; Y, Y') = \widetilde{D}(X, X'; Y, Y') / \approx$$

are called morphisms (resp. finite-dimensional morphisms).

By $M_m(X, X'; Y, Y')$ (resp. $\widetilde{M}_m(X, X'; Y, Y')$, $m \geq 0$, we denote the set of all morphisms from $M(X, X'; Y, Y')$ (resp. $\widetilde{M}$) which are represented by cotriads $(\nu, \chi) \in D(X, X'; Y, Y')$ (resp. $\widetilde{D}$).

By $\mathcal{C}(m, n)$, $m \geq n$, we denote the class of all pairs (f, U), where U is an open bounded subset of $\mathbb{R}^m$ and $f : (\overline{U}, \partial U) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ is a continuous map.

We say that $(f_0, U), (f_1, U)$ from $\mathcal{C}(m, n)$ are homotopic if there is a homotopy $h : (\overline{U} \times [0, 1], \partial U \times [0, 1]) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ such that $h(\cdot, i) = f_i$, $i = 0, 1$.

Let B^m be a closed ball centered at zero of radius $r = 1$, $(f, U) \in \mathcal{C}(m, n)$ where $n \leq m < 2n - 2$. Assume, without loss of generality, that $\overline{U} \subset B^m$ and that $\dim \partial U \leq m - 1$. Consider the following sequence of Abelian groups and homomorphisms

$$\pi^{n-1}(\partial U) \xrightarrow{\delta_1} \pi^n(\overline{U}, \partial U) \xleftarrow{j^\sharp} \pi^n(B^m, B^m \backslash U) \xrightarrow{i^\sharp} \pi^n(B^m, S^{m-1}) \xleftarrow{\delta_2} \pi^{n-1}(S^{m-1})$$

in which δ_1 denotes the coboundary homomorphism of the pair $(\overline{U}, \partial U)$, $j : (\overline{U}, \partial U) \to (B^m, B^m \backslash U)$, $i : (B^m, S^{m-1}) \to (B^m, B^m \backslash U)$ are the inclusions and δ_2 is the coboundary homomorphism of the pair (B^m, S^{m-1}). Clearly $j^\sharp$ is the excision isomorphism and δ_2 is an isomorphism in view of the contractibility of B^m and the exactness of the cohomotopy sequence of the pair (B^m, S^{m-1}). Let

$$\kappa = \delta_2^{-1} \circ i^\sharp \circ (j^\sharp)^{-1} \circ \delta_1$$

and let $\eta := [f|\partial U] \in \pi^{n-1}(\partial U)$, where $[f|\partial U]$ denotes the homotopy class of $f|\partial U$ and $\pi^{n-1}(\partial U)$ denotes $(n-1)$-th cohomotopy group of ∂U. Without loss of generality we have identified here $[\partial U; \mathbb{R}^n \setminus 0]$ with $\pi^{n-1}(\partial U)$.

Definition 1.12. The generalized degree of f on U is the element

$$\deg(f, U) := \kappa(\eta) \in \pi^{n-1}(S^{m-1}) \cong \Pi_{m-n}.$$

Let U be an open subset of $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^k$. Consider the problem of the bifurcation of solutions to the inclusion

$$0 \in \Phi(z, \lambda), \tag{1.1}$$

where $\Phi : U \multimap \mathbb{R}^n$ is a multifield corresponding to a multimap $F : U \multimap \mathbb{R}^n$, i.e., $\Phi(z) = z - F(z)$.

Let us make the following assumptions:

(1) $\Phi \in \widetilde{M}_n(U; \mathbb{R}^n)$ is a morphism such that $0 \in \Phi(0, \lambda)$ for all $\lambda \in \Lambda := \{\lambda \in \mathbb{R}^k \mid (0, \lambda) \in U\}$.

We define the set of nontrivial solutions to (1.1) as

$$S := \{(z, \lambda) \in U \setminus \Lambda \times \{0\} \mid z \neq 0,\ 0 \in \Phi(z, \lambda)\},$$

and suppose that

(2) the set of bifurcation points

$$B(\Phi) := \{(0, \lambda) \in \Lambda \times \{0\} \mid (0, \lambda) \in \overline{S}\}$$

is compact.

In order to define the bifurcation index of Φ we need some auxiliary objects. Let us consider an arbitrary continuous function $\alpha : \Lambda \to [0, \infty)$ such that, for $(0, \lambda) \notin B(\Phi)$,

$$0 < \alpha(\lambda) < d((0, \lambda), \partial U \cup \overline{S})$$

and

$$\alpha(\lambda) = 0$$

for $(\lambda, 0) \in B(\Phi)$. For instance we may put

$$\alpha(\lambda) = \min\left\{1, \frac{1}{2}d((0,\lambda), \partial U \cup \overline{S})\right\}.$$

Next we set

$$X := \{(z,\lambda) \in \mathbb{R}^m \mid \lambda \in \Lambda,\ \|z\| = \alpha(\lambda)\},$$
$$X^+ := \{(z,\lambda) \in \mathbb{R}^m \mid \lambda \in \Lambda,\ \|z\| < \alpha(\lambda)\}.$$

Observe that $X^+ \cup X \subset U$ and put $X^- := U \backslash \overline{X}^+$. It is easy to see that $S \subset X^-$ and $B(\Phi) \subset X$.

Let $f : U \to \mathbb{R}$ be a continuous function such that

$$f(z,\lambda) = \begin{cases} < 0 & \text{for } (z,\lambda) \in X^-, \\ = 0 & \text{for } (z,\lambda) \in X \ , \\ > 0 & \text{for } (z,\lambda) \in X^+. \end{cases}$$

Now we consider a morphism Ψ from $\widetilde{M}_n(U;\mathbb{R}^{n+1})$ such that, for all $(z,\lambda) \in U$, $\Psi(z,\lambda) = \Phi(z,\lambda) \times \{f(z,\lambda)\}$.

Since, by (ii), the set of zeros of Ψ is compact, there is an open bounded set U' such that $\overline{U'} \subset U$ and $0 \notin \Psi(z,\lambda)$ for $(z,\lambda) \in U \setminus U'$. Therefore $(\Psi, U') \in \widetilde{M}(m, n+1)$.

Definition 1.13. The bifurcation index $\mathrm{Bi}\,(\Phi)$ of Φ is defined by the following formula

$$\mathrm{Bi}\,(\Phi) := \deg(\Psi, U') \in \Pi_{k-1}.$$

We will need the following version of the global bifurcation result of Kryszewski (see [5, 20]).

In addition to above assumptions let us suppose that

(3) there is an open set $U_1 \supset U$ and a morphism $\Phi_1 \in \widetilde{M}_n(U_1;\mathbb{R}^n)$ such that $\Phi_1|U = \Phi$ and $0 \in \Phi_1(0,\lambda)$ for all $(0,\lambda) \in \{0\} \times U_1 \cap \mathbb{R}^k$. Let

$$S_1 := \{(z,\lambda) \in U_1 \,|\, z \neq 0,\ 0 \in \Phi_1(z,\lambda)\}.$$

Lemma 1.14. *Let K be a compact subset of U_1 such that $B(\Phi) \subset K$ and $\{0\} \times K \cap (\mathbb{R}^k \backslash \Lambda) = \emptyset$ (e.g. $K = B(\Phi)$). If $\mathrm{Bi}\,(\Phi) \neq 0$, then there is a nonempty connected set $C \subset S_1 \backslash K$ such that $\overline{C} \cap K \neq \emptyset$ and at least one of the followings occurs:*

(i) *C is unbounded;*

(ii) $\overline{C} \cap \partial U_1 \neq \emptyset$;

(iii) *there is a point $\lambda_* \in \mathbb{R}^k \backslash \Lambda$ such that $(0,\lambda_*) \in U_1$ and $(0,\lambda_*) \in \overline{C}$. Thus Φ_1 has bifurcation points outside U connected to K in $\overline{S}_1$.*

2. Main result

We shall study the periodic problem for a family of differential inclusions of the following form:

$$\begin{aligned} &z'(t) \in F(t, z(t), \mu), \\ &z(0) = z(T), \end{aligned} \tag{2.1}$$

under assumptions that

(H_1) $F : \mathbb{R} \times \mathbb{R}^n \times \Lambda \to Kv(\mathbb{R}^n)$ is a T-periodic multimap ($T > 0$), satisfying the upper Carathéodory conditions and the sublinear growth condition;

(H_2) for each $\mu \in \Lambda$ problem (2.1) admits a solution $z : [0,T] \to \mathbb{R}^n$ with $z(0) = z(T) = 0$.

By a solution of problem (2.1) we mean a pair (z, μ), satisfying inclusion (2.1) a.e. on $[0,T]$, where $z \in C([0,T]; \mathbb{R}^n)$ is a T-periodic absolutely continuous function, $\mu \in \Lambda$. From (H_2) it follows that $(0, \mu)$ is a solution to problem (2.1) for each $\mu \in \Lambda$. These solutions are called trivial. Let us denote by S the set of all nontrivial solutions of problem (2.1).

Let $\mathbb{R}^n = \mathbb{R}^{n-2} \times \mathbb{R}^2$ be a metric space. Denote by q the operator of projection on $\mathbb{R}^2$ and $p = i - q$, where i is the identity map. The elements of $\mathbb{R}^2$ and $\mathbb{R}^{n-2}$ are denoted by ξ and ζ, respectively. Let φ, ρ be polar coordinates in $\mathbb{R}^2$.

We consider the multivalent Riemann surface

$$\Pi = \{(\varphi, \rho) : \varphi \in (-\infty, \infty), \rho \in (0, \infty)\}.$$

On $\Pi \times \mathbb{R}$ we define a continuously differentiable in the first argument and continuous in the second argument function $W(\xi, \mu)$ such that

$$\frac{\partial W(\varphi, \rho, \mu)}{\partial \varphi} > 0, \quad (\varphi, \rho) \in \Pi, \ \mu \in \Lambda, \tag{2.2}$$

$$W(\varphi + 2\pi, \rho, \mu) = W(\varphi, \rho, \mu) + 2\pi, \quad (\varphi, \rho) \in \Pi, \ \mu \in \Lambda. \tag{2.3}$$

On $\mathbb{R}^{n-2} \times \Lambda$ let $V(\zeta, \mu)$ be a continuously differentiable in the first argument and continuous in the second argument function such that $\frac{\partial V(0,\mu)}{\partial \zeta} = 0$ and the following coercivity condition

$$\lim_{\|\zeta\| \to \infty} V(\zeta, \mu) = +\infty \tag{2.4}$$

holds true.

For each $\mu \in \Lambda$, choose $\rho_1 := \rho_1(\mu)$, $\rho_2 := \rho_2(\mu)$ such that $0 \le \rho_1 < \rho_2$ and for $\vartheta_0 := \min V(\zeta, \mu)$, take $\vartheta := \vartheta(\mu)$ such that $\vartheta > \vartheta_0$. We define the following domain

$$\Omega_\mu(\vartheta, \rho_1, \rho_2) = \{z \in \mathbb{R}^n : V(pz, \mu) < \vartheta, \rho_1 < \|qz\| < \rho_2\}.$$

We assume that on $[0,T]$ continuous functions $\alpha(t,\mu)$, $\beta(t,\mu)$ are given such that, for each $\mu \in \Lambda$ and a.e. $t \in [0,T]$ the following holds:

$$\sup_{z \in \Omega_\mu(\vartheta,\rho_1,\rho_2)} \sup_{y \in F(t,z,\mu)} \left\langle \frac{\partial W(qz,\mu)}{\partial qz}, qy \right\rangle < \alpha(t,\mu), \tag{2.5}$$

$$\inf_{z \in \Omega_\mu(\vartheta,\rho_1,\rho_2)} \inf_{y \in F(t,z,\mu)} \left\langle \frac{\partial W(qz,\mu)}{\partial qz}, qy \right\rangle > \beta(t,\mu). \tag{2.6}$$

Let us give the following definition.

Definition 2.1. A pair of functions $\{V(\zeta,\mu), W(\xi,\mu)\}$ with properties (2.2)-(2.6) is called the multivalent guiding function (MGF) for inclusion (2.1) on $\Omega_\mu(\vartheta, \rho_1, \rho_2)$ if the following conditions hold true:

$$\sup_{t \in [0,T]} \sup_{y \in F(t,z,\mu)} \frac{|\langle qy, qz \rangle|}{\|qz\|} < \frac{\rho_2 - \rho_1}{2T}, \quad z \in \Omega_\mu(\vartheta, \rho_1, \rho_2), \tag{2.7}$$

$$\left\langle \frac{\partial V(pz,\mu)}{\partial pz}, py \right\rangle < 0, \qquad y \in F(t,z,\mu), \ V(pz,\mu) \ge \vartheta, \ \|qz\| \le \rho_2, \tag{2.8}$$

$$2\pi(N_\mu - 1) < \int_0^T \alpha(\tau,\mu) d\tau, \quad \int_0^T \beta(\tau,\mu) d\tau < 2\pi N_\mu, \tag{2.9}$$

where N_μ is an integer and $\alpha(t,\mu), \beta(t,\mu)$ are functions from (2.5), (2.6), respectively.

For all $\mu \in \Lambda$ and $\rho_0(\mu) = \rho_0 = (\rho_1 + \rho_2)/2$ we define

$$G_\mu(\vartheta, \rho_0) = \{z \in \mathbb{R}^n : V(pz, \mu) < \vartheta,\ \|qz\| < \rho_0\},$$
$$\partial G_\mu(\vartheta, \rho_0) = \partial G_\zeta(\vartheta) \times \overline{G_\xi(\rho_0)} \cup \overline{G_\zeta(\vartheta)} \times \partial G_\xi(\rho_0).$$

Let us define the map $\nabla V : \mathbb{R}^{n-2} \times \Lambda \to \mathbb{R}^{n-2}$ by

$$\nabla V(\zeta, \mu) = \frac{\partial V(\zeta, \mu)}{\partial \zeta}.$$

We assume that for fixed $r > \eta > 0$, $\mu_0 \in \Lambda$ the following condition is satisfied

$$\nabla V(pz, \mu) \neq 0 \tag{2.10}$$

for all $z \in \overline{G_\mu(\vartheta, \rho_0)}$ and $\mu : r - \eta \leq |\mu - \mu_0| \leq r + \eta$.

Now we are in position to formulate the main result of this paper.

Theorem 2.2. *Suppose that conditions* (H$_1$) *and* (H$_2$) *are satisfied. Let* $\{V(\zeta, \mu), W(\xi, \mu)\}$ *be MGF for inclusion* (2.1) *on* $\Omega\mu(\vartheta, \rho_1, \rho_2)$ *for each* $\mu : |\mu - \mu_0| \geq r$.

Then one of the following cases occurs:

(i) *there exists a sequence* $\{(y_n, \mu_n)\}_{n=1}^{\infty}$, $\mu_n \to \overline{\mu}$, $|\overline{\mu} - \mu_0| = r$, $y_n \in \mathbb{R}^n$, $y_n \neq y_m$ *for* $n \neq m$, *and a sequence* (z_n) *of solutions of problem* (2.1) *for* $\mu = \mu_n$ *such that* $z_n(0) = z_n(T) = y_n \to 0$ *and* $z_n \to \overline{z}$, *where* $\overline{z}$ *is a solution of problem* (2.1) *for* $\mu = \overline{\mu}$ *such that* $\overline{z}(0) = \overline{z}(T) = 0$;

(ii) *there exists a connected set* C *of points* (y, μ) *with* $y \neq 0$ *such that*

- $(0, \overline{\mu}) \in \overline{C}$ *where* $|\overline{\mu} - \mu_0| < r$,
- C *is unbounded or* $\overline{C} \cap \partial U \neq \emptyset$ *or* $(0, \widetilde{\mu}) \in \overline{C}$ *for some* $\widetilde{\mu} \in \Lambda : |\widetilde{\mu} - \mu_0| > r$,
- *each point* $(y, \mu) \in \overline{C}$ *corresponds to a solution* $z : [0, T] \to \mathbb{R}^n$ *of inclusion* (2.1) *with* $z(0) = z(T) = y$. *In particular, there is a sequence* $(z_n)_{n=1}^{\infty}$ *of solutions to inclusion* (2.1) *for* $\mu = \mu_n$, $z_n(0) = z_n(T) = y_n$, *where* $\mu_n \to \overline{\mu}$ *in* Λ *with* $\|\overline{\mu} - \mu_0\| < r$, *converging to a solution* $\overline{z}$ *to inclusion* (2.1) *for* $\mu = \overline{\mu}$, $\overline{z}(0) = \overline{z}(T) = 0$.

Proof.

Claim 1. First of all we shall show that the trajectory $z(t)$, starting at $\partial G_\mu(\vartheta, \rho_0)$, satisfies the following estimate

$$z(t) \in G_\mu(\vartheta, \rho_2) \tag{2.11}$$

for $t \in (0, T]$. We consider a component $\zeta(t)$ of the trajectory $z(t)$. If $\zeta(0)$ is an interior point of $G_\zeta(\vartheta)$, then there exists $\varepsilon_1 > 0$, such that

$$\zeta(t) \in G_\zeta(\vartheta), \quad t \in (0, \varepsilon_1). \tag{2.12}$$

Let us take $\zeta(0) \in \partial G_\zeta(\vartheta)$, i.e., $V(\zeta(0), \mu) = \vartheta$. Since $\|\xi(0)\| \leq \rho_0 < \rho_2$, from (2.8) it follows that

$$\langle \nabla V(\zeta(0), \mu), py \rangle < 0 \quad \text{for all } \ y \in F(0, \zeta(0), \xi(0), \mu).$$

Then for small $t > 0$ we have $V(\zeta(t), \mu) < \vartheta$. So for some $\varepsilon_1 > 0$ estimate (2.12) holds true.

Considering the component $\xi(t)$, obviously for some $\varepsilon_2 > 0$ we obtain

$$\xi(t) \in G_\xi(\rho_2), \quad t \in (0, \varepsilon_2). \tag{2.13}$$

From (2.12) and (2.13) it follows that $z(t) \in G_\mu(\vartheta, \rho_2),\ 0 < t < \min\{\varepsilon_1, \varepsilon_2\}$. It means that there exists a positive number

$$t_* = \sup\{t > 0 : \quad z(t) \in G_\mu(\vartheta, \rho_2)\}.$$

Notice that inclusion (2.11) is equivalent to the following estimate: $t_* > T$.

Since $z(t_*) \in \partial G_\mu(\vartheta, \rho_2)$, we have $V(\zeta(t_*), \mu) = \vartheta$ or $\|\xi(t_*)\| = \rho_2$. From (2.8) it follows that $V(\zeta(t_*), \mu) < \vartheta$. Therefore $\|\xi(t_*)\| = \rho_2$ and from $\|\xi(0)\| \le \rho_0$ we obtain

$$\|\xi(t_*)\| - \|\xi(0)\| \ge \rho_2 - \rho_0 = (\rho_2 - \rho_1)/2.$$

Let $\varphi(t)$, $\rho(t)$ be polar coordinates of $\xi(t)$. Then

$$\rho(t_*) - \rho(0) \ge (\rho_2 - \rho_1)/2.$$

Therefore,

$$\max_{t\in[0,t_*]} \|\rho'(t)\| \ge (\rho_2 - \rho_1)/2t_*. \tag{2.14}$$

On the other hand, since

$$\|\rho'(t)\| = \frac{\langle qy, \xi(t)\rangle}{\|\xi(t)\|}, \quad y \in F(t, \zeta(t), \xi(t), \mu),$$

and $z(t) \in G_\mu(\vartheta, \rho_2)$ for $t \in (0, t_*)$, from (2.7) it follows the estimate

$$\max_{t\in[0,t_*]} \|\rho'(t)\| < (\rho_2 - \rho_1)/2T. \tag{2.15}$$

Comparing (2.14) and (2.15) we see that $t_* > T$. Therefore, any trajectory $z(\cdot)$, starting at $\partial G_\mu(\vartheta, \rho_0)$ for $t \in (0, T]$, satisfies the estimate $z(t) \in G_\mu(\vartheta, \rho_2)$.

Claim 2. Let us take $z(0) \in \overline{G_\zeta(\vartheta)} \times \partial G_\xi(\rho_0)$ and

$$\rho(0) = \rho_0 = (\rho_1 + \rho_2)/2.$$

Since $z(t) \in G_\mu(\vartheta, \rho_2)$ for $t \in (0, T]$, we have the estimate

$$\max_{[0,t_*]} |\rho'(t)| < (\rho_2 - \rho_1)/2T.$$

Therefore,

$$\rho(t) > \rho(0) - (\rho_2 - \rho_1)t/2T, \quad t \in (0, T]$$

and we obtain

$$\rho(t) > \rho_1, \quad t \in (0, T].$$

Then

$$z(t) \in \Omega_\mu(\vartheta, \rho_1, \rho_2), \quad t \in (0, T].$$

Let us denote $\omega(t, \mu) = W(\xi(t), \mu)$. The map $\nabla W : \mathbb{R}^2 \times \Lambda \to \mathbb{R}^2$ is defined as

$$\nabla W(\xi, \mu) = \frac{\partial W(\xi, \mu)}{\partial \xi}.$$

Then for each $\mu \in \Lambda$

$$\omega'(t, \mu) = \langle \nabla W(\xi(t), \mu), qy\rangle,$$

where $y \in F(t, z, \mu)$ and

$$\beta(t, \mu) < \omega'(t, \mu) < \alpha(t, \mu).$$

Now by using the integral representation of the function $\omega(t, \mu)$ we obtain

$$\int_0^T \beta(\tau, \mu)d\tau < \omega(T, \mu) - \omega(0, \mu) < \int_0^T \alpha(\tau, \mu)d\tau. \tag{2.16}$$

From (2.9) it follows that

$$2\pi(N_\mu - 1) < \omega(T, \mu) - \omega(0, \mu) < 2\pi N_\mu.$$

Then we have that $\xi(T) \neq \xi(0)$ for $\mu :\ r + \varepsilon/2 \le |\mu - \mu_0| < r + \varepsilon$, $1/2 < \lambda < 1$ and $z_0 \in \partial G_\mu(\vartheta, \rho_0)$.

Claim 3. Let us define a map $f_V : \mathbb{R}^{n-2} \times \Lambda \to \mathbb{R}^{n-2}$ by

$$f_V(\zeta, \mu) = \begin{cases} \nabla V(\zeta, \mu), & \text{if } \|\nabla V(\zeta, \mu)\| \leq 1, \\ \frac{\nabla V(\zeta,\mu)}{\|\nabla V(\zeta,\mu)\|}, & \text{if } \|\nabla V(\zeta, \mu)\| > 1, \end{cases}$$

and a map $P : [0, T] \times \mathbb{R}^{n-2} \times \Lambda \to \mathbb{R}^{n-2}$ by the formula

$$P(t, \zeta_0, \mu) = \zeta(t) - \zeta_0,$$

where $\zeta : [0, T] \to \mathbb{R}^{n-2}$ is the unique solution to the problem $\zeta'(t) = f_V(\zeta, \mu)$, $\zeta(0) = \zeta_0$. It is clearly a well-defined single-valued (continuous) map since f_V is bounded and it satisfies the Lipschitz condition with respect to the second variable.

Let $\Phi : [0, T] \times \mathbb{R}^n \times \Lambda \multimap \mathbb{R}^n$ be given by

$$\Phi(t, z, \mu) = \{z(t) - z_0 \mid z'(s) \in F(s, z(s), \mu) \,\text{a.e.}\, s \in [0, T], z(0) = z_0, pz(0) = \zeta_0\}.$$

Assume that there is $\varepsilon : 0 < \varepsilon \leq \eta/2$ such that, for all $\mu : r + \varepsilon/2 \leq |\mu - \mu_0| < r + \varepsilon$ and $z_0 \in \overline{G_\mu(\vartheta, \rho_0)}$

$$0 \notin \Phi(T, z_0, \mu). \tag{2.17}$$

If condition (2.17) is not satisfied, then there are sequences $\mu_n \to \overline{\mu} \in \Lambda$ and $y_n \to 0$ in $\mathbb{R}^n$ such that $\mu_n \neq \mu_m, y_n \neq y_m$ for $n \neq m$, $|\overline{\mu} - \mu_0| = r$ and

$$0 \in \Phi(T, y_n, \mu_n).$$

Hence there is a sequence of solutions $z_n : [0, T] \to \mathbb{R}^n$ to problem (2.1) for $\mu = \mu_n$ such that $z_n(0) = z_n(T) = z_n$. Notice that the Gronwall inequality implies $z_n \to \overline{z}$ in $C([0, T]; \mathbb{R}^n)$. Then $\overline{z}$ is the solution of problem (2.1) for $\mu = \overline{\mu}$ such that $\overline{z}(0) = \overline{z}(T) = 0$.

Claim 4. For each $\mu \in \Lambda : r + \varepsilon/2 \leq |\mu - \mu_0| < r + \varepsilon$; $z_0 \in \partial G_\zeta(\vartheta) \times \overline{G_\xi(\rho_0)}$ and $t \in [0, T]$, we shall show that

$$0 \neq P(t, \zeta_0, \mu).$$

Indeed, take the solution $\zeta(t)$ to the problem $\zeta'(t) = f_V(\zeta(t), \mu)$, $\zeta(0) \in \partial G_\zeta(\vartheta)$. Then

$$\begin{aligned} V(\zeta(t), \mu) - V(\zeta(0), \mu) &= \int_0^t \langle \nabla V(\zeta(s), \mu), \zeta'(s) \rangle ds \\ &= \int_0^t \langle \nabla V(\zeta(s), \mu), f_V(\zeta(s), \mu) \rangle ds > 0. \end{aligned}$$

Thus $\zeta(t) \neq \zeta(0) = \zeta_0$ and

$$0 \neq P(t, \zeta_0, \mu).$$

Claim 5. For $t \in [0, T]$, consider a map $h_t : \mathbb{R}^{n-2} \times \Lambda \times [0, 1] \to \mathbb{R}^{n-2}$,

$$h_t(\zeta_0, \mu, \lambda) = (1 - \lambda)\nabla V(\zeta_0, \mu) + \lambda P(t, \zeta_0, \mu).$$

We shall show that there is $\tau \in [0, T]$ such that, for $\mu \in \Lambda : r + \varepsilon/2 \leq |\mu - \mu_0| < r + \varepsilon$; $\zeta(0) \in \partial G_\zeta(\vartheta)$ and $\lambda \in [0, 1]$,

$$0 \neq h_\tau(\zeta_0, \mu, \lambda).$$

Indeed, the continuity of f_V, V, and (2.10) imply that there is $\tau > 0$ such that for $\zeta(0) \in \partial G_\zeta(\vartheta)$, $\zeta_0' \in \mathbb{R}^{n-2}$ such that $|V(\zeta_0, \mu) - V(\zeta_0', \mu)| \leq \tau$, we have

$$\langle \nabla V(\zeta_0, \mu), f_V(\zeta_0', \mu) \rangle > 0.$$

Now suppose that for $\mu \in \Lambda : r + \varepsilon/2 \leq |\mu - \mu_0| < r + \varepsilon$; $\zeta(0) \in \partial G_\zeta(\vartheta)$ and $\lambda \in [0,1]$

$$h_\tau(\zeta_0, \mu, \lambda) = 0.$$

Then from (2.10) and Claim 4 for each $\lambda \in (0,1)$ we obtain

$$\zeta(\tau) - \zeta_0 = \frac{\lambda - 1}{\lambda} \nabla V(\zeta_0, \mu),$$

where $\zeta' = f_V(\zeta, \mu)$ on $[0,T]$ and $\zeta(0) = \zeta_0$. Since $|f_V| \leq 1$, it is clear that, for all $\theta \in [0, \tau]$, $|V(\zeta(\theta), \mu) - V(\zeta_0, \mu)| \leq \tau$. Thus

$$0 > \langle \zeta(\tau) - \zeta_0, \nabla V(\zeta_0, \mu) \rangle = \int_0^\tau \langle f_V(\zeta(\theta), \mu), \nabla V(\zeta_0, \mu) \rangle d\theta > 0,$$

a contradiction.

Claim 6. Now let

$$k(\lambda) = \begin{cases} 1, & \text{if } \lambda \in [0, \frac{1}{2}), \\ 2 - 2\lambda, & \text{if } \lambda \in [\frac{1}{2}, 1]; \end{cases}$$

and

$$t(\lambda) = \begin{cases} 2(T - \tau)\lambda + \tau, & \text{if } \lambda \in [0, \frac{1}{2}), \\ T, & \text{if } \lambda \in [\frac{1}{2}, 1]. \end{cases}$$

Let us consider a multimap $\Psi' : \mathbb{R}^n \times \Lambda \times [0,1] \multimap \mathbb{R}^n$, given by

$$\Psi'(z_0, \mu, \lambda) = \{z(t(\lambda)) - z_0 \mid z'(\theta) \in k(\lambda) f_V(pz(\theta), \mu) + (1 - k(\lambda)) F(\theta, z(\theta), \mu)\},$$

where $z(0) = z_0$, $pz(0) = \zeta_0$. It is clear that

$$\Psi'(z_0, \mu, \lambda) = \begin{cases} h_\tau(\zeta_0, \mu, 1) = P(\tau, \zeta_0, \mu), & \text{if } \lambda = 0, \\ \Phi(T, z_0, \mu) := \Phi_T, & \text{if } \lambda = 1. \end{cases}$$

In view of condition (2.17) and Claim 5, if $\mu : \ r + \varepsilon/2 \leq |\mu - \mu_0| < r + \varepsilon$ and $z_0 \in \overline{G_\mu(\vartheta, \rho_0)}$, then

$$0 \notin \Psi'(z_0, \mu, \lambda), \quad \lambda = 0, 1.$$

Now we show that also for $z_0 \in \partial G_\mu(\vartheta, \rho_0)$

$$0 \notin \Psi'(z_0, \mu, \lambda) \quad \text{for all } \lambda \in (0,1),$$

i.e., each $\psi' \in \Psi'(z_0, \mu, \lambda)$ is non-zero. By the definition of Ψ',

$$\psi' = z(t(\lambda)) - z_0,$$

where the function $z : [0,T] \to \mathbb{R}^n$ is such that $z(0) = z_0$, $pz(0) = \zeta_0$ and

$$z'(\theta) \in k(\lambda) f_V(pz(\theta), \mu) + (1 - k(\lambda)) F(\theta, z(\theta), \mu),$$

i.e., $z'(\theta) = k(\lambda) f_V(pz(\theta), \mu) + (1 - k(\lambda)) y(\theta)$, where $y(\theta) \in F(\theta, z(\theta), \mu)$. Then, for $0 < \lambda \leq 1/2$ by Claim 4 we obtain

$$\psi' = P(t(\lambda), \zeta_0, \mu) \neq 0.$$

If $1/2 < \lambda < 1$, then

$$V(p(\psi' + z_0), \mu) - V(pz_0, \mu) \geq k(\lambda) \int_0^T \langle \nabla V(pz(\theta), \mu), f_V(pz(\theta), \mu) \rangle d\theta > 0$$

for $z_0 \in \partial G_\zeta(\vartheta) \times \overline{G_\xi(\rho_0)}$. Hence $p\psi' \neq 0$.

Now we shall show that for $z_0 \in \overline{G_\zeta(\vartheta)} \times \partial G_\xi(\rho_0)$ and $\mu :\ r + \varepsilon/2 \leq |\mu - \mu_0| < r + \varepsilon$, $1/2 < \lambda < 1$,

$$q\psi' \neq 0.$$

We have

$$\omega(T,\mu) - \omega(0,\mu) = \int_0^T \omega'(\tau,\mu)d\tau = \int_0^T \langle \nabla W(\xi(\tau),\mu), q\widetilde{y}\rangle d\tau = (1-k(\lambda)) \int_0^T \langle \nabla W(\xi(\tau),\mu), qy\rangle d\tau,$$

where $\widetilde{y} \in k(\lambda) f_V(pz(\tau),\mu) + (1-k(\lambda))F(\tau, z(\tau),\mu))$, $y \in F(\tau, z(\tau),\mu)$. Then by (2.16) we have

$$(1-k(\lambda))2\pi(N_\mu - 1) < \omega(T,\mu) - \omega(0,\mu) < (1-k(\lambda))2\pi N_\mu.$$

It follows that $\xi(T) \neq \xi(0)$, i.e., $q\psi' \neq 0$ $\mu :\ r+\varepsilon/2 \leq |\mu-\mu_0| < r+\varepsilon$, $1/2 < \lambda < 1$ and $z_0 \in \overline{G_\zeta(\vartheta)} \times \partial G_\xi(\rho_0)$.

Claim 7. Finally, we consider a multimap $\Psi : \mathbb{R}^n \times \Lambda \times [0,1] \multimap \mathbb{R}^n$ given by

$$\Psi(z_0,\mu,\lambda) = \begin{cases} h_\tau(pz_0,\mu,2\lambda), & \text{if } \lambda \in [0,1/2], \\ \Psi'(z_0,\mu,2\lambda-1), & \text{if } \lambda \in (1/2,1]. \end{cases}$$

In view of assumptions (2.10), (2.17), and Claims 5 and 6 for $\mu :\ r + \varepsilon/2 \leq |\mu - \mu_0| < r + \varepsilon$ and $z_0 \in \overline{G_\mu(\vartheta,\rho_0)}$ we obtain

$$0 \notin \Psi(z_0,\mu,\lambda), \quad \lambda = 0, 1.$$

For $\mu :\ r + \varepsilon/2 \leq |\mu - \mu_0| < r + \varepsilon$ and $z_0 \in \partial G_\mu(\vartheta,\rho_0)$ we also have

$$0 \notin \Psi(z_0,\mu,\lambda), \quad \lambda \in [0,1].$$

Then

$$\mathrm{Bi}\,(\Phi_T) = \mathrm{Bi}\,(\nabla V) \neq 0.$$

The assertion follows from Lemma 1.14. □

Acknowledgment

The authors are deeply grateful to professor V. Obukhovskii for useful discussions.

This research is supported by the joint Taiwan NSC - Russia RFBR grant 14-01-92004, the RFBR grants 14-01-00468, 16-01-00386, RSF grant 14-21-00066 (in Voronezh State University), MOST 103-2923-E-037-001-MY3, MOST 104-2221-E-230-004 and Kaohsiung Medical University 'Aim for the Top Universities Grant, grant No. KMU-TP103F00'.

References

[1] P. S. Aleksandrov, B. A. Pasynkov, *Vvedenie v teoriyu razmernosti: Vvedenie v teoriyu topologicheskikh prostranstv i obshchuyu teoriyu razmernosti*, (Russian), [Introduction to dimension theory: An introduction to the theory of topological spaces and the general theory of dimension], Izdat. "Nauka", Moscow, (1973). 1

[2] Y. G. Borisovich, B. D. Gel'man, A. D. Myshkis, V. V. Obukhovskiĭ, *Introduction to the theory of multivalued maps and differential inclusions*, (Russian) 2nd Ed., Librokom, Moscow, (2011). 1

[3] K. Deimling, *Multivalued differential equations*, de Gruyter Series in Nonlinear Analysis and Applications, Walter de Gruyter and Co., Berlin, (1992).

[4] A. Fryszkowski, *Fixed point theory for decomposable sets*, Topological Fixed Point Theory and Its Applications, Kluwer Academic Publishers, Dordrecht, (2004).

[5] L. Górniewicz, *Topological fixed point theory of multivalued mappings*, Second edition, Topological Fixed Point Theory and Its Applications, Springer, Dordrecht, (2006). 1, 1, 1

[6] M. Kamenskii, V. V. Obukhovskiĭ, P. Zecca, *Condensing multivalued maps and semilinear differential inclusions in Banach spaces*, De Gruyter Series in Nonlinear Analysis and Applications, Walter de Gruyter and Co., Berlin, (2001). 1

[7] S. V. Kornev, *On the method of multivalent guiding functions for periodic solution of differential inclusions*, Autom. Remote Control, **64** (2003), 409–419. 1

[8] S. V. Kornev, *Nonsmooth integral directing functions in the problems of forced oscillations*, Translation of Avtomat. i Telemekh, (2015), 31–43, Autom. Remote Control, **76** (2015), 1541–1550. 1

[9] S. V. Kornev, *The method of generalized integral guiding function in the periodic problem of differential inclusions*, (Russian) The Bulletin of Irkutsk State University. Mathematics., **13** (2015), 16–31. 1

[10] S. V. Kornev, *Multivalent guiding function in a problem on existence of periodic solutions of some classes of differential inclusions*, Izv. Vyssh. Uchebn. Zaved. Mat., **11** (2016), 14–26. 1

[11] S. V. Kornev, *On asymptotics of solutions for differential inclusions with nonconvex right-hand side*, (Russian) The Bulletin of Voronezh State University. Phisics. Mathematics, **1** (2016), 96–104. 1

[12] S. V. Kornev, V. V. Obukhovskiĭ, *On nonsmooth multivalent guiding functions*, (Russian) Differ. Uravn., **39** (2003), 1497–1502, translation in Differ. Equ., **39** (2003), 1578–1584. 1

[13] S. V. Kornev, V. V. Obukhovskiĭ, *On some developments of the method of integral guiding functions*, Funct. Differ. Equ., **12** (2005), 303–310. 1

[14] S. V. Kornev, V. V. Obukhovskiĭ, *Asymptotic behavior of solutions of differential inclusions and the method of guiding functions*, Translation of Differ. Uravn., **51** (2015), 700705, Differ. Equ., **51** (2015), 711–716. 1

[15] S. V. Kornev, V. V. Obukhovskiĭ, J.-C. Yao, *On asymptotics of solutions for a class of functional differential inclusions*, Discuss. Math. Differ. Incl. Control Optim., **34** (2014), 219–227. 1

[16] S. V. Kornev, V. V. Obukhovskiĭ, P. Zecca, *Guiding functions and periodic solutions for inclusions with causal multioperators*, Appl. Anal., (2016), 1–11. 1

[17] M. A. Krasnosel'skiĭ, *The operator of translation along the trajectories of differential equations*, Translations of Mathematical Monographs, Translated from the Russian by Scripta Technica, American Mathematical Society, Providence, R.I., (1968). 1

[18] M. A. Krasnosel'skiĭ, A. I. Perov, *On a certain principle of existence of bounded, periodic and almost periodic solutions of systems of ordinary differential equations*, (Russian) Dokl. Akad. Nauk SSSR, **123** (1958), 235–238.

[19] M. A. Krasnosel'skiĭ, P. P. Zabreĭko, *Geometrical methods of nonlinear analysis*, Translated from the Russian by Christian C. Fenske, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, (1984). 1

[20] W. Kryszewski, *Homotopy Properties of Set-Valued in Mappings*, University Nicholas Copernicus Publishing, Toruń, (1997). 1, 1, 1

[21] Z. Liu, N. V. Loi, V. V. Obukhovskiĭ, *Existence and global bifurcation of periodic solutions to a class of differential variational inequalities*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **23** (2013), 10 pages. 1

[22] N. V. Loi, Z. Liu, V. V. Obukhovskiĭ, *On an A-bifurcation theorem with application to a parameterized integro-differential system*, Fixed Point Theory, **16** (2015), 127–141.

[23] N. V. Loi, V. V. Obukhovskiĭ, J.-C. Yao, *A bifurcation of solutions of nonlinear Fredholm inclusions involving CJ-multimaps with applications to feedback control systems*, Set-Valued Var. Anal., **21** (2013), 247–269.

[24] N. V. Loi, V. V. Obukhovskiĭ, J.-C. Yao, *A multiparameter global bifurcation theorem with application to a feedback control system*, Fixed Point Theory, **16** (2015), 353–370. 1

[25] V. V. Obukhovskiĭ, M. Kamenskiĭ, S. V. Kornev, Y.-C. Liou, *On asymptotics of solutions for some classes of differential inclusions via the generalized guiding functions method*, submitted to J. Nonlin. Conv. Anal., (2016). 1

[26] V. V. Obukhovskiĭ, N. V. Loi, S. V. Kornev, *Existence and global bifurcation of solutions for a class of operator-differential inclusions*, Differ. Equ. Dyn. Syst., **20** (2012), 285–300. 1

[27] V. V. Obukhovskiĭ, P. Zecca, N. V. Loi, S. V. Kornev, *Method of guiding functions in problems of nonlinear analysis*, Lecture Notes in Mathematics, Springer, Heidelberg, (2013). 1

[28] D. I. Rachinskiĭ, *Multivalent guiding functions in forced oscillation problems*, Nonlinear Anal., **26** (1996), 631–639. 1