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Quenching for a parabolic system with general singular terms

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Abstract

In this paper, we study a parabolic system with general singular terms and positive Dirichlet boundary conditions. Some sufficient conditions for finite-time quenching and global existence of the solutions are obtained, and the blow-up of time-derivatives at the quenching point is verified. Furthermore, under some appropriate hypotheses, we prove that the quenching point is only origin and quenching of the system is non-simultaneous. Moreover, the estimate of quenching rate of the corresponding solution is established in this article.

Keywords: Quenching, quenching set, quenching rate, singular term, parabolic system.
2010 MSC: 35B50, 35K45, 35K55.

1. Introduction and main results

In the present work, we consider the following parabolic system coupled with general singular terms subject to positive Dirichlet boundary conditions

$$\begin{cases} u_t(x,t) = \Delta u - f(v(x,t)), & (x,t) \in \Omega \times (0,T), \\ v_t(x,t) = \Delta v - g(u(x,t)), & (x,t) \in \Omega \times (0,T), \\ u = v = 1, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and the initial data satisfying

$$u_0, v_0 \in C^2(\Omega) \cap C^1(\bar{\Omega});\ u_0, v_0 = 1,\ x \in \partial\Omega;\ 0 < u_0, v_0 \leq 1,\ x \in \bar{\Omega}. \tag{1.2}$$

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To begin our study better, we also assume that positive functions $f(v):(0,1]\to(0,+\infty)$ and $g(u):(0,1]\to(0,+\infty)$ verify the following conditions:

(H_1) $f(v)$ and $g(u)$ are locally Lipschitz on $(0,1]$;

(H_2) $f'(v)<0$ for $v\in(0,1]$ and $g'(u)<0$ for $u\in(0,1]$;

(H_3) $\lim_{v\to 0^+} f(v)=+\infty$ and $\lim_{u\to 0^+} g(u)=+\infty$;

(H_4) $f''(v)\geq 0$ for $v\in(0,1]$ and $g''(u)\geq 0$ for $u\in(0,1]$.

Because of the singular nonlinearity in the absorption terms of (1.1), the finite-time quenching phenomena may occur for the model. We say the solution (u,v) of the problem (1.1) quenches, if (u,v) exists in the classical sense and is positive for all $0\leq t<T$, and also satisfies

$$\inf_{t\to T}\min_{0\leq x\leq 1}\{u(x,t),v(x,t)\}=0.$$

If this happens, then T will be called as quenching time. Clearly at quenching time T, a singularity develops in the absorption term, consequently the classical solution can no longer exist. Throughout this paper, the notion here as usual, $f\sim g$ means that there exists finite positive constants c_1,c_2 such that $c_1g\leq f\leq c_2g$.

Since the study of quenching phenomena was begun in 1975 by Kawarada [7], a lot of works have been contributed to this subject. For example, Zheng and Wang in [18] studied the coupled parabolic system

$$\begin{cases} u_t(x,t)=u_{xx}-v^{-p}, & (x,t)\in\Omega\times(0,T),\\ v_t(x,t)=v_{xx}-u^{-q}, & (x,t)\in\Omega\times(0,T),\\ u=v=1, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in\bar{\Omega}. \end{cases} \tag{1.3}$$

Their mainly results read as follows:

(1) If $p,q\geq 1$, then any quenching in (1.3) is simultaneous; if $p<1\leq q$, then any quenching in (1.3) is non-simultaneous with u being strictly positive; and if $p,q<1$, then there exists (u_0,v_0) such that simultaneous quenching occurs.

(2) If quenching is non-simultaneous and, for instance v is the unique quenching component, then $v(0,t)\sim(T-t)$. Otherwise,

   (a) $u(0,t)\sim(T-t)^{\frac{(p-1)}{(pq-1)}}, v(0,t)\sim(T-t)^{\frac{q-1}{pq-1}}$ if $p,q>1$ or $p,q<1$;

   (b) $u(0,t),v(0,t)\sim(T-t)^{\frac{1}{2}}$ if $p=q=1$;

   (c) $u(0,t)\sim|\log(T-t)|^{\frac{-1}{q-1}}, v(0,t)\sim(T-t)|\log(T-t)|^{\frac{q}{q-1}}$ if $q>p=1$.

Salin in [16] considered the semilinear parabolic equation

$$\begin{cases} u_t(x,t)=u_{xx}+\lg(\alpha u), & (x,t)\in(-l,l)\times(0,T),\\ u(\pm l,t)=1, & 0\leq t\leq T,\\ u(x,0)=u_0(x), & x\in[-l,l], \end{cases}$$

and derived that the quenching rate is $\lim_{t\to T}(1+\frac{1}{T-t}\int_0^{u(x,t)}\frac{\mathrm{d}s}{\log(\alpha s)})=0$.

Afterwards, Mu et al. in [14] studied the following reaction-diffusion system with logarithmic singularity,

$$\begin{cases} u_t(x,t)=\Delta u+\lg(\alpha v), & (x,t)\in\Omega\times(0,T),\\ v_t(x,t)=\Delta v+\lg(\beta u), & (x,t)\in\Omega\times(0,T),\\ u=v=1, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in\bar{\Omega}, \end{cases} \tag{1.4}$$

where $0 < \alpha, \beta < 1$. They proved that if $u_0(x) \le v_0(x), \alpha < \beta$, then any quenching in (1.4) is non-simultaneous with v being strictly positive; and if $u_0(x) \ge v_0(x), \alpha > \beta$, then any quenching in (1.4) is non-simultaneous with u being strictly positive. Besides, if quenching is non-simultaneous and, for instance, v is the quenching component, then when $t \to T^-$, $v(0,t) \sim (T-t)$. Furthermore, the blow-up of time-derivatives at the quenching point is also proved. For more research on quenching phenomena for parabolic system with Neumann boundary conditions, we refer readers to [3, 5, 15, 17, 19], and some advances in quenching phenomena those days, we refer readers to [1, 2, 8–11] and references therein. In addition, for some research on decay, see [6, 12, 13] and corresponding references therein.

Motivated by those papers and references therein, the main purpose of this paper is to study the quenching phenomena of parabolic system (1.1) coupled with general singular terms under proper assumptions to get some more general conclusions.

To state our results conveniently, we firstly introduce some notions.

Let φ be the first eigenfunction with the first eigenvalue λ_1 of the problem

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0, & x \in \Omega, \\ \varphi = 0, & x \in \partial\Omega, \end{cases}$$

normalized by $\int_\Omega \varphi(x)dx = 1, \varphi(x) > 0$ in Ω.

Theorem 1.1. *Solutions of* (1.1) *quenches in finite-time for any initial data provided that* λ_1 *small enough.*

As many authors who understand quenching, it is said that time-derivatives blow up while the solution itself remains bounded (see [2, 7]). In the rest of this paper, without any special explanation, we always assume that the initial data u_0, v_0 satisfy

$$\Delta u_0 - f(v_0) < 0, \Delta v_0 - g(u_0) < 0, x \in \Omega. \tag{1.5}$$

Then, we have the following results.

Theorem 1.2. *If* Ω *is a convex domain, then the solution of* (1.1) *quenches in finite-time and* (u_t, v_t) *blow up at this time.*

Theorem 1.3. *If* $\Omega = B_R = \{x \in \mathbb{R}^N : |x| < R\}$ *and* $R \ge \min(\sqrt{\frac{2N}{f(1)}}, \sqrt{\frac{2N}{g(1)}})$, *then the radial solution of* (1.1) *quenches in finite time for any initial data.*

Theorem 1.4. *If the diameter of* Ω *is small enough and the initial data satisfies* $0 < \epsilon \le u_0, v_0 \le 1$ *on* $\bar{\Omega}$, *then the solution of* (1.1) *does not quench in finite-time. For this case, we say that the solution* (u, v) *of* (1.1) *exists globally.*

Theorem 1.5. *If* Ω *is a convex domain, then the quenching set of* (1.1) *is a compact subset of* Ω. *In particular, if* Ω *is a ball centered at the origin with radius* R, *the radial initial data* (u_0, v_0) *satisfies both* (1.2) *and*

$$u_0'(r), v_0'(r) > 0 \text{ for } r \in (0, R] \text{ with } u_0''(0), v_0''(0) > 0,$$

then the origin is the only quenching point.

Theorem 1.6. *For all initial data* u_0, *if* $g(P)$ *is large enough, then there exists an open set (in the* C^2 *topology) of the initial data* v_0 *such that* v *quenches while* u *is strictly positive for all* $t \in [0, T]$, *where* $P = \max_{x\in\bar{\Omega}} u_0 \le 1$. *Furthermore, we have the quenching rate of* $v(0,t)$ *as follows*

$$v(0,t) \sim (T-t).$$

Theorem 1.7. *For all initial data* v_0, *if* $f(Q)$ *is large enough, then there exists an open set (in the* C^2 *topology) of the initial data* u_0 *such that* u *quenches while* v *is strictly positive for all* $t \in [0, T]$, *where* $Q = \max_{x\in\bar{\Omega}} v_0 \le 1$. *Furthermore, we have the quenching rate of* $u(0,t)$ *as follows*

$$u(0,t) \sim (T-t).$$

Remark 1.8. If $f(v) = v^{-p}$ or $-\log(\alpha v)$ and $g(u) = u^{-q}$ or $-\log(\beta u)$ with $p, q, \alpha, \beta > 0$, it is easy to see that $f(v), g(u)$ satisfy the conditions (H_1)-(H_4). Therefore, we extend the corresponding results of [14, 16, 18] to a more general system (1.1) in this paper.

The plan of this paper is organized as follows. In the next section, we obtain the sufficient condition for finite-time quenching and global existence. In Section 3, we obtain the quenching set. Moreover, under appropriate hypotheses, we prove that the solution of the system is non-simultaneous quenching, and estimate the quenching rate.

2. Finite-time quenching and global existence

In this section, we obtain the sufficient condition for finite-time quenching and quenching set, which reads in Theorems 1.1 - 1.3, and global existence of solutions is solved in Theorem 1.4.

Proof of Theorem 1.1. Let (u, v) be the solution of (1.1) with the maximal existence time T. By the maximum principle we have $0 \le u, v \le 1$ in $\Omega \times (0, T)$. Let

$$F_u(t) = \int_\Omega (1-u)\varphi dx, \quad F_v(t) = \int_\Omega (1-v)\varphi dx, \quad F(t) = F_u(t) + F_v(t), \quad t \in [0, T).$$

By the properties we assumed for functions $f(v), g(u)$ with $u, v \in (0, 1]$ and corresponding Taylor expansions, we can obtain

$$f(v) \ge \delta(1-v) + c_1, \quad g(u) \ge \delta(1-u) + c_2,$$

where δ, c_1, c_2 are positive constants determined by $f(v), g(u)$.

By a straight-forward computation, we have

$$\begin{aligned} F_u'(t) &= -\int_\Omega \Delta u \varphi dx + \int_\Omega f(v)\varphi dx \\ &= \int_\Omega \Delta(1-u)\varphi dx + \int_\Omega f(v)\varphi dx \\ &\ge -\lambda_1 \int_\Omega (1-u)\varphi dx + \delta \int_\Omega (1-v)\varphi dx + c_1 \\ &= -\lambda_1 F_u(t) + \delta F_v(t) + c_1, \end{aligned}$$

that is,

$$F_u'(t) \ge -\lambda_1 F_u(t) + \delta F_v(t) + c_1.$$

Similarly, we have

$$F_v'(t) \ge -\lambda_1 F_v(t) + \delta F_u(t) + c_2.$$

Consequently,

$$F'(t) \ge (\delta - \lambda_1)F(t) + C, \quad C = c_1 + c_2.$$

Notice $0 < F(t) < 2$ in $\bar{\Omega} \times (0, T)$ and λ_1 small enough, it is easy to obtain that $(\delta - \lambda_1)F(t) + C > 0$, hence

$$\frac{dF}{(\delta - \lambda_1)F(t) + C} \ge dt, \ t \in [0, T). \tag{2.1}$$

Integrating (2.1) from 0 to t, we have the problem

$$t \le \begin{cases} \frac{1}{\delta - \lambda_1} \ln \frac{(\delta-\lambda_1)F(t)+C}{(\delta-\lambda_1)F(0)+C}, & \delta \ne \lambda_1, \\ \frac{1}{C}(F(t) - F(0)), & \delta = \lambda_1. \end{cases}$$

Now, letting $t \to T^-$, combining $\lim_{t\to T^-} F(t) \le 2$, we obtain

$$T \le \begin{cases} \frac{1}{\delta-\lambda_1} \ln \frac{2(\delta-\lambda_1)+C}{(\delta-\lambda_1)F(0)+C}, & \delta \neq \lambda_1, \\ \frac{1}{C}(2-F(0)), & \delta = \lambda_1. \end{cases} \tag{2.2}$$

Clearly, the right-hand side of (2.2) is greater than zero, which shows finite-time quenching of the solutions in the system (1.1). The proof of Theorem 1.1 is complete. □

Next, we prove Theorem 1.2. We start with the following lemmas.

Lemma 2.1. *Assume that the initial data satisfy* (1.5), *then* $u_t, v_t < 0$ *for* $(x,t) \in \Omega \times (0,T)$. *Furthermore, we have for any* $\eta > 0$, *there exists* $c > 0$ *such that*

$$u_t, v_t < -c, (x,t) \in \Omega_\eta \times [\eta, T),$$

where $\Omega_\eta = \{x \in \Omega : dist(x, \partial\Omega) > \eta\}$.

Proof. Let $I(x,t) = u_t(x,t),\ J(x,t) = v_t(x,t)$. Differentiating the system (1.1) with respect to t, we have

$$\begin{cases} I_t = \Delta I - f'(v)J, & (x,t) \in \Omega \times (0,T), \\ J_t = \Delta J - g'(u)I, & (x,t) \in \Omega \times (0,T), \\ I = J = 0, & (x,t) \in \partial\Omega \times (0,T), \\ I(x,0) < 0,\ J(x,0) < 0, & x \in \bar{\Omega}. \end{cases} \tag{2.3}$$

By the comparison principle, we have $I(x,t) = u_t(x,t) < 0, J(x,t) = v_t(x,t) < 0$. Therefore, (u,v) are strictly decreasing in time.

Consider the following auxiliary system

$$\begin{cases} w_t = \Delta w - f(v_0), & (x,t) \in \Omega \times (0,T), \\ z_t = \Delta z - g(u_0), & (x,t) \in \Omega \times (0,T), \\ w = z = 1, & (x,t) \in \partial\Omega \times (0,T), \\ w(x,0) = u_0(x),\ z(x,0) = v_0(x), & x \in \bar{\Omega}. \end{cases} \tag{2.4}$$

It is easy to see that (2.4) has a unique global solution, by (1.5) we have $w_t(x,t) < 0, z_t(x,t) < 0$, for $\Omega \times (0, +\infty)$. Let $\Phi = u - w, \Psi = v - z$. Therefore, we have

$$\Phi_t = u_t - w_t \le 0, \quad \Psi_t = v_t - z_t \le 0, \quad (x,t) \in \Omega \times (0,T).$$

If we choose $c = \min\{\min_{\Omega_\eta \times [\eta,T)} |w_t|, \min_{\Omega_\eta \times [\eta,T)} |z_t|\}$, then we have $u_t, v_t < -c, (x,t) \in \Omega_\eta \times [\eta, T)$. □

Lemma 2.2. *Assume that* Ω *is a convex domain and* (1.5) *holds. Then for any* $\eta > 0$, *there exists* $\varepsilon > 0$ *such that*

$$u_t \le -\varepsilon f(v), v_t \le -\varepsilon g(u), (x,t) \in \Omega_\eta \times [\eta, T).$$

Proof. Let $\tilde{u} = u_t + \varepsilon f(v),\ \tilde{v} = v_t + \varepsilon g(u),\ (x,t) \in \Omega_\eta \times [\eta, T)$. Then

$$\begin{aligned} \tilde{u}_t - \Delta \tilde{u} &= u_{tt} + \varepsilon f'(v) v_t - \Delta u_t - \varepsilon f''(v)|\nabla v|^2 - \varepsilon f'(v)\Delta v \\ &\le -f'(v)v_t + \varepsilon f'(v) v_t - \varepsilon f'(v)\Delta v \\ &= -f'(v)v_t - \varepsilon f'(v) g(u) \\ &= -f'(v)\tilde{v}. \end{aligned}$$

Similarly, we have

$$\tilde{v}_t - \Delta \tilde{v} \le -g'(u)\tilde{u}.$$

By Lemma 2.1, if η is small enough, then there exists $c > 0$ such that $u_t, v_t < -c$ for $(x,t) \in \Omega_\eta \times [\eta, T)$. Furthermore, we can select an $\varepsilon > 0$ small enough such that

$$\begin{cases} \tilde{u} = u_t + \varepsilon f(v) \le 0, & (x,t) \in \partial\Omega_\eta \times [\eta, T), \\ \tilde{v} = v_t + \varepsilon g(u) \le 0, & (x,t) \in \partial\Omega_\eta \times [\eta, T). \end{cases}$$

And initial data satisfy

$$\begin{cases} \tilde{u}(x,0) = \Delta u_0(x) - f(v_0(x)) + \varepsilon f(v_0(x)) \le 0, \\ \tilde{v}(x,0) = \Delta v_0(x) - g(u_0(x)) + \varepsilon g(u_0(x)) \le 0. \end{cases}$$

By the comparison principle, we derive that $\tilde{u} = u_t + \varepsilon f(v) < 0,\ \tilde{v} = v_t + \varepsilon g(u) < 0, (x,t) \in \Omega_\eta \times [\eta, T)$. □

Proof of Theorem 1.2. By Lemma 2.1, fixing η and integrating (2.4) from η to t, we get $0 < u(x,t) < u(x,\eta) - c(t-\eta) < 1 - c(t-\eta)$, namely, the quenching time $T < \frac{1}{c} + \eta$, therefore, under the condition (1.5) the solutions of (1.1) quench in finite-time. By Lemma 2.2, if $v(x,t) \to 0$ (respectively, $u(x,t) \to 0$) for $t \to T^-$, then $v_t \to -\infty$ (respectively, $u_t \to -\infty$). The proof of Theorem 1.2 is completed. □

Next, we will prove Theorems 1.3 and 1.4. We first introduce the following lemma and consider the radial solutions of (1.1) on $\Omega = B_R = \{x \in \mathbb{R}^N : |x| < R\}$.

Lemma 2.3. *Let (u,v) be the global solution of (1.1) with $(u_0, v_0) \equiv (1,1), u, v \ge b$ on $\bar{B}_R \times (0,\infty)$ for some $b \in (0,1)$. Then (u,v) approaches uniformly from above to a solution (U,V) of the steady-state problem*

$$\begin{cases} \Delta U = f(V), \Delta V = g(U), & x \in B_R, \\ U = V = 1, & x \in \partial B_R. \end{cases} \tag{2.5}$$

Proof. Since $(1,1)$ is a strict super-solution of the problem (1.1), by Lemma 2.1, we have $u_t, v_t < 0$ in $B_R \times (0,\infty)$. Define

$$W(x,t) = \int_{B_R} G(x,y)u(y,t)dy, \quad Z(x,t) = \int_{B_R} G(x,y)v(y,t)dy, \quad (x,t) \in B_R \times (0,\infty),$$

where $G(x,y)$ is the Green's function associated with the operator $-\Delta$ on B_R under Dirichlet boundary conditions. Hence

$$W_t = \int_{B_R} G(x,y)u_t(y,t)dy = \int_{B_R} G(x,y)\Delta u(y,t)dy - \int_{B_R} G(x,y)f(v(y,t))dy,$$
$$Z_t = \int_{B_R} G(x,y)v_t(y,t)dy = \int_{B_R} G(x,y)\Delta v(y,t)dy - \int_{B_R} G(x,y)g(u(y,t))dy,$$

namely,

$$W_t = 1 - u(x,t) - \int_{B_R} G(x,y)f(v(y,t))dy,$$
$$Z_t = 1 - v(x,t) - \int_{B_R} G(x,y)g(u(y,t))dy.$$

It follows from $u_t, v_t < 0$ that $G(x,y)f(v(y,t))$ and $G(x,y)g(u(y,t))$ are nondecreasing with respect to t. Thus the monotone convergence theorem with

$$b \le U(x) = \lim_{t\to\infty} u(x,t), \qquad b \le V(x) = \lim_{t\to\infty} v(x,t),$$

which implies that

$$\lim_{t\to\infty} W_t = 1 - U(x) - \int_{B_R} G(x,y)f(V(y))dy,$$

$$\lim_{t\to\infty} Z_t = 1 - V(x) - \int_{B_R} G(x,y)g(U(y))dy.$$

On the other hand, since W and Z are bounded, and $W_t, Z_t \le 0$ by $u_t, v_t < 0$, we have

$$\lim_{t\to\infty} W_t = 0, \quad \lim_{t\to\infty} Z_t = 0,$$

which yield

$$U(x) = 1 - \int_{B_R} G(x,y)f(V(y))dy, \quad V(x) = 1 - \int_{B_R} G(x,y)g(U(y))dy.$$

Consequently, (U,V) is a solution of the problem (2.5), and the uniform convergence is guaranteed by Dini's theorem. The proof of Lemma 2.3 is complete. □

Proof of Theorem 1.3. Consider the auxiliary problem

$$\begin{cases} \underline{u}_t(x,t) = \Delta\underline{u} - f(\underline{v}(x,t)), & (x,t) \in \Omega\times(0,T), \\ \underline{v}_t(x,t) = \Delta\underline{v} - g(\underline{u}(x,t)), & (x,t) \in \Omega\times(0,T), \\ \underline{u} = \underline{v} = \epsilon, & (x,t) \in \partial\Omega\times(0,T), \\ \underline{u}(x,0) = \underline{v}(x,0) = \epsilon, & x \in \bar{\Omega}. \end{cases} \tag{2.6}$$

It is easy to see that the solution of (2.6) is a sub-solution of (1.1). By the comparison principle, we have $\underline{u} \le u, \underline{v} \le v$, it suffices to prove that $(\underline{u}, \underline{v})$ is global. Let ϕ satisfy

$$\begin{cases} \Delta\phi - K = 0, & x \in B_R, \\ \phi = \epsilon, & x \in \partial B_R, \end{cases}$$

where $K > \max\{f(\epsilon), g(\epsilon)\}$, and B_R is the ball centered at origin with radius R. So we have

$$\phi(x) = \frac{K(|x|^2 - R^2)}{2N} + \epsilon \ \text{ with } \ \min_{\bar{B}_R} \phi(x) = \epsilon - \frac{KR^2}{2N},$$

and $K \ge \max\{f(\phi), g(\phi)\}$ by taking R small enough. Therefore, (ϕ,ϕ) is a time-independent sub-solution of (2.6) for $\Omega \subset B_R$. By Lemma 2.3, the global solutions of (1.1) exist provided that Ω small enough. The proof of Theorem 1.3 is completed. □

Proof of Theorem 1.4. Consider the auxiliary system

$$\begin{cases} \bar{u}_t(x,t) = \Delta\bar{u} - f(\bar{v}(x,t)), & (x,t) \in \Omega\times(0,T), \\ \bar{v}_t(x,t) = \Delta\bar{v} - g(\bar{u}(x,t)), & (x,t) \in \Omega\times(0,T), \\ \bar{u} = \bar{v} = 1, & (x,t) \in \partial\Omega\times(0,T), \\ \bar{u}(x,0) = \bar{v}(x,0) = 1, & x \in \bar{\Omega}. \end{cases}$$

By the comparison principle, we have $u \le \bar{u}, v \le \bar{v}$.

We first consider the following system

$$\begin{cases} \Delta\bar{u}^* = f(1), & \text{in } B_R, \\ \Delta\bar{v}^* = g(1), & \text{in } B_R, \\ \bar{u}^* = \bar{v}^* = 1, & \text{on } \partial B_R. \end{cases}$$

By the Green's function, the solution $(\bar{u}^*, \bar{v}^*)$ denotes as the following

$$\begin{cases} \bar{u}^* = \frac{f(1)(|x|^2 - R^2)}{2N} + 1, \\ \bar{v}^* = \frac{g(1)(|x|^2 - R^2)}{2N} + 1, \end{cases}$$

and

$$\begin{cases} \min\bar{u}^* = \frac{-f(1)R^2}{2N} + 1, \\ \min\bar{v}^* = \frac{-g(1)R^2}{2N} + 1. \end{cases}$$

Clearly, $(\bar{u}^*, \bar{v}^*)$ is a super-solution of (1.1). By Lemma 2.3, the solution (u,v) of (1.1) is global only if $\bar{u}^*, \bar{v}^* > 0$. Thus, the proof of Theorem 1.4 is complete. □

3. Quenching set and non-simultaneous quenching

In this section, we firstly obtain the quenching set. Secondly, we prove the quenching is non-simultaneous and establish the non-simultaneous quenching rates of corresponding solutions.

Proof of Theorem 1.5. We will employ a similar method as it in [4, 18] to prove this theorem. Without loss of generality, assume

$$\frac{\partial u_0}{\partial n},\quad \frac{\partial v_0}{\partial n} > 0, \quad \text{on } \partial\Omega, \tag{3.1}$$

where n is the outward normal on $\partial\Omega$; otherwise, we can work with the initial data $(u,v)|_{t'=t-\tau} = 0$ for any small $\tau > 0$.

Take $y_0 \in \partial\Omega$, and assume for simplicity $y_0 = 0$ with the outward normal $(1, 0, ..., 0)$ at y_0. Define

$$\Omega_a^+ = \Omega \cap \{x_1 > a\}, \quad \Omega_a^- = \{(x_1, x')|(2a - x_1, x') \in \Omega_a^+\}$$

with $a < 0, x' = (x_2, ..., x_N)$. Clearly, Ω_a^- is the reflection of Ω_a^+ with respect to the hyperplane $x_1 = a$.

Consider the functions

$$\Gamma(x,t) = u(x_1, x', t) - u(2a - x_1, x', t), \quad \Upsilon(x,t) = v(x_1, x', t) - v(2a - x_1, x', t)$$

in $\Omega_a^- \times (0,T)$. Since $f(v)$ and $g(u)$ are locally Lipschitz on $(0,1]$, a simple computation yields

$$\Gamma_t - \Delta\Gamma = -[f(v(x_1, x', t)) - f(v(2a - x_1, x', t))] = \rho_1\Gamma,$$

$$\Upsilon_t - \Delta\Upsilon = -[g(u(x_1, x', t)) - g(u(2a - x_1, x', t))] = \rho_2\Upsilon,$$

where ρ_1, ρ_2 are nonnegative and bounded in $\Omega_a^- \times (0,t)$ for any fixed $t \in (0,T)$. In addition, $\Gamma = \Upsilon = 0$ on $x_1 = a$ and $\Gamma = u(x_1, x', t) - 1 < 0, \Upsilon = v(x_1, x', t) - 1 < 0$ on $(\partial\Omega_a^- \cap \{x_1 < a\}) \times (0,T)$. Finally, $\Gamma(x,0), \Upsilon(x,0) < 0$ for $x \in \Omega_a^-$ provided $|a|$ is small enough by (3.1). By the maximum principle, $\Gamma, \Upsilon < 0$ in $\Omega_a^- \times (0,T)$ and $2\frac{\partial u}{\partial x_1} = \frac{\partial \Gamma}{\partial x_1} > 0, 2\frac{\partial v}{\partial x_1} = \frac{\partial \Upsilon}{\partial x_1} > 0$ on $x_1 = a$. By the arbitrariness of a, it follows that

$$\frac{\partial u}{\partial x_1}, \frac{\partial v}{\partial x_1} > 0, \ \text{in } \Omega_{a_0}^+ \times (0,T), \tag{3.2}$$

provided $|a_0|$ is small enough.

Now introduce the functions

$$\chi = u_{x_1} - \varepsilon(x_1 - a_0), \quad \psi = v_{x_1} - \varepsilon(x_1 - a_0)$$

in $\Omega_a^+ \times (0,T)$ with $\varepsilon > 0$ to be determined. We have

$$\chi_t - \Delta\chi + f'(v)\psi = -\varepsilon(x_1 - a_0)f'(v) \geq 0,$$

$$\psi_t - \Delta\psi + g'(u)\chi = -\varepsilon(x_1 - a_0)g'(u) \geq 0.$$

Additionally, $\chi, \psi > 0$ on $x_1 = a_0$ by (3.2), and $\chi(x,0), \psi(x,0) > 0$ by (3.1). Furthermore, we claim that $\chi, \psi > 0$ on $(\partial\Omega_{a_0}^+ \cap \partial\Omega) \times (0,T)$. Indeed, let $(\bar{u}, \bar{v})$ solve

$$\begin{cases} \bar{u}_t = \Delta\bar{u}, \quad \bar{v}_t = \Delta\bar{v}, & (x,t) \in \Omega \times (0,T), \\ \bar{u} = \bar{v} = 1, & (x,t) \in \partial\Omega \times (0,T), \\ \bar{u}(x,0) = u_0(x), \quad \bar{v}(x,0) = v_0(x), & x \in \bar{\Omega}. \end{cases}$$

Then $u \leq \bar{u}, v \leq \bar{v}$ by the comparison principle. Consequently

$$\frac{\partial u}{\partial n} \geq \frac{\partial \bar{u}}{\partial n} \geq b_0 > 0, \quad \frac{\partial v}{\partial n} \geq \frac{\partial \bar{v}}{\partial n} \geq c_0 > 0, \quad \text{on } \partial\Omega \times (0,T).$$

It follows that

$$\chi \geq b_0 \cos < n, x_1 > -\varepsilon(x_1 - a_0) > 0, \quad \psi \geq c_0 \cos < n, x_1 > -\varepsilon(x_1 - a_0) > 0$$

for $x \in \partial\Omega^+_{a_0} \cap \partial\Omega$ with ε small enough. Therefore, by the maximum principle, we have $\chi, \psi > 0$ in $\Omega^+_{a_0} \times (0, T)$. In particular,

$$u_{x_1} \geq \varepsilon(x_1 - a_0), v_{x_1} \geq \varepsilon(x_1 - a_0), \text{ for } (x_1, x', t) \in [a_0, 0] \times \{x' = 0\} \times (0, T). \tag{3.3}$$

Integrating (3.3) with respect to x_1, we get

$$u(x_1, 0, t) \geq u(a_0, 0, t) + \frac{1}{2}\varepsilon(x_1 - a_0)^2,$$

$$v(x_1, 0, t) \geq v(a_0, 0, t) + \frac{1}{2}\varepsilon(x_1 - a_0)^2.$$

Thus, any point $x = (x_1, x') \in (a_0, 0) \times \{x' = 0\}$ cannot be a quenching point. The above argument shows that a_0 can be chosen independent of the initial point $y_0 \in \partial\Omega$. By varying y_0 along with $\partial\Omega$, we conclude that there is an Ω-neighborhood Ω' of $\partial\Omega$ such that any point $x \in \Omega'$ is not a quenching point.

Since Ω is a ball centered at the origin with radius R, the radial initial data (u_0, v_0) satisfies both (1.2) and $u_0'(r), v_0'(r) > 0$ for $r \in (0, R]$ with $u_0''(0), v_0''(0) > 0$, then we can follow the proof of Lemma 2.2 in [4] to conclude that the origin is the only quenching point. The proof of Theorem 1.5 is complete. □

Proof of Theorem 1.6. Consider the system (2.3) again with $v_0 = \xi < 1$. Then

$$\begin{cases} I_t = \Delta I - f'(v)J, & (x,t) \in \Omega \times (0,T), \\ J_t = \Delta J - g'(u)I, & (x,t) \in \Omega \times (0,T), \\ I = J = 0, & (x,t) \in \partial\Omega \times (0,T), \\ I(x,0) = \Delta u_0(x) - f(v_0(x)) < 0, & x \in \bar{\Omega}, \\ J(x,0) = -g(u_0(x)) < 0, & x \in \bar{\Omega}. \end{cases}$$

Set $P = \max_{x \in \bar{\Omega}} u_0 \leq 1$. Therefore, taking ξ sufficient small, by the comparison principle, we have $u_t(x,t) \leq -g(P)$ and $v_t(x,t) \leq -g(P)$ for any $(x,t) \in \Omega \times (0,T)$. Therefore, $v(x,t) \geq g(P)(T-t)$. Then substituting it into equation (1.1), we can obtain $u_t \geq \Delta u - f(A(T-t))$, where $A = g(P)$.

Consider the following problem with the solution $\underline{u}(t)$,

$$\begin{cases} \underline{u}'(t) = -f(A(T-t)), & t \in (0,T), \\ \underline{u}(0) = \min_{x \in \bar{\Omega}} u_0(x) = m, & t \in (0,T), \end{cases}$$

then we have

$$u(x,t) \geq \underline{u}(t) = m - \int_0^t f(A(T-s))ds. \tag{3.4}$$

If ξ is small enough, by (3.4) we have

$$u(x,T) \geq m - \lim_{t \to T} \int_0^t f(A(T-s))ds > 0.$$

Something important that we have to remark here is above arguments are still working with any initial data v_0, which is close to $v_0 = \xi$ in the C^2 topology, so the details are omitted. Thus we have proved that v quenches while u is strictly positive for all $t \in [0,T]$.

Next, we will give the estimate for $v(0,t)$. By the estimate of (2.2), we have

$$v_t(0,t) \leq -\varepsilon g(u(0,t)) \leq -\varepsilon g(P) < 0.$$

Thus, there exists a positive constant c_1 such that

$$v(0,T) - v(0,t) \leq -c_1(T-t),$$

that is

$$v(0,t) \geq c_1(T-t). \tag{3.5}$$

On the other hand, from the condition (1.5), we have $v_x(x,t) \geq 0$, so $\Delta v(0,t) \geq 0$ and $v_t \geq -g(u(0,t))$. Thus, $v_t(0,t) \geq -g(c_0) = -C_1 (C_1 > 0)$, where $c_0 = \inf_{0\leq t\leq T} \min_{x\in\bar{\Omega}} u(x,t) > 0$. Since $c_0 > 0, v(0,T) = 0$ and T is quenching time, so

$$v(0,t) \leq C_1(T-t). \tag{3.6}$$

Combining (3.5) and (3.6), we can get $v(0,t) \sim (T-t)$. To this end, the proof of Theorem 1.6 is complete. □

Proof of Theorem 1.7. Since the process of proof is similar to that of Theorem 1.6, we omit it here. □

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