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Further result on $\mathscr{H}_\infty$ state estimation of static neural networks with interval time-varying delay

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Abstract

This paper considers the $\mathscr{H}_\infty$ state estimation problem of static neural networks with interval time-varying delay. By constructing a suitable Lyapunov-Krasovskii functional, the single-integral and double-integral terms in the time derivative of the Lyapunov functional are handled by utilizing the inverses of first-order and squared reciprocally convex parameters techniques. An improved delay dependent criterion is established such that the error system is globally asymptotically stable with $\mathscr{H}_\infty$ performance. The desired estimator gain matrix and the optimal performance index are obtained via solving a convex optimization problem subject to linear matrix inequalities. Two numerical examples are given to illustrate the effectiveness of the proposed method. ©2016 all rights reserved.

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1. Introduction

In recent years, neural networks have been gaining increasing research attention because of their extensive applications in many areas such as reconstruction of moving image, signal processing, the tasks of pattern recognition, associative memories, fixed-point computations, and so on [6, 8, 24]. It is well-known that due to the finite speed limit of information processing and the inherent communication time of neurons, time delay is usually encountered in the implementation of networks. The manifestation of time delay in neurons may lead to undesirable dynamic network behaviors such as oscillation, instability or other poor

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performances. As a result, numerous results about stability analysis, synchronization and state estimation for neural networks with time delay have been reported [2, 4, 5, 13, 14, 17, 21, 25, 30–32, 36, 37].

On the other hand, based on the difference of choosing the local field states of neurons or the neuron states as basic variables, neural networks can be classified as local field neural networks or static neural networks [34]. Since a static neural network can be transferred equivalently to a local field neural network in the case of satisfying certain assumptions, more attention has been drawn to local field neural networks. However, in many applications, these assumptions cannot always be satisfied [19]. Thus, it is necessary to investigate static neural networks, the corresponding results for static neural networks are relatively few [7, 11, 12, 15, 18, 26, 33, 35]. In addition, in [11, 12] the foregoing stability criteria for static neural networks with time-varying delay are only applied into the case when the lower bound of the delay is zero. In fact, there exists a special type of time delay in practical engineering systems, i.e., interval time-varying delay $h_1 \leq h(t) \leq h_2$ and h_1 is not restricted to be zero, which commonly exists in networked control systems.

In general, a neural network is a highly interconnected network with a large number of neurons. As described in [12], the neuron states are not often completely available in the network output, it is usually the case that only partial information about the states of the nodes is available in the network outputs. Therefore, in order to understand the network behavior better, the neural state estimation problem has been gaining considerable amount of interest in recent years. For example, in [29], the authors first investigated the state estimation problem and derived some delay-independent state estimation conditions. In [10], the authors discussed the delay dependent state estimation problem for delayed static neural network by a delay partition approach. Further, Zhang and Yu [38] studied the issue of the exponential state estimation for Markovian jumping neural networks with time varying discrete and distributed delays. Mathiyalagan et al. [22] investigated the problem of robust exponential stability and $\mathscr{H}_\infty$ control for switched neutral-type neural networks. Vadivel et al. [28] addressed the issue of robust state estimation for a class of fuzzy neural networks with time-varying delays and parameter uncertainties. Besides, when designing a neural network or implementing it by VLSI in practice, the energy-to-energy gain from exogenous disturbances to the estimation error may be restricted less than a prescribed level. Therefore, the $\mathscr{H}_\infty$ state estimation is another important issue and has been investigated by some researchers [1, 7, 9, 11, 12, 16, 20].

Among the existing results, for the purpose of conservative reduction, many techniques such as free weighting matrix, delay decomposition and Jensen's integral inequalities have been employed in terms of linear matrix inequalities (LMIs). However, a few disadvantages of these research works still need to be concerned. When constructing the Lyapunov-Krasovskii functional, most of the developed approaches in those do not make full use of the information about the time-varying delay $h(t)$ and only consider the upper bound of the delay. In addition, the previous convex method also only applies the inverses of first-order technique to tackle the time-varying delay. Therefore, it remains a space to further improve the results reported in [1, 7, 9, 11, 12, 16, 20], which motivates this work.

In this paper, we focus on the $\mathscr{H}_\infty$ state estimation problem for delayed neural networks. The purpose of the problem is to design an $\mathscr{H}_\infty$ state estimator via the available output measurements such that the dynamics of the estimation errors system is asymptotically stable, and with a prescribed $\mathscr{H}_\infty$ noise attenuation level. The main contributions of this paper are listed as follows: First, much better performance can be achieved by the developed approach, which is benefited from the proposed the inverses of first-order and squared reciprocally convex parameters techniques. Second, compared with existing relevant results, the criteria in this paper not only lead to less conservative $\mathscr{H}_\infty$ state estimation conditions, but also have smaller computational burden since our theoretical proof is not concerned with any delay-decomposing method or free-weighting matrix method. As a result, the gain matrix of the state estimator and the optimal performance index can be simultaneously obtained by solving a convex optimization problem under the constraint of LMIs. In addition, for simplicity, the distributed delay is not considered in this paper. However, it is not difficult to extend the proposed approach to the $\mathscr{H}_\infty$ state estimation of static neural networks with both discrete and distributed delays [16]. Two numerical examples are given to illustrate the effectiveness of the proposed method.

Notation: In this presentation, the following notations will be used. $\Re^n$ denotes the n-dimensional

Euclidean vector space, and $\Re^{m\times n}$ is the set of all $m\times n$ real matrix. $*$ denotes the symmetric part. For symmetric matrices X and Y, $X > Y$ means that the matrix $X-Y$ is positive definite, whereas $X\geq Y$ means that the matrix $X-Y$ is nonnegative. I_n, 0_n and $0_{m\times n}$ denote $n\times n$ identity matrix, $n\times n$ and $m\times n$ zero matrices, respectively. $X^{\perp}$ denotes a basis for the null-space of X. col $\{x_1,x_2,\dots,x_n\}$ means $[x_1^T,x_2^T,\dots,x_n^T]^T$. The subscript $'T'$ represents the transpose, and $diag\{\cdots\}$ denotes the block diagonal matrix. For any matrix X, Sym $\{X\}$ means $X+X^T$. $X_{[f(t)]}\in\Re^{m\times n}$ means that the elements of matrix $X_{[f(t)]}$ include the scalar value of $f(t)$, i.e., $X_{[f_0]}=X_{[f(t)=f_0]}$.

2. Preliminaries

Consider the following static neural network with interval time-varying delay:

$$\begin{cases}\dot{x}(t)=-Ax(t)+g(Wx(t-h(t))+I)+B_1\omega(t),\\ y(t)=Cx(t)+Dx(t-h(t))+B_2\omega(t),\\ z(t)=Hx(t),\\ x(t)=\varphi(t),\ \forall t\in[-h_2,0],\end{cases}\tag{2.1}$$

where $x(t)=[x_1(t),x_2(t),\dots,x_n(t)]^T\in\Re^n$ denotes the neuron state vector, n is the number of neurons, $\omega(t)\in\Re^q$ is a noise disturbance belonging to $\ell_2[0,\infty)$, $y(t)\in\Re^m$ is the network measurement, and $z(t)\in\Re^p$, to be estimated, is a linear combination of the states. $A=diag\{a_1,a_2,\dots,a_n\}\in\Re^{n\times n}$ with $a_i>0, i=1,2,\dots,n$ is a positive diagonal matrix, W is the delayed interconnection weight matrix, and B_1, B_2, C, D and H are known real constant matrices with appropriate dimensions. $g(x(t))=[g_1(x_1(t)),g_2(x_2(t)),\dots,g_n(x_n(t))]^T\in\Re^n$ is a continuous activation function, $I=[I_1,I_2,\dots,I_n]^T$ is constant input vector, and $\varphi(t)$ is an initial condition defined on $[-h_2,0]$. $h(t)$ denotes the interval time-varying delay which satisfies

$$h_1\leq h(t)\leq h_2,\quad h_{12}=h_2-h_1,\quad \dot{h}(t)\leq u,$$

where h_1 and h_2 are known positive scalars, and u is a constant.

In addition, it is assumed that each neuron activation function in (2.1), $g_i(\cdot)$, $i=1,2,\dots,n$, satisfies the following condition:

Assumption 2.1. The neuron activation functions $g_i(\cdot)$, $i=1,2,\dots,n$ are continuous, bounded and satisfy

$$k_i^-\leq\frac{g_i(s_1)-g_i(s_2)}{s_1-s_2}\leq k_i^+,\quad \forall s_1,s_2\in\Re, s_1\neq s_2,$$

where k_i^- and k_i^+ are known real constants.

As mentioned before, the major objective of this paper is to present an efficient algorithm to deal with the $\mathscr{H}_\infty$ state estimation problem of the static neural networks (2.1). Now we consider the following state estimator for the neural networks (2.1):

$$\begin{cases}\dot{\hat{x}}(t)=-A\hat{x}(t)+g(W\hat{x}(t-h(t))+I)+L(y(t)-C\hat{x}(t)-D\hat{x}(t-h(t))),\\ \hat{z}(t)=H\hat{x}(t),\\ x(t)=0,\quad \forall t\in[-h_2,0],\end{cases}\tag{2.2}$$

where $\hat{x}(t)\in\Re^n$ denotes the estimated state, and $\hat{z}(t)\in\Re^q$ denotes the estimated measurements of $z(t)$. L is the estimator gain matrix to be determined.

Let the error signals be $e(t)=x(t)-\hat{x}(t)$ and $\bar{z}(t)=z(t)-\hat{z}(t)$. Then from (2.1) and (2.2), we easily obtain the error system as follows:

$$\begin{cases}\dot{e}(t)=-(A+LC)e(t)-LDe(t-h(t))+f(We(t-h(t)))+(B_1-LB_2)\omega(t),\\ \bar{z}(t)=He(t),\end{cases}\tag{2.3}$$

where $e(t)=[e_1(t),e_2(t),\dots,e_n(t)]^T\in\Re^n$ is the state vector of the transformed system and $f(We(t))=$

$g(Wx(t)+I)-g(W\hat{x}(t)+I)$.

The $\mathscr{H}_\infty$ performance state estimation problem is stated as follows. For a prescribed level $\gamma>0$ of noise attenuation, it is to find a suitable state estimator (2.2) such that:

(1) the estimation error system (2.3) with $\omega(t)\equiv 0$ is globally asymptotically stable;

(2) under the zero-initial condition, $||\bar{z}||_2<\gamma||\omega||_2$ holds for all nonzero $\omega(t)\in \ell_2[0,\infty)$, where $||\bar{z}||_2=\sqrt{\int_0^\infty ||\bar{z}(t)||^2dt}$ and $||\bar{\omega}||_2=\sqrt{\int_0^\infty ||\bar{\omega}(t)||^2dt}$.

Now, any of the following lemmas will play an important role in the derivation of the main results.

Lemma 2.1 (Park et al.[23])**.** *Let $f_1,f_2,\cdots,f_N:\Re^m\longrightarrow\Re$ have positive values in an open subset D of R^m. Then, the reciprocally convex combination of f_i over D satisfies*

$$\min_{\{a_i|a_i>0,\sum_i a_i=1\}}\sum_i \frac{1}{a_i}f_i(t)=\sum_i f_i(t)+\max_{g_{i,j}(t)}\sum_{i\neq j} g_{i,j}(t),$$

subject to

$$\{g_{i,j}:R^m\longrightarrow R, g_{i,j}(t)\triangleq g_{i,j}(t),\begin{bmatrix} f_i(t) & g_{i,j}(t)\\ g_{i,j}(t) & f_i(t)\end{bmatrix}\geq 0\}.$$

Lemma 2.2 (Schur complement, Boyd et al. [3])**.** *Let M, P, Q be given matrices such that $Q>0$, then*

$$\begin{bmatrix} P & M^T\\ * & -Q\end{bmatrix}<0\Leftrightarrow P+M^TQ^{-1}M<0.$$

Lemma 2.3 (Tian et al. [27])**.** *Ξ_1, Ξ_2 and Ξ are constant matrices of appropriate dimensions and $0\leq h_1\leq h(t)\leq h_2$, then*

$$(h(t)-h_1)\Xi_1+(h_2-h(t))\Xi_2+\Xi<0,$$

if and only if

$$(h_2-h_1)\Xi_1+\Xi<0 \quad and \quad (h_2-h_1)\Xi_2+\Xi<0.$$

3. Main results

Theorem 3.1. *For given scalars $0<h_1<h_2$ and u, matrices $K_1=diag\{k_1^-,\ldots,k_n^-\}$, and $K_2=diag\{k_1^+,\ldots,k_n^+\}$, the $\mathscr{H}_\infty$ performance state estimation problem is solvable if there exist positive define matrices $P=[P_{ij}]_{5\times5}$, $Q=[Q_{ij}]_{3\times3}$, $M=[M_{ij}]_{3\times3}$, $R=[R_{ij}]_{3\times3}$, X_1, X_2, S_1, S_2, U_1, U_2, Z_1, Z_2, positive diagonal matrices $\Delta_i(i=1,2,\ldots,4)$, $\Lambda_i=diag(\lambda_{i1},\ldots,\lambda_{in}), i=1,2,\ldots,6$, and matrices $T_k(k=1,2,\ldots,6)$, N, G with appropriate dimensions such that the following LMIs hold:*

$$\begin{bmatrix} \Xi^*_{[h(t)=h_1]} & \hat{B} & \hat{H}\\ * & -\gamma^2 I & 0\\ * & * & -I\end{bmatrix}<0, \tag{3.1}$$

$$\begin{bmatrix} \Xi^*_{[h(t)=h_2]} & \hat{B} & \hat{H}\\ * & -\gamma^2 I & 0\\ * & * & -I\end{bmatrix}<0, \tag{3.2}$$

$$\begin{bmatrix} 2Z_1 & 0 & T_1 & 0\\ * & Z_1 & 0 & T_2\\ * & * & 2Z_1 & 0\\ * & * & * & Z_1\end{bmatrix}>0, \begin{bmatrix} 2Z_2 & 0 & T_3 & 0\\ * & Z_2 & 0 & T_4\\ * & * & 2Z_2 & 0\\ * & * & * & Z_2\end{bmatrix}>0, \tag{3.3}$$

$$\begin{bmatrix} S_1 & T_5 \\ * & S_1 \end{bmatrix} > 0, \quad \begin{bmatrix} S_2 + \frac{h_{12}^2}{2} Z_1 & T_6 \\ * & S_2 + \frac{h_{12}^2}{2} Z_2 \end{bmatrix} > 0, \tag{3.4}$$

where

$$\begin{aligned}
\Xi^*_{[h(t)=h_1]} &= \Xi - \Psi^T_{1[h_1]} \Xi_1 \Psi_{1[h_1]} - \Psi^T_{2[h_1]} \Xi_2 \Psi_{2[h_1]}, \\
\Xi^*_{[h(t)=h_2]} &= \Xi - \Psi^T_{1[h_2]} \Xi_2 \Psi_{1[h_2]} - \Psi^T_{2[h_2]} \Xi_2 \Psi_{2[h_2]}, \\
\Xi &= \begin{bmatrix} [E_{ij}]_{7\times 7} & \Pi \\ * & \Phi \end{bmatrix}, \\
[E_{ij}]_{7\times 7} &= \begin{bmatrix} E_{11} & -GD & E_{13} & E_{14} & 0 & P_{12} & P_{13} \\ * & E_{22} & E_{23} & E_{24} & E_{25} & 0 & 0 \\ * & * & E_{33} & E_{34} & 0 & E_{36} & P_{23} \\ * & * & * & E_{44} & 0 & P_{23}^T & E_{47} \\ * & * & * & * & E_{55} & 0 & 0 \\ * & * & * & * & * & E_{66} & 0 \\ * & * & * & * & * & * & -R_{22} \end{bmatrix},
\end{aligned}$$

with

$$\begin{aligned}
E_{11} &= -P_{11}A - A^T P_{11} + P_{14} + P_{14}^T + M_{11} + h_1^2 X_1 - X_2 + h_{12}^2 S_1 - h_1^2 U_1 \\
&\quad - 2W^T K_1 \Delta_1 K_2 W - NA - GC - AN^T - C^T G^T, \\
E_{13} &= -P_{14} + P_{15} - A^T P_{12} + P_{24}^T + X_2, \quad E_{14} = -P_{15} - A^T P_{13} + P_{34}^T, \\
E_{22} &= -(1-u)Q_{11} - 2S_2 + T_6 + T_6^T - 2W^T K_1 \Delta_2 K_2 W, \\
E_{23} &= S_2 - T_6^T, \quad E_{24} = -T_6 + S_2, \quad E_{25} = -(1-u)Q_{12}, \\
E_{33} &= -P_{24} - P_{24}^T + P_{25} + P_{25}^T + Q_{11} - M_{11} + R_{11} - X_2 - h_1^2 U_2 - S_2 - 2W^T K_1 \Delta_3 K_2 W, \\
E_{34} &= -P_{25} - P_{34}^T + P_{35}^T + T_6, \\
E_{36} &= P_{22} + Q_{12} - W^T K_1 \Lambda_3 W + W^T K_2 \Lambda_4 W - M_{12} + R_{12}, \\
E_{44} &= -P_{35} - P_{35}^T - R_{11} - S_2 - 2K_1 W^T \Delta_4 K_2 W, \\
E_{47} &= P_{33} - R_{12} - W^T K_1 \Lambda_5 W + W^T K_2 \Lambda_6 W, \\
E_{55} &= -(1-u)Q_{22}, \quad E_{66} = Q_{22} - M_{22} + R_{22}, \\
\Pi &= \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & 0 & \Pi_{15} & \Pi_{16} & \Pi_{17} & \Pi_{18} \\ 0 & \Pi_{22} & 0 & 0 & 0 & 0 & 0 & \Pi_{28} \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & 0 & \Pi_{35} & \Pi_{36} & \Pi_{37} & 0 \\ 0 & \Pi_{42} & 0 & \Pi_{44} & -P_{45}^T & -P_{55} & -P_{55} & 0 \\ 0 & \Pi_{52} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Pi_{63} & 0 & P_{24} & P_{25} & P_{25} & 0 \\ 0 & 0 & 0 & \Pi_{74} & P_{34} & P_{35} & P_{35} & 0 \end{bmatrix}, \\
\Pi_{11} &= M_{13} + W^T K_2 \Delta_1 + W^T K_1 \Delta_1, \quad \Pi_{12} = P_{11} + N, \\
\Pi_{15} &= -A^T P_{14} + P_{44} + h_1 U_1, \quad \Pi_{16} = -A^T P_{15} + P_{45}, \quad \Pi_{17} = -A^T P_{15} + P_{45}, \\
\Pi_{18} &= M_{12} - W^T K_1 \Lambda_1 W + W^T K_2 \Lambda_2 W - N - AN^T - C^T G^T, \\
\Pi_{22} &= -(1-u)Q_{13} + W^T K_2 \Delta_2 + W^T K_1 \Delta_2, \quad \Pi_{28} = -D^T G^T, \\
\Pi_{32} &= P_{12}^T, \quad \Pi_{33} = Q_{13} - M_{13} + R_{13} + W^T K_1 \Delta_3 + W^T K_2 \Delta_3, \\
\Pi_{35} &= -P_{44} + P_{45}^T + h_1 U_2, \quad \Pi_{36} = -P_{45} + P_{55}, \quad \Pi_{37} = -P_{45} + P_{55}, \\
\Pi_{42} &= P_{13}^T, \quad \Pi_{44} = W^T K_2 \Delta_4 + W^T K_1 \Delta_4 - R_{13}, \quad \Pi_{52} = -(1-u)Q_{23}, \\
\Pi_{63} &= Q_{23} - M_{23} + R_{23} + W^T \Lambda_3 - W^T \Lambda_4, \\
\Pi_{74} &= -R_{23} + W^T \Lambda_5 - W^T \Lambda_6,
\end{aligned}$$

$$\Phi=\begin{bmatrix}\Phi_{11}&0&0&0&0&0&0&\Phi_{18}\\ *&\Phi_{22}&0&0&P_{14}&P_{15}&P_{15}&N^T\\ *&*&\Phi_{33}&0&0&0&0&0\\ *&*&*&\Phi_{44}&0&0&0&0\\ *&*&*&*&\Phi_{55}&0&0&0\\ *&*&*&*&*&-S_1&-T_5&0\\ *&*&*&*&*&*&-S_1&0\\ *&*&*&*&*&*&*&\Phi_{88}\end{bmatrix},$$

$$\Phi_{11}=M_{33}-2\Delta_1,\quad \Phi_{18}=M_{23}^T+\Lambda_1W-\Lambda_2W,\quad \Phi_{22}=-(1-u)Q_{33}-2\Delta_2,$$

$$\Phi_{33}=Q_{33}-M_{33}+R_{33}-2\Delta_3,\quad \Phi_{44}=-R_{33}-2\Delta_4,\quad \Phi_{55}=-X_1-U_1-U_2,$$

$$\Phi_{88}=M_{22}+h_1^2X_2+h_{12}^2S_2+\frac{h_1^4}{4}(U_1+U_2)+\frac{h_{12}^4}{4}(Z_1+Z_2)-N-N^T,$$

$$\Xi_1=\begin{bmatrix}Z_1&T_1+T_2\\ *&Z_1\end{bmatrix},\quad \Xi_2=\begin{bmatrix}Z_2&T_3+T_4\\ *&Z_2\end{bmatrix},$$

$$\Psi_{1[h(t)=h_1]}=\begin{bmatrix}0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&-I_n&0_n&0_n\\ 0_n&h_{12}I_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&-I_n&0_n\end{bmatrix},$$

$$\Psi_{1[h(t)=h_2]}=\begin{bmatrix}0_n&0_n&h_{12}I_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&-I_n&0_n&0_n\\ 0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&-I_n&0_n\end{bmatrix},$$

$$\Psi_{2[h(t)=h_1]}=\begin{bmatrix}0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&I_n&0_n&0_n\\ 0_n&0_n&0_n&-h_{12}I_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&I_n&0_n\end{bmatrix},$$

$$\Psi_{2[h(t)=h_2]}=\begin{bmatrix}0_n&-h_{12}I_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&I_n&0_n&0_n\\ 0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&0_n&I_n&0_n\end{bmatrix},$$

$$\hat{B}=col[(NB_1-GB_2)^T\ \ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ \ (NB_1-GB_2)^T],$$

$$\hat{H}=col[H^T\ \ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n\ 0_n.$$

Furthermore, the gain matrix L is designed as $L=N^{-1}G$.

Proof. The proof is divided into two parts. We first show that (2.3) holds for all nonzero $\omega(t)$ under zero-initial conditions. Then, the globally asymptotical stability of the error system (2.3) with $\omega(t)=0$ will be proven. Consider the Lyapunov functional candidate as follows:

$$V(e_t)=\sum_{i=1}^{10}V_i(e_t),$$

$$\begin{aligned}V_1(e_t)=&\ \eta_1^T(t)P\eta_1(t)+2\sum_{i=1}^{n}[\lambda_{1i}\int_0^{W_ie(t)}(f_i(s)-k_i^-s)ds+\lambda_{2i}\int_0^{W_ie(t)}(k_i^+s-f_i(s))ds]\\ &+2\sum_{i=1}^{n}[\lambda_{3i}\int_0^{W_ie(t-h_1)}(f_i(s)-k_i^-s)ds+\lambda_{4i}\int_0^{W_ie(t-h_1)}(k_i^+s-f_i(s))ds]\\ &+2\sum_{i=1}^{n}[\lambda_{5i}\int_0^{W_ie(t-h_2)}(f_i(s)-k_i^-s)ds+\lambda_{6i}\int_0^{W_ie(t-h_2)}(k_i^+s-f_i(s))ds],\end{aligned}$$

$$V_2(e_t)=\int_{t-h(t)}^{t-h_1}\eta_2^T(s)Q\eta_2(s)ds+\int_{t-h_1}^{t}\eta_2^T(s)M\eta_2(s)ds+\int_{t-h_2}^{t-h_1}\eta_2^T(s)R\eta_2(s)ds,$$

$$V_3(e_t)=h_1\int_{-h_1}^{0}\int_{t+\beta}^{t}e^T(s)X_1e(s)dsd\beta,$$

$$V_4(e_t)=h_{12}\int_{-h_2}^{-h_1}\int_{t+\beta}^{t}e^T(s)S_1e(s)dsd\beta,$$

$$
\begin{aligned}
V_5(e_t) &= h_1 \int_{-h_1}^{0} \int_{t+\beta}^{t} \dot{e}^T(s) X_2 \dot{e}(s) ds d\beta, \\
V_6(e_t) &= h_{12} \int_{-h_2}^{-h_1} \int_{t+\beta}^{t} \dot{e}^T(s) S_2 \dot{e}(s) ds d\beta, \\
V_7(e_t) &= \frac{h_1^2}{2} \int_{-h_1}^{0} \int_{r}^{0} \int_{t+\beta}^{t} \dot{e}^T(s) U_1 \dot{e}(s) ds d\beta dr, \\
V_8(e_t) &= \frac{h_1^2}{2} \int_{-h_1}^{0} \int_{-h_1}^{r} \int_{t+\beta}^{t} \dot{e}^T(s) U_2 \dot{e}(s) ds d\beta dr, \\
V_9(e_t) &= \frac{h_{12}^2}{2} \int_{-h_2}^{-h_1} \int_{r}^{-h_1} \int_{t+\beta}^{t} \dot{e}^T(s) Z_1 \dot{e}(s) ds d\beta dr, \\
V_{10}(e_t) &= \frac{h_{12}^2}{2} \int_{-h_2}^{-h_1} \int_{-h_2}^{r} \int_{t+\beta}^{t} \dot{e}^T(s) Z_2 \dot{e}(s) ds d\beta dr,
\end{aligned} \tag{3.5}
$$

where $\eta_1(t) = col\{e(t), e(t-h_1), e(t-h_2), \int_{t-h_1}^{t} x(s)ds, \int_{t-h_2}^{t-h_1} e(s)ds\}$ and $\eta_2(t) = col\{e(t), \dot{e}(t), f(We(t))\}$. Letting $\alpha = (h(t)-h_1)/h_{12}$, $\beta = (h_2 - h(t))/h_{12}$ and the time derivative of $V(e_t)$ along the trajectory of system (2.3) is given by

$$
\begin{aligned}
\dot{V}_1(e_t) =& 2\zeta^T(t) P \dot{\zeta}(t) + 2[f(We(t)) - K_1 We(t)]^T \Lambda_1 W \dot{e}(t) + 2[K_2 We(t) - f(We(t))]^T \Lambda_2 \\
& W\dot{e}(t) + 2[f(We(t-h_1)) - K_1 We(t-h_1)]^T \Lambda_3 W \dot{e}(t-h_1) + 2[K_2 We(t-h_1) \\
& - f(We(t-h_1))]^T \Lambda_4 W \dot{e}(t-h_1) + 2[f(We(t-h_2)) - K_1 We(t-h_2)]^T \Lambda_5 W \dot{e}(t-h_2) \\
& + 2[K_2 We(t-h_2) - f(We(t-h_2))]^T \Lambda_6 W \dot{e}(t-h_2),
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\dot{V}_2(e_t) =& \eta_2^T(t-h_1) Q \eta_2(t-h_1) - (1-\dot{h}(t)) \eta_2^T(t-h(t)) Q \eta_2(t-h(t)) \\
& + \eta_2^T(t) M \eta_2(t) - \eta_2^T(t-h_1) M \eta_2(t-h_1) \\
& + \eta_2^T(t-h_1) R \eta_2(t-h_1) - \eta_2^T(t-h_2) R \eta_2(t-h_2),
\end{aligned} \tag{3.7}
$$

$$
\dot{V}_3(e_t) \le h_1^2 e^T(t) X_1 e(t) - \int_{t-h_1}^{t} e^T(s) ds X_1 \int_{t-h_1}^{t} e(s) ds, \tag{3.8}
$$

$$
\begin{aligned}
\dot{V}_4(e_t) &= h_{12}^2 e^T(t) S_1 e(t) - h_{12} \int_{t-h(t)}^{t-h_1} e^T(s) S_1 e(s) ds - h_{12} \int_{t-h_2}^{t-h(t)} e^T(s) S_1 e(s) ds \\
&\le h_{12}^2 e^T(t) S_1 e(t) - \frac{1}{\alpha} \int_{t-h(t)}^{t-h_1} e^T(s) ds S_1 \int_{t-h(t)}^{t-h_1} e(s) ds \\
&\quad - \frac{1}{\beta} \int_{t-h_2}^{t-h(t)} e^T(s) ds S_1 \int_{t-h_2}^{t-h(t)} e(s) ds,
\end{aligned} \tag{3.9}
$$

$$
\dot{V}_5(e_t) \le h_1^2 \dot{e}^T(t) X_2 \dot{e}(t) - \int_{t-h_1}^{t} \dot{e}^T(s) ds X_2 \int_{t-h_1}^{t} \dot{e}(s) ds, \tag{3.10}
$$

$$
\begin{aligned}
\dot{V}_6(e_t) &= h_{12}^2 \dot{e}^T(t) S_2 \dot{e}(t) - h_{12} \int_{t-h(t)}^{t-h_1} \dot{e}^T(s) S_2 \dot{e}(s) ds - h_{12} \int_{t-h_2}^{t-h(t)} \dot{e}^T(s) S_2 \dot{e}(s) ds \\
&\le h_{12}^2 e^T(t) S_1 e(t) - \frac{1}{\alpha} \int_{t-h(t)}^{t-h_1} \dot{e}^T(s) ds S_2 \int_{t-h(t)}^{t-h_1} \dot{e}(s) ds \\
&\quad - \frac{1}{\beta} \int_{t-h_2}^{t-h(t)} \dot{e}^T(s) ds S_2 \int_{t-h_2}^{t-h(t)} \dot{e}(s) ds,
\end{aligned} \tag{3.11}
$$

$$\begin{aligned}\dot{V}_7(e_t) &= \frac{h_1^4}{4}\dot{e}^T(t)U_1\dot{e}(t) - \frac{h_1^2}{2}\int_{-h_1}^{0}\int_{t+\beta}^{t}\dot{e}^T(s)U_1\dot{e}(s)dsd\beta \\ &\leq \frac{h_1^4}{4}\dot{e}^T(t)U_1\dot{e}(t) - (h_1e^T(t) - \int_{t-h_1}^{t}e^T(s)ds)U_1(h_1e(t) - \int_{t-h_1}^{t}e(s)ds),\end{aligned} \tag{3.12}$$

$$\begin{aligned}\dot{V}_8(e_t) &= \frac{h_1^4}{4}\dot{e}^T(t)U_2\dot{e}(t) - \frac{h_1^2}{2}\int_{-h_1}^{0}\int_{t-h_1}^{t+\beta}\dot{e}^T(s)U_2\dot{e}(s)dsd\beta \\ &\leq \frac{h_1^4}{4}\dot{e}^T(t)U_2\dot{e}(t) - (\int_{t-h_1}^{t}e^T(s)ds - h_1e^T(t-h_1))U_2(\int_{t-h_1}^{t}e(s)ds - h_1e(t-h_1)),\end{aligned} \tag{3.13}$$

$$\begin{aligned}\dot{V}_9(e_t) &= \frac{h_{12}^4}{4}\dot{e}^T(t)Z_1\dot{e}(t) - \frac{h_{12}^2}{2}\int_{-h_2}^{-h_1}\int_{t+\beta}^{t-h_1}\dot{e}^T(s)Z_1\dot{e}(s)dsd\beta \\ &= \frac{h_{12}^4}{4}\dot{e}^T(t)Z_1\dot{e}(t) - \frac{h_{12}^2}{2}(h_2 - h(t))\int_{t-h(t)}^{t-h_1}\dot{e}^T(s)Z_1\dot{e}(s)dsd\beta \\ &\quad - \frac{h_{12}^2}{2}\int_{-h(t)}^{-h_1}\int_{t+\beta}^{t-h_1}\dot{e}^T(s)Z_1\dot{e}(s)dsd\beta - \frac{h_{12}^2}{2}\int_{-h_2}^{-h(t)}\int_{t+\beta}^{t-h(t)}\dot{e}^T(s)Z_1\dot{e}(s)dsd\beta \\ &\leq \frac{h_{12}^2}{4}\dot{e}^T(t)Z_1\dot{e}(t) - \frac{h_{12}^2}{2}\frac{\beta}{\alpha}\int_{t-h(t)}^{t-h_1}\dot{e}^T(s)dsZ_1\int_{t-h(t)}^{t-h_1}\dot{e}(s)ds - \frac{1}{\alpha^2}\int_{-h(t)}^{-h_1}\int_{t+\beta}^{t-h_1}\dot{e}^T(s)dsd\beta \\ &\quad Z_1\int_{-h(t)}^{-h_1}\int_{t+\beta}^{t-h_1}\dot{e}(s)dsd\beta - \frac{1}{\beta^2}\int_{-h_2}^{-h(t)}\int_{t+\beta}^{t-h(t)}\dot{e}^T(s)dsd\beta Z_1\int_{-h_2}^{-h(t)}\int_{t+\beta}^{t-h(t)}\dot{e}(s)dsd\beta,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\dot{V}_{10}(e_t) &= \frac{h_{12}^4}{4}\dot{e}^T(t)Z_2\dot{e}(t) - \frac{h_{12}^2}{2}\int_{-h_2}^{-h_1}\int_{t-h_2}^{t+\beta}\dot{e}^T(s)Z_2\dot{e}(s)dsd\beta \\ &= \frac{h_{12}^4}{4}\dot{e}^T(t)Z_2\dot{e}(t) - \frac{h_{12}^2}{2}(h(t) - h_1)\int_{t-h_2}^{t-h(t)}\dot{e}^T(s)Z_2\dot{e}(s)dsd\beta \\ &\quad - \frac{h_{12}^2}{2}\int_{-h(t)}^{-h_1}\int_{t-h(t)}^{t+\beta}\dot{e}^T(s)Z_2\dot{e}(s)dsd\beta - \frac{h_{12}^2}{2}\int_{-h_2}^{-h(t)}\int_{t-h_2}^{t+\beta}\dot{e}^T(s)Z_2\dot{e}(s)dsd\beta \\ &\leq \frac{h_{12}^4}{4}\dot{e}^T(t)Z_2\dot{e}(t) - \frac{h_{12}^2}{2}\frac{\alpha}{\beta}\int_{t-h_2}^{t-h(t)}\dot{e}^T(s)dsZ_2\int_{t-h_2}^{t-h(t)}\dot{e}(s)ds - \frac{1}{\alpha^2}\int_{-h(t)}^{-h_1}\int_{t-h(t)}^{t+\beta}\dot{e}^T(s)dsd\beta \\ &\quad Z_2\int_{-h(t)}^{-h_1}\int_{t-h(t)}^{t+\beta}\dot{e}(s)dsd\beta - \frac{1}{\beta^2}\int_{-h_2}^{-h(t)}\int_{t-h_2}^{t+\beta}\dot{e}^T(s)dsd\beta Z_2\int_{-h_2}^{-h(t)}\int_{t-h_2}^{t+\beta}\dot{e}(s)dsd\beta,\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}\xi(t) =& col\{e(t), e(t-h(t), e(t-h_1), e(t-h_2), \dot{e}(t-h(t)), \dot{e}(t-h_1), \dot{e}(t-h_2), f(We(t)), f(We(t-h(t))) \\ &, f(We(t-h_1)), f(We(t-h_2)), \int_{t-h_1}^{t}e(s)ds, \int_{t-h(t)}^{t-h_1}e(s)ds, \int_{t-h_2}^{t-h(t)}e(s)ds, \dot{e}(t)\}.\end{aligned}$$

From Lemma 2.1, we can infer if there exist matrices T_5 and T_6 such that (3.4) holds, then it holds that

$$\begin{aligned}&-\frac{1}{\alpha}\int_{t-h(t)}^{t-h_1}e^T(s)dsS_1\int_{t-h(t)}^{t-h_1}e(s)ds - \frac{1}{\beta}\int_{t-h_2}^{t-h(t)}e^T(s)dsS_1\int_{t-h_2}^{t-h(t)}e(s)ds \\ &\leq -\begin{bmatrix}\int_{t-h(t)}^{t-h_1}e(s)ds \\ \int_{t-h_2}^{t-h(t)}e(s)ds\end{bmatrix}^T\begin{bmatrix}S_1 & T_5 \\ * & S_1\end{bmatrix}\begin{bmatrix}\int_{t-h(t)}^{t-h_1}e(s)ds \\ \int_{t-h_2}^{t-h(t)}e(s)ds\end{bmatrix},\end{aligned} \tag{3.16}$$

and

$$
\begin{aligned}
&-\frac{1}{\alpha}\int_{t-h(t)}^{t-h_1}\dot{e}^T(s)dsS_2\int_{t-h(t)}^{t-h_1}\dot{e}(s)ds-\frac{1}{\beta}\int_{t-h_2}^{t-h(t)}\dot{e}^T(s)dsS_2\int_{t-h_2}^{t-h(t)}\dot{e}(s)ds\\
&\quad-\frac{h_{12}^2}{2}\frac{\beta}{\alpha}\int_{t-h(t)}^{t-h_1}\dot{e}^T(s)dsZ_1\int_{t-h(t)}^{t-h_1}\dot{e}(s)ds-\frac{h_{12}^2}{2}\frac{\alpha}{\beta}\int_{t-h_2}^{t-h(t)}\dot{e}^T(s)dsZ_2\int_{t-h_2}^{t-h(t)}\dot{e}(s)ds\\
&\quad\leq-\begin{bmatrix}\int_{t-h(t)}^{t-h_1}\dot{e}(s)ds\\ \int_{t-h_2}^{t-h(t)}\dot{e}(s)ds\end{bmatrix}^T\begin{bmatrix}S_2 & T_6\\ * & S_2\end{bmatrix}\begin{bmatrix}\int_{t-h(t)}^{t-h_1}\dot{e}(s)ds\\ \int_{t-h_2}^{t-h(t)}\dot{e}(s)ds\end{bmatrix}.
\end{aligned}
\tag{3.17}
$$

Note that if $h(t)=h_1$ or $h(t)=h_2$, we have

$$
\int_{t-h(t)}^{t-h_1}\dot{e}(s)ds=\int_{t-h(t)}^{t-h_1}e(s)ds=0 \quad\text{or}\quad \int_{t-h_2}^{t-h(t)}\dot{e}(s)ds=\int_{t-h_2}^{t-h(t)}e(s)ds=0,
$$

respectively. So inequalities (3.16) and (3.17) still hold.

Using a similar manner, we can derive the upper bounds of the second-order reciprocally convex combinations in (3.14) and (3.15) for the matrices T_1, T_2, T_3, T_4 satisfying (3.3) as

$$
\begin{aligned}
&-\frac{1}{\alpha^2}\int_{-h(t)}^{-h_1}\int_{t+\beta}^{t-h_1}\dot{e}^T(s)dsd\beta Z_1\int_{-h(t)}^{-h_1}\int_{t+\beta}^{t-h_1}\dot{e}(s)dsd\beta\\
&\quad-\frac{1}{\beta^2}\int_{-h_2}^{-h(t)}\int_{t+\beta}^{t-h(t)}\dot{e}^T(s)dsd\beta Z_1\int_{-h_2}^{-h(t)}\int_{t+\beta}^{t-h(t)}\dot{e}(s)dsd\beta\\
&\quad\leq-\begin{bmatrix}\int_{-h(t)}^{-h_1}\int_{t+\beta}^{t-h_1}\dot{e}(s)dsd\beta\\ \int_{-h_2}^{-h(t)}\int_{t+\beta}^{t-h(t)}\dot{e}(s)dsd\beta\end{bmatrix}^T\begin{bmatrix}Z_1 & T_1+T_2\\ * & Z_1\end{bmatrix}\begin{bmatrix}\int_{-h(t)}^{-h_1}\int_{t+\beta}^{t-h_1}\dot{e}(s)dsd\beta\\ \int_{-h_2}^{-h(t)}\int_{t+\beta}^{t-h(t)}\dot{e}(s)dsd\beta\end{bmatrix},
\end{aligned}
\tag{3.18}
$$

and

$$
\begin{aligned}
&-\frac{1}{\alpha^2}\int_{-h(t)}^{-h_1}\int_{t-h(t)}^{t+\beta}\dot{e}^T(s)dsd\beta Z_2\int_{-h(t)}^{-h_1}\int_{t-h(t)}^{t+\beta}\dot{e}(s)dsd\beta\\
&\quad-\frac{1}{\beta^2}\int_{-h_2}^{-h(t)}\int_{t-h_2}^{t+\beta}\dot{e}^T(s)dsd\beta Z_2\int_{-h_2}^{-h(t)}\int_{t-h_2}^{t+\beta}\dot{e}(s)dsd\beta\\
&\quad\leq-\begin{bmatrix}\int_{-h(t)}^{-h_1}\int_{t-h(t)}^{t+\beta}\dot{e}(s)dsd\beta\\ \int_{-h_2}^{-h(t)}\int_{t-h_2}^{t+\beta}\dot{e}(s)dsd\beta\end{bmatrix}^T\begin{bmatrix}Z_2 & T_3+T_4\\ * & Z_2\end{bmatrix}\begin{bmatrix}\int_{-h(t)}^{-h_1}\int_{t-h(t)}^{t+\beta}\dot{e}(s)dsd\beta\\ \int_{-h_2}^{-h(t)}\int_{t-h_2}^{t+\beta}\dot{e}(s)dsd\beta\end{bmatrix}.
\end{aligned}
\tag{3.19}
$$

When $h(t)=h_1$ or $h(t)=h_2$, we have

$$
\int_{-h(t)}^{-h_1}\int_{t+\beta}^{t-h_1}\dot{e}(s)dsd\beta=\int_{-h(t)}^{-h_1}\int_{t-h(t)}^{t+\beta}\dot{e}(s)dsd\beta=0,
$$

or

$$
\int_{-h_2}^{-h(t)}\int_{t+\beta}^{t-h(t)}\dot{e}(s)dsd\beta=\int_{-h_2}^{-h(t)}\int_{t-h_2}^{t+\beta}\dot{e}(s)dsd\beta=0,
$$

respectively. So the relations (3.18) and (3.19) still hold.

Under Assumption 2.1, it is not difficult to see for any positive diagonal matrices $\Delta_i (i=1,2,\ldots,4)$, the following inequality holds:

$$
\begin{aligned}
0 \leq &2[f(We(t))-K_1We(t)]^T\Delta_1[K_2We(t)-f(We(t))]+2[f(We(t-h(t)))\\
&-K_1We(t-h(t))]^T\Delta_2[K_2We(t-h(t))-f(We(t-h(t)))]+2[f(We(t-h_1))\\
&-K_1We(t-h_1)]^T\Delta_3[K_2We(t-h_1)-f(We(t-h_1))]+2[f(We(t-h_2))\\
&-K_1We(t-h_2)]^T\Delta_4[K_2We(t-h_2)-f(We(t-h_2))].
\end{aligned}
$$

Furthermore, for any matrix N with appropriate dimension, the following zero equation holds:

$$2[e^T(t)+\dot{e}^T(t)]N[-\dot{e}(t)-(A+LC)e(t)-LDe(t-h(t))+f(We(t-h(t)))+(B_1-LB_2)\omega(t)]=0.$$

And the following inequality holds

$$2[e^T(t)+\dot{e}^T(t)]N(B_1-LB_2)\omega(t)\le\gamma^{-2}\xi^T(t)\hat{B}\hat{B}^T\xi(t)+\gamma^2\omega^T(t)\omega(t). \tag{3.20}$$

From the conditions (3.6) to (3.20), it can be seen that

$$\begin{aligned}\dot{V}(e_t)\le&\xi^T(t)\{\Xi+\gamma^{-2}\hat{B}\hat{B}^T-\Psi^T_{1[h(t)]}\underbrace{\begin{bmatrix} Z_1 & T_1+T_2\\ * & Z_1\end{bmatrix}}_{\Xi_1}\Psi_{1[h(t)]}\\ &-\Psi^T_{2[h(t)]}\underbrace{\begin{bmatrix} Z_2 & T_3+T_4\\ * & Z_2\end{bmatrix}}_{\Xi_2}\Psi_{2[h(t)]}\}\xi(t)+\gamma^2\omega^T(t)\omega(t),\end{aligned} \tag{3.21}$$

where

$$\Psi_{1[h(t)]}=\begin{bmatrix} 0_n & 0_n & (h(t)-h_1)I_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & -I_n & 0_n & 0_n\\ 0_n & (h_2-h(t))I_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & -I_n & 0_n\end{bmatrix},$$

$$\Psi_{2[h(t)]}=\begin{bmatrix} 0_n & -(h(t)-h_1)I_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & I_n & 0_n & 0_n\\ 0_n & 0_n & 0_n & -(h_2-h(t))I_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & I_n & 0_n\end{bmatrix}.$$

Define

$$\mathcal{J}=\int_0^\infty[\bar{z}^T(t)\bar{z}(t)-\gamma^2\omega^T(t)\omega(t)]dt.$$

Then, one has

$$\begin{aligned}\mathcal{J}&=\int_0^\infty[\bar{z}^T(t)\bar{z}(t)-\gamma^2\omega^T(t)\omega(t)+\dot{V}(e_t)-\dot{V}(e_t)]dt\\ &\le\int_0^\infty[\bar{z}^T(t)\bar{z}(t)-\gamma^2\omega^T(t)\omega(t)+\dot{V}(e_t)]dt\\ &\le\int_0^\infty\xi^T(t)\tilde{\Xi}\xi(t)dt,\end{aligned}$$

where

$$\tilde{\Xi}=\Xi+\gamma^{-2}\hat{B}\hat{B}^T-\Psi^T_{1[h(t)]}\Xi_1\Psi_{1[h(t)]}-\Psi^T_{2[h(t)]}\Xi_2\Psi_{2[h(t)]}+\hat{H}\hat{H}^T.$$

By Lemma 2.3, the following matrix inequality:

$$\Xi+\gamma^{-2}\hat{B}\hat{B}^T-\Psi^T_{1[h(t)]}\Xi_1\Psi_{1[h(t)]}-\Psi^T_{2[h(t)]}\Xi_2\Psi_{2[h(t)]}+\hat{H}\hat{H}^T<0,$$

is equivalent to the following matrix inequalities

$$\Xi+\gamma^{-2}\hat{B}\hat{B}^T-\Psi^T_{1[h_1]}\Xi_1\Psi_{1[h_1]}-\Psi^T_{2[h_1]}\Xi_2\Psi_{2[h_1]}+\hat{H}\hat{H}^T<0, \tag{3.22}$$

$$\Xi+\gamma^{-2}\hat{B}\hat{B}^T-\Psi^T_{1[h_2]}\Xi_1\Psi_{1[h_2]}-\Psi^T_{2[h_2]}\Xi_2\Psi_{2[h_2]}+\hat{H}\hat{H}^T<0. \tag{3.23}$$

By applying Schur complement lemma to (3.22) and (3.23) we obtain (3.1) and (3.2), respectively. Therefore, one has $\mathcal{J}<0$, which implies $||\bar{z}||_2\le\gamma||\omega||_2$ for any nonzero $\omega(t)\in\ell_2[0,\infty)$.

Now we show the globally asymptotical stability of the estimation error system (2.3) with $\omega(t)=0$. For convenience, it is rewritten as

$$\dot{e}(t)=-(A+LC)e(t)-LDe(t-h(t))+f(We(t-h(t))). \tag{3.24}$$

We still consider the Lyapunov functional (3.5) and calculate its time-derivative along the solutions of (3.24). Similar to the proof of (3.21), one can obtain that

$$\begin{aligned}\dot{V}(e_t) \leq& \xi^T(t)(\Xi - \Psi^T_{1[h(t)]}\Xi_1\Psi_{1[h(t)]} - \Psi^T_{2[h(t)]}\Xi_2\Psi_{2[h(t)]})\xi(t)\\ \leq& 0\end{aligned}$$

is still guaranteed by (3.1)-(3.4).

According to the Lyapunov stability theory, the error system (2.3) with $\omega(t) = 0$ is globally asymptotically stable. This completes the proof. □

Remark 3.2. It should be noted that the proposed Lyapunov-Krasovskii functional in this paper is more generalized, since the $V_9(e_t)$ and $V_{10}(e_t)$ were not considered in [1, 12, 20]. Therefore, the stability results may be more applicable.

Remark 3.3. It can be seen from the proof that the terms

$$-\frac{h_{12}^2}{2}\int_{-h_2}^{-h_1}\int_{t+\beta}^{t-h_1}\dot{e}^T(s)Z_1\dot{e}(s)dsd\beta,$$

and

$$-\frac{h_{12}^2}{2}\int_{-h_2}^{-h_1}\int_{t-h_2}^{t+\beta}\dot{e}^T(s)Z_2\dot{e}(s)dsd\beta,$$

are handled by the reciprocally convex combination technique [23] and Jenson inequality. Its advantage is that better performance can be derived by Theorem 3.1 than the results obtained in [16, 20]. It will be illustrated by the numerical examples.

Remark 3.4. In [29], the delay-independent state estimation condition that does not consider the size of delay is very conservative, especially in the case of short delay. While in our paper the delay-dependent is exactly contrary, when the value of delay is small enough, the delay-dependent ones can be equivalent to the delay-independent ones. Thus, we consider the delay-dependent one is more general to some extent.

Remark 3.5. Letting $\alpha = \frac{h(t)-h_1}{h_{12}}$, $\beta = \frac{h_2-h(t)}{h_{12}}$ yield coefficients $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{\beta}{\alpha}$ and $\frac{\alpha}{\beta}$ in the time derivative of the Lyapunov functional, where double integral terms include $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, and triple integral terms include $\frac{\beta}{\alpha}$ and $\frac{\alpha}{\beta}$, by Lemma 2.1 we can derive the bound for the combinations of $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, and the combinations of $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{\beta}{\alpha}$ and $\frac{\alpha}{\beta}$, respectively.

Remark 3.6. In (3.18) and (3.19), the relations $\frac{1}{\alpha^2} = \frac{(\alpha+\beta)^2}{\alpha^2}$ and $\frac{1}{\beta^2} = \frac{(\alpha+\beta)^2}{\beta^2}$ are employed and the proof procedure is similar to that of Assumption 2.1 in [23]. The proposed method of the second-order reciprocally convex combination is very effective in reducing the conservatism of the state estimation condition.

Remark 3.7. In [12] and [10], in order to convert nonlinear matrix inequality into LMIs, the fact $-PR^{-1}P \leq -2P + R$ $(R \geq 0)$ is used. In this paper, we use zero equality to avoid this problem, which can give much flexibility in solving LMIs. The effectiveness of this method will be shown in the following numerical examples.

4. Illustrative example

In this section, two examples are provided to illustrate the advantage of Theorem 3.1 over some recent results.

Example 4.1. Consider a class of static neural networks (2.1), the parameters given in [1] is:

$$A = diag\{0.96, 0.8, 1.48\},\quad W = \begin{bmatrix} 0.5 & 0.3 & -0.36 \\ 0.1 & 0.12 & 0.5 \\ -0.42 & 0.78 & 0.9 \end{bmatrix},\quad H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix},\quad B_1 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.0 \end{bmatrix},$$

$$I = [0 \ \ 0 \ \ 0]^T, \quad B_2 = -0.1, \quad C = [1 \ \ 0 \ \ -2], \quad D = [0.5 \ \ 0 \ \ -1], \quad K_1 = diag\{0,0,0\}, \quad K_2 = diag\{1,1,1\}.$$

Here the activation function is assumed to be $g(x(t)) = \tanh(x(t))$, and the time-varying delay is taken as $h(t) = 0.5 + 0.5\cos(t)$. When $h_1 = 0.5$, $h_2 = 1$ and $u = 0.5$, by solving the LMI in Theorem 3.1, the optimal $\mathscr{H}_\infty$ performance index $\gamma = 0.7800$, while the optimal $\mathscr{H}_\infty$ performance index obtained by [1] is $\gamma = 0.9784$. Furthermore, as described in [29], we take noise distraction $\omega(t) = 0.01e^{-0.0005t}\sin(0.02t)$, $t \geq 0$. By using MATLAB LMI Toolbox the gain matrix is obtained as $L = [-0.1261, -0.6261, -0.4078]^T$. Fig.1 shows that the trajectories of true state $x_1(t)$, $x_2(t)$ and $x_3(t)$ and their estimations $\hat{x}_1(t)$, $\hat{x}_2(t)$ and $\hat{x}_3(t)$ with initial values $[-0.5, -0.7, 0.6]^T$ and $[-3, 0.5, -0.9]^T$, respectively. The response of the error $e_1(t)$, $e_2(t)$ and $e_3(t)$ are also given in Fig.1. Therefore, the simulation results illustrate the effectiveness of Theorem 3.1 for the design of guaranteed performance $\mathscr{H}_\infty$ state estimator of the delayed neural network.

In addition, to compare our method with the existing results in [12] and [1], we let $h_1 = 0$ in Theorem 3.1. Then, for different values h_2 and u, the optimal $\mathscr{H}_\infty$ performance index γ can be obtained by Theorem 3.1. The comparison results are listed in Table 1. It can be clearly seen from Table 1 that much better performance is achieved by our approach.

Example 4.2. Consider the delayed static neural networks (2.1) with the following parameters:

$$A = diag\{1.06, 1.42, 0.88\}, \quad W = \begin{bmatrix} -0.32 & 0.85 & -1.36 \\ 1.10 & 0.41 & -0.50 \\ 0.42 & 0.82 & -0.95 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0.5 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix},$$

$$I = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & -0.5 & 0.6 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 & 0.2 \\ 0 & 0 & 0.5 \end{bmatrix},$$

$$B_2 = [0.4 \ \ -0.3]^T, \quad K_1 = diag\{0,0,0\}, \quad K_2 = diag\{1,1,1\}.$$

When $h_1 = 0$, for different values h_2 and u, the optimal $\mathscr{H}_\infty$ performance index γ can be obtained by using the method proposed in this paper. The comparison results are listed in Table 2. It is clear that our results are significant better than those existing in [11, 12].

As similar to the above, the activation function is assumed to be $g(x(t)) = \tanh(x(t))$, and the time-varying delay is taken as $h(t) = 0.5 + 0.5\sin(t)$ and the noise distraction is taken as $\omega(t) = 0.01e^{-0.0005t}\sin(0.02t)$ $(t \geq 0)$, respectively. When the optimal $\mathscr{H}_\infty$ performance index $\gamma = 0.6554$, by using MTLAB LMI Toolbox the gain matrix is obtained as:

$$L = \begin{bmatrix} 0.7608 & -0.3685 \\ 0.7944 & -0.3985 \\ 0.6035 & -0.1430 \end{bmatrix}.$$

Fig. 2 shows that the trajectories of true state $x_1(t)$, $x_2(t)$, $x_3(t)$ and their estimations $\hat{x}_1(t)$, $\hat{x}_2(t)$ and $\hat{x}_3(t)$ with initial values $[1, -2, -0.7]^T$ and $[1.2, -0.8, 0.5]^T$, respectively. And the response of the error $e_1(t)$, $e_2(t)$ and $e_3(t)$ are given simultaneously. Thus, the simulation results illustrate the effectiveness of Theorem 3.1 for the design of guaranteed performance $\mathscr{H}_\infty$ state estimator of the delayed neural network.

Table 1: Comparison of the optimal $\mathscr{H}_\infty$ performance index γ with different (h_2, u).

Methods	$(0.8, 0.4)$	$(0.9, 0.7)$	$(1.1, 0.5)$
[26]	1.2989	1.3164	1.6441
[35]	0.9145	1.0123	1.4252
our results	0.8742	0.9127	1.2033

Table 2: Comparison of the optimal $\mathscr{H}_\infty$ performance index γ with different (h_2, u).

Methods	$(0.8, 0.6)$	$(0.9, 0.8)$	$(1, 0.5)$	$(1.1, 0.4)$
[25]	0.4631	0.8121	1.2142	3.5407
[26]	0.3868	0.4704	0.7594	1.9421
our results	0.3221	0.4178	0.6554	1.2108

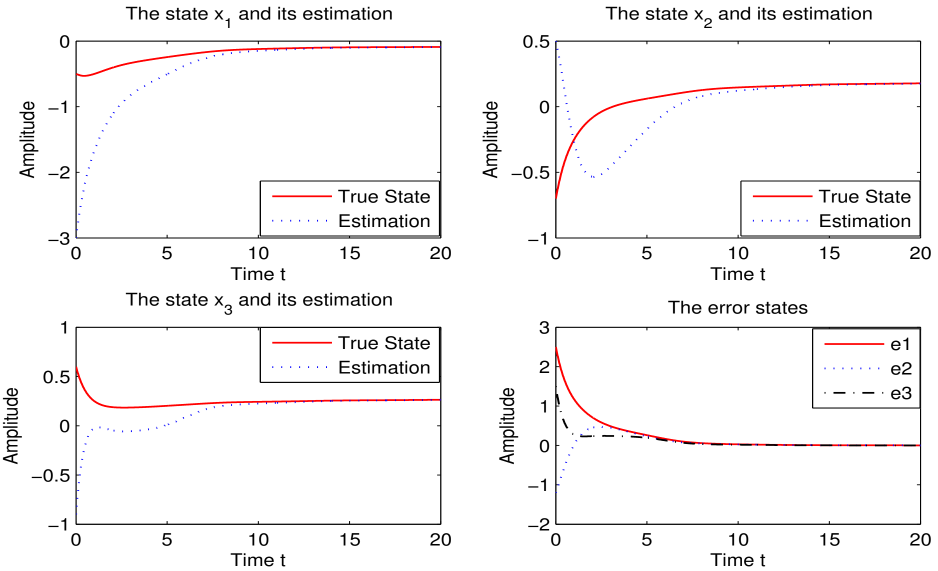

Figure 1: State responses and error trajectories in Example 4.1.

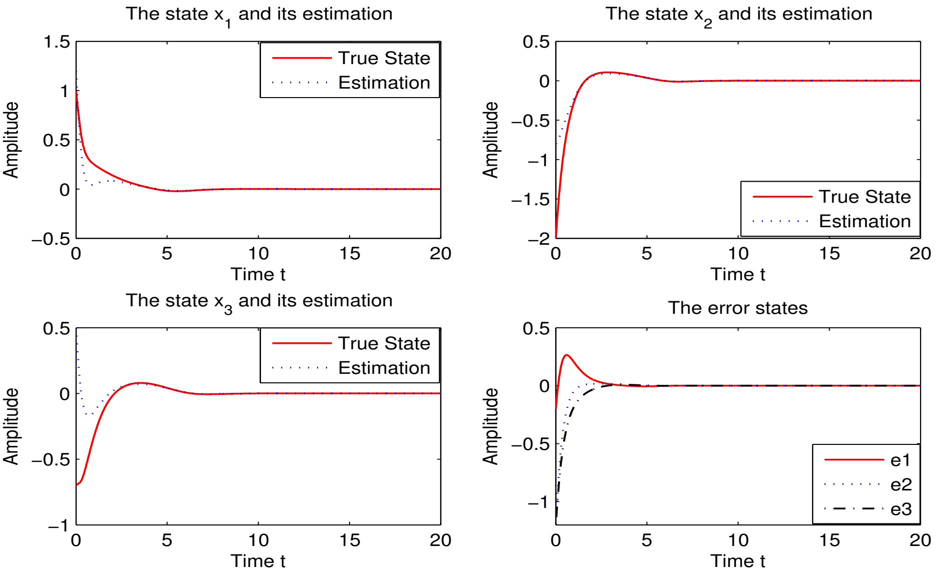

Figure 2: State responses and error trajectories in Example 4.2.

5. Conclusion

In this paper, the problem of stability analysis for a class of static recurrent neural networks with interval time-varying delay is considered. By constructing a properly augmented Lyapunov-Krasovskii functional containing triple integral terms and utilizing the inverses of first-order and squared reciprocally convex parameters techniques and zero equality, new improved delay-dependent stability criteria are proposed to guarantee the asymptotic stability of the concerned networks with the framework of linear matrix inequalities (LMIs). Finally, two numerical examples are given to illustrate the effectiveness of the proposed methods.

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