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Fixed points of mixed non-monotone tripled operators in ordered Banach spaces and applications

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Abstract

This paper is concerned with a class of mixed non-monotone tripled operators under the general conditions of ordering relations in ordered Banach spaces. By means of the cone theory and technique of equivalent norms, the existence and uniqueness of fixed points for such tripled operators are established. The proof method in this paper is different from those used in the former relevant theorems. At last, an application is presented to illustrate our result. We extend some previous existing results.

Keywords: Non-monotone tripled operator, cone theory, equivalent norms, fixed points, Banach spaces.
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1. Introduction and preliminaries

In this paper, we shall focus on the existence and uniqueness of fixed points for mixed non-monotone tripled operators in ordered Banach spaces. Recently, many research papers have appeared concerning with the existence and uniqueness of fixed points for nonlinear operators, such as [1, 2, 4, 5, 7, 8, 12, 14, 15, 17], in which the upper–lower solutions conditions or compactness conditions were usually used.

By using the cone theory, the authors in [10, 13] considered the existence of unique solution for nonlinear operator equations with single element in Banach spaces and applied to a class of abstract semilinear evolution equations with noncompact semigroup and the IVP of nonlinear second order integro-differential equations in Banach spaces. The existence and uniqueness of solutions for (system of) mixed non-monotone binary operator equations were stated in [9, 11, 16]. However, fewer papers dealt with non-monotone tripled operators in ordered Banach spaces via the generating and normal cone theory; see for instance [6].

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Motivated by the above works, we establish the existence and uniqueness of fixed points for mixed non-monotone tripled operators in ordered Banach spaces by means of the cone theory and technique of equivalent norms, which is different method from [6, 9–11, 13, 16]. Moreover, the two-side restriction conditions on the ordering relations of nonlinear operators here are more extensive. The interesting point is that we do not require the upper–lower solutions. Therefore, we extend essentially the corresponding results in [6, 10, 11, 13, 16]. Finally, the results in this paper are applied to the system of nonlinear Volterra integral equations in Banach spaces.

Let $(E, \|\cdot\|)$ be a real Banach space and P be a cone in E. The cone P defines a partial ordering in E by

$$x \leq y \Longleftrightarrow y - x \in P, \ \forall\, x, \ y \in E.$$

For more details about the cone and partial ordering, we refer the reader to [1–3, 5]. θ denotes the zero element in E.

A cone P is called normal if there exists a constant $N > 0$, such that for any $x, \ y \in E, \ \theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. A cone P is called generating if $E = P - P$, i.e., each element $x \in E$ can be represented by $x = y - z$, where $y, \ z \in P$. A cone P is said to be solid if its interior $\overset{o}{P}$ is non-empty.

The product space $E \times E \times E$ is also a real Banach space under the norm

$$\|(x, y, z)\| = \max\{\|x\|, \ \|y\|, \ \|z\|\}, \quad \forall\, (x, y, z) \in E \times E \times E.$$

Let $\tilde{P} = \{(x, y, z) \in E \times E \times E \mid x \geq \theta, \ y \leq \theta, \ z \geq \theta\}$. It is easy to verify that $\tilde{P}$ is a cone in $E \times E \times E$, which defines a partial ordering in $E \times E \times E$ by

$$(x_1, y_1, z_1) \leq (x_2, y_2, z_2) \Longleftrightarrow x_1 \leq x_2, \ y_1 \geq y_2, \ z_1 \leq z_2.$$

Lemma 1.1 ([2]). *P is generating if and only if there exists a constant $\tau > 0$, such that each element $x \in E$ can be represented by $x = y - z$, and $\|y\| \leq \tau\|x\|, \ \|z\| \leq \tau\|x\|$, where $y, \ z \in P$.*

The following two lemmas can be inferred from the similar methods in [2].

Lemma 1.2.

(i) *$\tilde{P}$ is generating if and only if P is generating.*

(ii) *$\tilde{P}$ is normal if and only if P is normal; and the two normal constants are equal.*

Lemma 1.3. *$\tilde{P}$ is generating if and only if there exists a constant $\tau > 0$, such that for each $(x, y, z) \in E \times E \times E$, there exist $(x_1, y_1, z_1), \ (x_2, y_2, z_2) \in \tilde{P}$ with $(x, y, z) = (x_1, y_1, z_1) - (x_2, y_2, z_2)$, and $\|(x_i, y_i, z_i)\| \leq \tau\|(x, y, z)\| (i = 1, 2)$.*

2. Main results

We shall present some results in this section to obtain the existence and uniqueness of fixed points for mixed non-monotone tripled operators in ordered Banach spaces.

Theorem 2.1. *Let P be a normal and generating cone in E, $A : E \times E \times E \to E \times E \times E$ be a nonlinear operator, and $L : E \times E \times E \to E$, $L(x, y, z) = x$. If there exist $u_0 \in P$, three positive bounded linear operators $B_1, B_2, B_3 : E \to E$ with $r(B_1 + B_2 + B_3) < 1$ (where $r(\cdot)$ is the spectral radius of bounded linear operator) and two positive integers n_1, n_2, such that for any $x, y, z, \tilde{x}, \tilde{y}, \tilde{z} \in E, \ x \geq \tilde{x}, \ y \leq \tilde{y}, \ z \geq \tilde{z}$, we have*

$$\begin{aligned} -B_1^{n_2}(x - \tilde{x}) - B_2^{n_2}(\tilde{y} - y) - B_3^{n_2}(z - \tilde{z}) - u_0 &\leq LA^{n_1}(x, y, z) - LA^{n_1}(\tilde{x}, \tilde{y}, \tilde{z}) \\ &\leq B_1^{n_2}(x - \tilde{x}) + B_2^{n_2}(\tilde{y} - y) + B_3^{n_2}(z - \tilde{z}) + u_0. \end{aligned} \tag{2.1}$$

Then the set $\Omega = \Big\{x \in E \mid x = \lambda LA^{n_1}(x, x, x), \ \lambda \in (0, 1)\Big\}$ is bounded.

Proof. In view of $r(B_1+B_2+B_3)<1$, we know there exists a positive constant α, which satisfies

$$\lim_{n\to\infty}\|(B_1+B_2+B_3)^n\|^{\frac{1}{n}}=r(B_1+B_2+B_3)<\alpha<1,$$

and then we choose a positive integer n_0, such that

$$\|(B_1+B_2+B_3)^n\|<\alpha^n<1,\quad n\geq n_0. \tag{2.2}$$

We now denote $\tilde{B}=B_1+B_2+B_3$ and define

$$\|x\|_1=\|x\|+\frac{\|\tilde{B}x\|}{\alpha}+\frac{\|\tilde{B}^2x\|}{\alpha^2}+\cdots+\frac{\|\tilde{B}^{n_0-1}x\|}{\alpha^{n_0-1}},\quad \forall\ x\in E. \tag{2.3}$$

It is clear that $\|\cdot\|_1$ is a norm in E, and from (2.3), we can easily get that

$$\|x\|\leq\|x\|_1\leq\left(1+\frac{\|\tilde{B}\|}{\alpha}+\frac{\|\tilde{B}^2\|}{\alpha^2}+\cdots+\frac{\|\tilde{B}^{n_0-1}\|}{\alpha^{n_0-1}}\right)\|x\|,\quad \forall\ x\in E. \tag{2.4}$$

Thus the two norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent. For any $x\in E$, (2.2) and (2.3) imply that

$$\begin{aligned}\|\tilde{B}x\|_1&=\alpha\left[\frac{\|\tilde{B}x\|}{\alpha}+\frac{\|\tilde{B}^2x\|}{\alpha^2}+\cdots+\frac{\|\tilde{B}^{n_0-1}x\|}{\alpha^{n_0-1}}+\frac{\|\tilde{B}^{n_0}x\|}{\alpha^{n_0}}\right]\\&\leq\alpha\ \|x\|_1,\end{aligned}$$

therefore, we have

$$\|\tilde{B}\|_1\leq\alpha<1. \tag{2.5}$$

Noting that P is a generating cone, by Lemma 1.1, there exists a $\tau>0$, such that each element $x\in E$ can be represented by

$$x=y-z,\quad y,z\in P,\quad \text{and}\quad \|y\|\leq\tau\|x\|,\quad \|z\|\leq\tau\|x\|. \tag{2.6}$$

This immediately yields that

$$-(y+z)\leq x\leq y+z. \tag{2.7}$$

Let

$$\|x\|_2=\inf\Big\{\ \|u\|_1\mid u\in P,\quad \text{and such that }\ -u\leq x\leq u\Big\}. \tag{2.8}$$

From (2.7), we find that $\|x\|_2$ is well-defined for any $x\in E$. Clearly, $\|\cdot\|_2$ is also a norm in E. According to (2.6)-(2.8) and (2.4), we have

$$\|x\|_2\leq\|y+z\|_1\leq 2\tau\left(1+\frac{\|\tilde{B}\|}{\alpha}+\frac{\|\tilde{B}^2\|}{\alpha^2}+\cdots+\frac{\|\tilde{B}^{n_0-1}\|}{\alpha^{n_0-1}}\right)\|x\|_1,\quad \forall\ x\in E. \tag{2.9}$$

Moreover, for any $u\in P$ which satisfies $-u\leq x\leq u$, we notice that $\theta\leq x+u\leq 2u$. This shows that $\|x\|_1\leq\|x+u\|_1+\|-u\|_1\leq(2N+1)\|u\|_1$, where N denotes the normal constant of P under the norm $\|\cdot\|_1$. Since u is arbitrary, we obtain

$$\|x\|_1\leq(2N+1)\|x\|_2,\quad \forall\ x\in E. \tag{2.10}$$

The two inequalities (2.9) and (2.10) indicate that the norms $\|\cdot\|_2$ and $\|\cdot\|_1$ are equivalent.

For any $x,\ y\in E$ and $u\in P$ which satisfy $-u\leq x-y\leq u$, we can easily verify that

$$x\geq u_1,\quad y\geq u_1,\quad u_1=\frac{1}{2}(x+y-u). \tag{2.11}$$

We set

$$u_2 = \frac{1}{2}(x - y + u), \quad u_3 = \frac{1}{2}(y - x + u),$$

then $x - u_1 = u_2,\ y - u_1 = u_3,\ u_2 + u_3 = u$, and we can deduce from (2.1) and (2.11) that

$$-B_3^{n_2} u_2 - u_0 \le LA^{n_1}(x, x, x) - LA^{n_1}(x, x, u_1) \le B_3^{n_2} u_2 + u_0, \tag{2.12}$$

$$-B_2^{n_2} u_2 - B_3^{n_2} u_3 - u_0 \le LA^{n_1}(x, u_1, y) - LA^{n_1}(x, x, u_1) \le B_2^{n_2} u_2 + B_3^{n_2} u_3 + u_0, \tag{2.13}$$

$$-B_1^{n_2} u_2 - B_2^{n_2} u_3 - u_0 \le LA^{n_1}(x, u_1, y) - LA^{n_1}(u_1, y, y) \le B_1^{n_2} u_2 + B_2^{n_2} u_3 + u_0, \tag{2.14}$$

$$-B_1^{n_2} u_3 - u_0 \le LA^{n_1}(y, y, y) - LA^{n_1}(u_1, y, y) \le B_1^{n_2} u_3 + u_0. \tag{2.15}$$

By (2.12)-(2.15), then we get

$$-(B_1^{n_2} + B_2^{n_2} + B_3^{n_2})u - 4u_0 \le LA^{n_1}(x, x, x) - LA^{n_1}(y, y, y) \le (B_1^{n_2} + B_2^{n_2} + B_3^{n_2})u + 4u_0,$$

which implies that

$$-\tilde{B}^{n_2} u - 4u_0 \le LA^{n_1}(x, x, x) - LA^{n_1}(y, y, y) \le \tilde{B}^{n_2} u + 4u_0. \tag{2.16}$$

Since $B_i^{n_2}$ $(i = 1, 2, 3)$ are positive operators and $u_0 \in P$, we can derive from (2.16), (2.8), and (2.5) that

$$\begin{aligned} \|LA^{n_1}(x, x, x) - LA^{n_1}(y, y, y)\|_2 &\le \|\tilde{B}^{n_2} u\|_1 + 4\|u_0\|_1 \\ &\le \|\tilde{B}\|_1^{n_2} \|u\|_1 + 4\|u_0\|_1 \\ &\le \alpha^{n_2} \|u\|_1 + 4\|u_0\|_1, \quad \forall\ x, y \in E. \end{aligned}$$

By the arbitrariness of u again, we have

$$\|LA^{n_1}(x, x, x) - LA^{n_1}(y, y, y)\|_2 \le \alpha^{n_2} \|x - y\|_2 + 4\|u_0\|_1, \quad \forall\ x, y \in E. \tag{2.17}$$

Hence, for any $x \in \Omega$, we obtain $\|x\|_2 = \lambda \|LA^{n_1}(x, x, x)\|_2 \le \alpha^{n_2} \|x\|_2 + 4\|u_0\|_1 + \|LA^{n_1}(\theta, \theta, \theta)\|_2$, thus

$$\|x\|_2 \le \frac{4\|u_0\|_1 + \|LA^{n_1}(\theta, \theta, \theta)\|_2}{1 - \alpha^{n_2}}. \tag{2.18}$$

Further, (2.4), (2.10), and (2.18) yield that

$$\begin{aligned} \|x\| \le \|x\|_1 &\le (2N + 1)\|x\|_2 \\ &\le (2N + 1) \frac{4\|u_0\|_1 + \|LA^{n_1}(\theta, \theta, \theta)\|_2}{1 - \alpha^{n_2}}, \end{aligned}$$

i.e., the set Ω is bounded. The proof is complete. □

Remark 2.2. If $u_0 = \theta$ in (2.1), by virtue of (2.17) and the Banach contraction mapping principle, we know that the operator equation $LA^{n_1}(x, x, x) = x$ has a unique solution x^* in E and for any $x_0 \in E$, the iterative sequence $x_n = LA^{n_1}(x_{n-1}, x_{n-1}, x_{n-1})$ converges to x^* $(n \to \infty)$. Hence this theorem can involve the conclusions of Theorem 2.2 in [6].

Remark 2.3. When the nonlinear operator LA^{n_1} in Theorem 2.1 is completely continuous or condensing, the famous Leray-Schauder fixed point theorem implies that LA^{n_1} has at least a fixed point in the closed ball $B_R = \{x \in E \mid \|x\| \le R\}$ of E, where $R = \sup\{\|x\| \mid x \in E,\ x = \lambda LA^{n_1}(x, x, x),\ \lambda \in (0, 1)\}$.

Theorem 2.4. *Let P be a normal and generating cone in E, $A_i (i = 1, 2, 3) : E \times E \times E \to E \times E \times E$ be nonlinear operators, and $L_i (i = 1, 2, 3) : E \times E \times E \to E$ defined by $L_1(x, y, z) = x, L_2(x, y, z) = y, L_3(x, y, z) = z$. If there exist $(u_0, -v_0, w_0) \in \tilde{P}$, three positive bounded linear operators $B_1, B_2, B_3 : E \to E$*

with $r(B_i) < 1 (i = 1, 2, 3)$, *and two positive integers* n_1, n_2, *such that for any* $x, y, z, \tilde{x}, \tilde{y}, \tilde{z} \in E,\ x \ge \tilde{x},\ y \le \tilde{y},\ z \ge \tilde{z}$, *we have*

$$- B_1^{n_2}(x - \tilde{x}) - u_0 \le L_1 A_1^{n_1}(x, y, z) - L_1 A_1^{n_1}(\tilde{x}, \tilde{y}, \tilde{z}) \le B_1^{n_2}(x - \tilde{x}) + u_0, \tag{2.19}$$

$$- B_2^{n_2}(\tilde{y} - y) - v_0 \le L_2 A_2^{n_1}(x, y, z) - L_2 A_2^{n_1}(\tilde{x}, \tilde{y}, \tilde{z}) \le B_2^{n_2}(\tilde{y} - y) + v_0, \tag{2.20}$$

$$- B_3^{n_2}(z - \tilde{z}) - w_0 \le L_3 A_3^{n_1}(x, y, z) - L_3 A_3^{n_1}(\tilde{x}, \tilde{y}, \tilde{z}) \le B_3^{n_2}(z - \tilde{z}) + w_0. \tag{2.21}$$

Then the set

$$\Omega = \Big\{(x, y, z) \in E \times E \times E \mid (x, y, z) = \lambda\Big(L_1 A_1^{n_1}(x, y, z), L_2 A_2^{n_1}(x, y, z), L_3 A_3^{n_1}(x, y, z)\Big), \lambda \in (0, 1)\Big\},$$

is bounded.

Proof. Define the operator $L:\ E \times E \times E \to E \times E \times E$ by

$$L(x, y, z) = \Big(L_1 A_1^{n_1}(x, y, z),\ L_2 A_2^{n_1}(x, y, z),\ L_3 A_3^{n_1}(x, y, z)\Big), \quad \forall\ (x, y, z) \in E \times E \times E. \tag{2.22}$$

For any $x, y, z, \tilde{x}, \tilde{y}, \tilde{z} \in E,\ x \ge \tilde{x},\ y \le \tilde{y},\ z \ge \tilde{z}$, from (2.19)-(2.22), we get

$$\begin{aligned} \hat{L}(x, y, z) - \hat{L}(\tilde{x}, \tilde{y}, \tilde{z}) =& \Big(L_1 A_1^{n_1}(x, y, z) - L_1 A_1^{n_1}(\tilde{x}, \tilde{y}, \tilde{z}), L_2 A_2^{n_1}(x, y, z) \\ & - L_2 A_2^{n_1}(\tilde{x}, \tilde{y}, \tilde{z}), L_3 A_3^{n_1}(x, y, z) - L_3 A_2^{n_1}(\tilde{x}, \tilde{y}, \tilde{z})\Big) \\ \le& \Big(B_1^{n_2}(x - \tilde{x}) + u_0, -B_2^{n_2}(\tilde{y} - y) - v_0, B_3^{n_2}(z - \tilde{z}) + w_0\Big) \\ =& \Big(B_1^{n_2}(x - \tilde{x}), B_2^{n_2}(y - \tilde{y}), B_3^{n_2}(z - \tilde{z})\Big) + (u_0, -v_0, w_0). \end{aligned} \tag{2.23}$$

Similarly, we have

$$\hat{L}(x, y, z) - \hat{L}(\tilde{x}, \tilde{y}, \tilde{z}) \ge -\Big(B_1^{n_2}(x - \tilde{x}), B_2^{n_2}(y - \tilde{y}), B_3^{n_2}(z - \tilde{z})\Big) - (u_0, -v_0, w_0). \tag{2.24}$$

We also define the operator $B:\ E \times E \times E \to E \times E \times E$ as follows

$$\hat{B}(x, y, z) = (B_1 x, B_2 y, B_3 z) \quad \forall\ (x, y, z) \in E \times E \times E. \tag{2.25}$$

Thus (2.23)-(2.25) imply that

$$\begin{aligned} -\hat{B}^{n_2}(x - \tilde{x}, y - \tilde{y}, z - \tilde{z}) - (u_0, -v_0, w_0) \le& \hat{L}(x, y, z) - \hat{L}(\tilde{x}, \tilde{y}, \tilde{z}) \\ \le& \hat{B}^{n_2}(x - \tilde{x}, y - \tilde{y}, z - \tilde{z}) + (u_0, -v_0, w_0). \end{aligned} \tag{2.26}$$

It follows from $r(B_i) < 1$ $(i = 1, 2, 3)$ and (2.25) that we can select a positive constant β, which satisfies $\lim\limits_{n\to\infty} \|\hat{B}^n\|^{\frac{1}{n}} = r(\hat{B}) < \max\{r(B_1), r(B_2), r(B_3)\} < \beta < 1$. Thus there exists a positive integer n_0, such that

$$\|\hat{B}^n\| < \beta^n < 1, \quad n \ge n_0.$$

For any $(x, y, z) \in E \times E \times E$, we define

$$\|(x, y, z)\|_1 = \|(x, y, z)\| + \frac{\|\hat{B}(x, y, z)\|}{\beta} + \frac{\|\hat{B}^2(x, y, z)\|}{\beta^2} + \cdots + \frac{\|\hat{B}^{n_0-1}(x, y, z)\|}{\beta^{n_0-1}}. \tag{2.27}$$

Evidently, $\|\cdot\|_1$ is a norm in $E \times E \times E$. Using the similar method in Theorem 2.1 , we can show that the two norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent and

$$\|\hat{B}\|_1 \le \beta < 1. \tag{2.28}$$

Since P is a normal and generating cone, Lemma 1.2 implies that $\tilde{P}$ is also a normal and generating cone. By Lemma 1.3, there exists a $\tau > 0$, such that each element $(x, y, z) \in E \times E \times E$ can be represented by

$$(x, y, z) = (x_1, y_1, z_1) - (x_2, y_2, z_2), \quad (x_i, y_i, z_i) \in \tilde{P}, \text{ and } \|(x_i, y_i, z_i)\| \le \tau\|(x, y, z)\|, \ i = 1, 2. \tag{2.29}$$

From (2.29), we easily see that

$$-(x_1, y_1, z_1) - (x_2, y_2, z_2) \le (x, y, z) \le (x_1, y_1, z_1) + (x_2, y_2, z_2). \tag{2.30}$$

Now we set

$$\|(x, y, z)\|_2 = \inf\Big\{ \|(u, v, w)\|_1 \mid (u, v, w) \in \tilde{P}, \text{ and such that } -(u, v, w) \le (x, y, z) \le (u, v, w)\Big\}. \tag{2.31}$$

By (2.30), we observe that $\|(x, y, z)\|_2$ is well-defined for any $(x, y, z) \in E \times E \times E$. It is obvious that $\| \cdot \|_2$ is a norm in $E \times E \times E$. By (2.29)-(2.31) and (2.27), for any $(x, y, z) \in E \times E \times E$, we find

$$\begin{aligned}\|(x, y, z)\|_2 &\le \|(x_1, y_1, z_1) + (x_2, y_2, z_2)\|_1 \\ &\le 2\tau\left(1 + \frac{\|\hat{B}\|}{\beta} + \frac{\|\hat{B}^2\|}{\beta^2} + \cdots + \frac{\|\hat{B}^{n_0-1}\|}{\beta^{n_0-1}}\right)\|(x, y, z)\|_1.\end{aligned} \tag{2.32}$$

Also for any $(u, v, w) \in \tilde{P}$ with $-(u, v, w) \le (x, y, z) \le (u, v, w)$, we know $\theta \le (x, y, z)+(u, v, w) \le 2(u, v, w)$, which implies that

$$\|(x, y, z)\|_1 \le \|(x, y, z) + (u, v, w)\|_1 + \| - (u, v, w)\|_1 \le (2N' + 1)\|(u, v, w)\|_1,$$

where N' denotes the normal constant of $\tilde{P}$ under the norm $\| \cdot \|_1$. Since (u, v, w) is arbitrary, we get

$$\|(x, y, z)\|_1 \le (2N' + 1)\|(x, y, z)\|_2, \quad \forall\ (x, y, z) \in E \times E \times E. \tag{2.33}$$

From (2.32) and (2.33), we conclude that the norms $\| \cdot \|_2$ and $\| \cdot \|_1$ are equivalent.

For any $(x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}) \in E \times E \times E$ and $(u, v, w) \in \tilde{P}$ with the form $-(u, v, w) \le (x, y, z) - (\tilde{x}, \tilde{y}, \tilde{z}) \le (u, v, w)$, we obtain

$$x \ge u^*, \quad \tilde{x} \ge u^*, \quad u^* = \frac{1}{2}(x + \tilde{x} - u), \tag{2.34}$$

$$y \le v^*, \quad \tilde{y} \le v^*, \quad v^* = \frac{1}{2}(y + \tilde{y} - v), \tag{2.35}$$

$$z \ge w^*, \quad \tilde{z} \ge w^*, \quad w^* = \frac{1}{2}(z + \tilde{z} - w). \tag{2.36}$$

We can deduce from (2.26) and (2.34)-(2.36) that

$$-\hat{B}^{n_2}(u_5, v_5, w_5) - (u_0, -v_0, w_0) \le \hat{L}(x, y, z) - \hat{L}(u^*, v^*, w^*) \le \hat{B}^{n_2}(u_5, v_5, w_5) + (u_0, -v_0, w_0), \tag{2.37}$$

$$-\hat{B}^{n_2}(u_6, v_6, w_6) - (u_0, -v_0, w_0) \le \hat{L}(\tilde{x}, \tilde{y}, \tilde{z}) - \hat{L}(u^*, v^*, w^*) \le \hat{B}^{n_2}(u_6, v_6, w_6) + (u_0, -v_0, w_0), \tag{2.38}$$

where

$$\begin{aligned} &u_5 = \frac{x - \tilde{x} + u}{2}, \quad v_5 = \frac{y - \tilde{y} + v}{2}, \quad w_5 = \frac{z - \tilde{z} + w}{2}, \\ &u_6 = \frac{\tilde{x} - x + u}{2}, \quad v_6 = \frac{\tilde{y} - y + v}{2}, \quad w_6 = \frac{\tilde{z} - z + w}{2}, \\ &u_5 + u_6 = u, \qquad v_5 + v_6 = v, \qquad w_5 + w_6 = w. \end{aligned}$$

Subtracting (2.38) from (2.37), we arrive at

$$-\hat{B}^{n_2}(u,v,w)-2(u_0,-v_0,w_0)\leq \hat{L}(x,y,z)-\hat{L}(\tilde{x},\tilde{y},\tilde{z})\leq \hat{B}^{n_2}(u,v,w)+2(u_0,-v_0,w_0). \tag{2.39}$$

Since $\hat{B}^{n_2}(u,v,w)\in\tilde{P}$ and $(u_0,-v_0,w_0)\in\tilde{P}$, it follows from (2.39), (2.31), and (2.28) that

$$\|\hat{L}(x,y,z)-\hat{L}(\tilde{x},\tilde{y},\tilde{z})\|_2\leq \|\hat{B}^{n_2}(u,v,w)\|_1+2\|(u_0,-v_0,w_0)\|_1\leq \beta^{n_2}\|(u,v,w)\|_1+2\|(u_0,-v_0,w_0)\|_1.$$

For each $(x,y,z)\in E\times E\times E$ and arbitrary (u,v,w), by the definition of new norm $\|\cdot\|_2$, we get

$$\|\hat{L}(x,y,z)-\hat{L}(\tilde{x},\tilde{y},\tilde{z})\|_2\leq \beta^{n_2}\|(x,y,z)-(\tilde{x},\tilde{y},\tilde{z})\|_2+2\|(u_0,-v_0,w_0)\|_1. \tag{2.40}$$

Consequently, for any $(x,y,z)\in\Omega$, we find

$$\begin{aligned}\|(x,y,z)\|_2&=\lambda\|(L_1A_1^{n_1}(x,y,z),L_2A_2^{n_1}(x,y,z),L_3A_3^{n_1}(x,y,z))\|_2\\&=\lambda\|\hat{L}(x,y,z)\|_2\\&\leq \beta^{n_2}\|(x,y,z)\|_2+2\|(u_0,-v_0,w_0)\|_1+\|\hat{L}(\theta,\theta,\theta)\|_2,\end{aligned}$$

then the above inequality leads to

$$\|(x,y,z)\|_2\leq \frac{2\|(u_0,-v_0,w_0)\|_1+\|\hat{L}(\theta,\theta,\theta)\|_2}{1-\beta^{n_2}}. \tag{2.41}$$

From (2.27), (2.33), and (2.41), we infer that

$$\begin{aligned}\|(x,y,z)\|\leq\|(x,y,z)\|_1&\leq(2N'+1)\|(x,y,z)\|_2\\&\leq(2N'+1)\frac{2\|(u_0,-v_0,w_0)\|_1+\|\hat{L}(\theta,\theta,\theta)\|_2}{1-\beta^{n_2}},\end{aligned}$$

i.e., the set Ω is bounded. This finishes the proof. □

Remark 2.5. If $(u_0,-v_0,w_0)=(\theta,\theta,\theta)$, by (2.40) and the Banach contraction mapping principle, we also obtain the existence and uniqueness of solutions and the iterative approximation sequences for the following systems of operator equations

$$\begin{cases}L_1A_1^{n_1}(x,y,z)=x,\\L_2A_2^{n_1}(x,y,z)=y,\\L_3A_3^{n_1}(x,y,z)=z.\end{cases}$$

Hence we generalize the conditions and conclusions of Theorem 2.1 in [6] directly. Meanwhile, the existence of fixed points can also be proved by the Leray-Schauder fixed point theorem.

Remark 2.6. The two-side restriction conditions (2.1) and (2.19)-(2.21) with respect to the ordering relations of nonlinear operators in Theorems 2.1 and 2.4 are a kind of more general conditions than those in [6, 10, 11, 13, 16], from which we can not only deduce the existence but also the uniqueness of fixed points.

Remark 2.7. In Theorems 2.1 and 2.4, we consider the more general nonlinear mixed non-monotone tripled operators LA^{n_1} and $L_iA_i^{n_1}(i=1,2,3)$ not LA and $L_iA_i(i=1,2,3)$ to satisfy the conditions of ordering relations.

3. An application

In order to apply Theorem 2.4 in this paper, we discuss the system of nonlinear Volterra integral equations in Banach space E.

Let $I=[0,T]$ $(T>0)$ and $C[I,E]$ be a Banach space of all continuous functions from I into E with the norm $\|x\|_C=\sup\{\|x(t)\|\mid t\in I\}$. We also set $P_C=\{u\in C[I,E]\mid u(t)\geq\theta,\ t\in I\}$, then it is easy to check that P_C is a cone in $C[I,E]$. The partial ordering defined by P_C in $C[I,E]$ is still denoted by $\leq$.

Consider the following system of nonlinear Volterra integral equations in E and define the nonlinear operators $A_i: C[I,E]\times C[I,E]\times C[I,E]\to C[I,E]$ as follows

$$\begin{cases} x(t) = x_0(t) + \displaystyle\int_0^t f_1\left(t, s, x(s), \int_0^s k_1(s,\tau)g_1(\tau, y(\tau))\mathrm{d}\tau, \int_0^T k_2(s,\tau)g_2(\tau, z(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ \qquad \triangleq A_1(x(t), y(t), z(t)), \quad t \in I, \\ y(t) = y_0(t) + \displaystyle\int_0^t f_2\left(t, s, x(s), \int_0^s k_3(s,\tau)g_3(\tau, y(\tau))\mathrm{d}\tau, \int_0^T k_4(s,\tau)g_4(\tau, z(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ \qquad \triangleq A_2(x(t), y(t), z(t)), \quad t \in I, \\ z(t) = z_0(t) + \displaystyle\int_0^t f_3\left(t, s, x(s), \int_0^s k_5(s,\tau)g_5(\tau, y(\tau))\mathrm{d}\tau, \int_0^T k_6(s,\tau)g_6(\tau, z(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ \qquad \triangleq A_3(x(t), y(t), z(t)), \quad t \in I, \end{cases} \tag{3.1}$$

where $x_0,\ y_0,\ z_0 \in C[I, E]$, $f_i : D \times E \times E \times E \to E$ $(i = 1, 2, 3)$, $D = \{(t, s) \in I \times I \mid s \leq t\}$, and $k_1, k_3, k_5 \in C[D,\ R^+]$, $k_2, k_4, k_6 \in C[I \times I,\ R^+]$, $g_1, g_3, g_5 : D_0 \times E \to E$, $g_2, g_4, g_6 : I \times E \to E$, $D_0 = \{s \in I \mid s \leq t\ \forall\, t \in I\}$. For any $x, y, z \in C[I, E]$, $f_1\Big(t, s, x(s), \int_0^s k_1(s,\tau)g_1(\tau, y(\tau))\mathrm{d}\tau,\ \int_0^T k_2(s,\tau)g_2(\tau, z(\tau))\mathrm{d}\tau\Big) : D \to E$ is continuous and f_2, f_3 have the same meanings.

It is easy to see that $(x, y, z) \in C[I, E] \times C[I, E] \times C[I, E]$ is the solution of system (3.1) if and only if $(x, y, z) = (A_1(x, y, z), A_2(x, y, z), A_3(x, y, z))$.

Theorem 3.1. *Let P be a normal solid cone of E. Suppose that $g_i(\tau, y)$ $(\tau \in I, i = 1, 3, 5)$ are nondecreasing in y and $g_j(\tau, z)$ $(\tau \in I, j = 2, 4, 6)$ are nondecreasing in z. There exist $u_0, v_0, w_0 \in P_C$, the constants $M_k \geq 0$ $(k = 1, 2, 3)$ and $N_l \geq 0$ $(l = 1, 2, 3, 4, 5)$, such that for any $(t, s) \in D, x, y, z, \tilde{x}, \tilde{y}, \tilde{z} \in C[I, E], x \geq \tilde{x}, y \leq \tilde{y}, z \geq \tilde{z}$, we have*

$$-M_1(x - \tilde{x}) - u_0 \leq f_1(t, s, x, y, z) - f_1(t, s, \tilde{x}, \tilde{y}, \tilde{z}) \leq N_1(x - \tilde{x}) + u_0, \tag{3.2}$$

$$-M_2(\tilde{y} - y) - v_0 \leq f_2(t, s, x, y, z) - f_2(t, s, \tilde{x}, \tilde{y}, \tilde{z}) \leq N_2(\tilde{y} - y) + v_0, \tag{3.3}$$

$$-M_3(z - \tilde{z}) - w_0 \leq f_3(t, s, x, y, z) - f_3(t, s, \tilde{x}, \tilde{y}, \tilde{z}) \leq N_3(z - \tilde{z}) + w_0, \tag{3.4}$$

$$g_3(\tau, y) - g_3(\tau, \tilde{y}) \leq N_4(\tilde{y} - y), \tag{3.5}$$

$$g_6(\tau, z) - g_6(\tau, \tilde{z}) \leq N_5(z - \tilde{z}), \tag{3.6}$$

and

$$N_5 T \max\{M_3, N_3\} \max_{t\in I} \int_0^T k_6(t, s) ds < 1. \tag{3.7}$$

Then the set

$$\Omega = \Big\{(x, y, z) \in C[I,\ E] \times C[I,\ E] \times C[I,\ E] \mid (x, y, z) = \lambda\Big(A_1(x, y, z), A_2(x, y, z), A_3(x, y, z)\Big), \lambda \in (0, 1)\Big\}$$

is bounded.

Proof. For any $x, y, z, \tilde{x}, \tilde{y}, \tilde{z} \in C[I, E], x \geq \tilde{x}, y \leq \tilde{y}, z \geq \tilde{z}$ and $t \in I$, from (3.1) and (3.2), we get

$$\begin{aligned} -B_1(x(t) - \tilde{x}(t)) - \int_0^t u_0(s)\mathrm{d}s \leq & A_1(x(t), y(t), z(t)) - A_1(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \\ \leq & B_1(x(t) - \tilde{x}(t)) + \int_0^t u_0(s)\mathrm{d}s, \end{aligned} \tag{3.8}$$

where $B_1 x(t) = \displaystyle\int_0^t \max\{M_1, N_1\}x(s)\mathrm{d}s$. Equations (3.1), (3.3), and (3.5) again imply that

$$\begin{aligned} -B_2(\tilde{y}(t) - y(t)) - \int_0^t v_0(s)\mathrm{d}s \leq & A_2(x(t), y(t), z(t)) - A_2(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \\ \leq & B_2(\tilde{y}(t) - y(t)) + \int_0^t v_0(s)\mathrm{d}s, \end{aligned} \tag{3.9}$$

where $B_2y(t) = N_4 \max\{M_2, N_2\} \int_0^t \int_0^s k_3(s,\tau)y(\tau)d\tau$. We also find, by (3.1), (3.4), and (3.6) that

$$\begin{aligned} -B_3(z(t)-\tilde{z}(t)) - \int_0^t w_0(s)\mathrm{d}s \leq & A_3(x(t),y(t),z(t)) - A_3(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)) \\ & \leq B_3(z(t)-\tilde{z}(t)) + \int_0^t w_0(s)\mathrm{d}s, \end{aligned} \tag{3.10}$$

where $B_3z(t) = N_5 \max\{M_3, N_3\} \int_0^t \int_0^T k_6(s,\tau)z(\tau)d\tau ds$. Using the similar method in [10], we can show that $r(B_1) = r(B_2) = 0$. It is well-known that $\|B_3\| = N_5 \max\{M_3, N_3\} T \max_{t\in I} \int_0^T k_6(t,s)ds$, thus it follows from (3.7) that $r(B_3) \leq \|B_3\| < 1$.

Since P is a normal and solid cone of E, it is easy to prove that P_C is also normal and solid in $C[I,E]$ directly from [3]. Thus, by [2], we know P_C is a generating cone in $C[I,E]$. Therefore, the conclusion of Theorem 3.1 can deduce from (3.8)-(3.10) and Theorem 2.4 (in which $n_1 = n_2 = 1$). □

Remark 3.2. Because the conditions in Theorems 2.1 and 2.4 of this paper are more general, their applications to nonlinear differential and integral equations must be extensive under the two side ordering relations conditions. Moreover, this also means that our results can not be obtained by the proof methods and results in [6, 10, 11, 13, 16].

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