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Hyperstability of a quadratic functional equation on abelian group and inner product spaces

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Abstract

Using the fixed point approach, we prove some results on hyperstability of the following quadratic functional equation

$$f(x+y+z)+f(x-y)+f(x-z)+f(y-z)=3[f(x)+f(y)+f(z)],$$

in the class of functions from an abelian group into a Banach space.

Keywords: Hyperstability, quadratic functional equation, fixed point theorem.
2010 MSC: 39B82, 39B52, 47H14, 47H10.

1. Introduction

The main motivation for the investigation of the stability of functional equations was given by Ulam in 1940 in his talk at the university of Wisconsin (see [29]), where he presented the following unsolved problem, among others.

Let $(G_1, \cdot)$ be a group and let $(G_2, \cdot, d)$ be a metric group. Given $\delta > 0$, does there exist $\epsilon > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) \leq \delta$$

for all $x, y \in G_1$, then there is a homomorphism $h : G_1 \to G_2$ with

$$d(f(x), h(x)) \leq \epsilon$$

for all $x \in G_1$?

Ulam's problem was partially solved by Hyers in 1941 as follows:

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Theorem 1.1 ([17])**.** *Let E be a normed vector space, F a Banach space and suppose that the mapping $f : E \to F$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$, where ϵ is a constant. Then the limit

$$T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in E$, and T is the unique additive mapping satisfying

$$\|f(x) - T(x)\| \leq \epsilon$$

for all $x \in E$.

Bourgin [6], Aoki [1], Rassias [23], and Gajda [14] treated this problem for approximate additive mappings controlled by variables and unbounded functions.

Theorem 1.2. *Let $f : E \to F$ be a mapping from a real normed vector space E into a Banach space F satisfying the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in E\backslash\{0\}$, where θ and p are constants with $\theta > 0$ and $p \neq 1$. Then there exists a unique additive mapping $T : E \to F$ such that

$$\|f(x) - T(x)\| \leq \frac{\theta}{|1 - 2^{p-1}|} \|x\|^p$$

for all $x \in E\backslash\{0\}$.

Theorem 1.2 is due to Aoki [1] for $0 < p < 1$ (see also [23]); Gajda [14] for $p > 1$; Hyers [17] for $p = 0$. Moreover, Rassias [24] extended it to a linear mapping under the additional condition that f is continuous. In particular, Bourgin [6] had commended the stability bounded by function on C^*-algebra.

In 1994, Găvruta [15] generalized Rassias's result [24] by replacing $\theta(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$.

The stability problems for various functional equations have been extensively investigated by a number of researchers and there are many interesting results concerned with this problem (see [12, 18, 19, 21, 25–28]).

Bae [2] and Bae et al. [3, 4] proved the stability of the quadratic functional equation

$$f(x+y+z) + f(x-y) + f(x-z) + f(y-z) = 3[f(x) + f(y) + f(z)]. \tag{1.1}$$

Lemma 1.3 ([2])**.** *If vector spaces X and Y are common domain and range of the mapping f in both the functional equations* (*) : $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ *and* (1.1)*, then the functional equation* (1.1) *is equivalent to the functional equation* (*)*.*

We say that the functional equation $\mathfrak{D}$ is *hyperstable* if any function f satisfying the equation $\mathfrak{D}$ is approximately a true solution of $\mathfrak{D}$. The hyperstability term was used for the first time probably in [20]. However, it seems that the first result for the hyperstability concerned with the ring homomorphisms was published in [6]. The hyperstability of the some functional equations, among others those mentioned above were studied by many authors (cf., e.g., [5, 7–9, 11, 13, 16, 22]).

2. Auxiliary results

In this paper, $\mathbb{N}$ stands for the set of natural numbers, $\mathbb{Z}$ stands for the set of integers and $\mathbb{R}$ stands for the set of reals. Let $\mathbb{R}_+ := [0, \infty)$ be the set of nonnegative real numbers and Y^X denotes the family of all mappings from a nonempty set X into a nonempty set Y.

The proof's method of the main results is based on a fixed point theorem in [10, Theorem 1]. Our method can be considered to be an extension of the investigations in [2–4, 18, 21].

Now, we will take the following three hypotheses (all notations come from [10]).

(H1) U is a nonempty set, V is a Banach space, $f_1, ...f_k : U \to U$, and $L_1, ...L_k : U \to \mathbb{R}_+$ are given.

(H2) $\mathcal{T} : V^U \to V^U$ is an operator satisfying the inequality

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^{k} L_i(x) \, \|\xi(f_i(x)) - \mu(f_i(x))\|$$

for all $\xi, \mu \in V^U$, $x \in U$.

(H3) $\Lambda : \mathbb{R}_+^U \to \mathbb{R}_+^U$ is a linear operator defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x))$$

for all $\delta \in \mathbb{R}_+^U$, $x \in U$.

The mentioned fixed point theorem is stated in [10] as follows.

Theorem 2.1. *Let hypotheses* (H1)-(H3) *be valid and functions* $\varepsilon : U \to \mathbb{R}_+$ *and* $\varphi : U \to V$ *fulfill the following two conditions:*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \varepsilon(x), \quad x \in U,$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \quad x \in U.$$

Then, there exists a unique fixed point ψ *of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \quad x \in U.$$

Moreover,

$$\psi(x) = \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \quad x \in U.$$

The main purpose of this paper is to reformulate the work that is in [2–4] on an abelian group by using another fixed point method.

3. Main results

The following theorem is the main result of this paper. It has been motivated by the issue of Ulam stability, which concerns approximate solutions of quadratic functional equation (1.1).

Theorem 3.1. *Let* $(G, +)$ *be an abelian group and* E *be a Banach space. Let* $f : G \to E$, $\varphi : G^3 \to [0, \infty)$ *and* $u : \mathbb{Z}^* = \mathbb{Z}\backslash\{0\} \to [0, \infty)$ *be functions satisfying the following three conditions*

$$\mathcal{M} := \{m \in \mathbb{Z}^* : 2u(m-1) + 3u(m) + 3u(-m) + u(2m) + u(2m-1) < 1\} \neq \emptyset, \tag{3.1}$$

$$\varphi(tx, ty, tz) \leq u(t)\varphi(x, y, z), \tag{3.2}$$

$$\|f(x + y + z) + f(x - y) + f(x - z) + f(y - z) - 3[f(x) + f(y) + f(z)]\| \leq \varphi(x, y, z) \tag{3.3}$$

for all $x, y, z \in G$, $t \in \{m-1, m, -m, 2m, 2m-1\}$ *and* $m \in \mathcal{M}$. *Then there exists a unique function* $Q : G \to E$ *satisfying* (1.1) *and*

$$\|f(x) - Q(x)\| \leq \phi(x), \tag{3.4}$$

where

$$\phi(x) := \inf_{m\in\mathcal{M}} \frac{\varphi(mx, (m-1)x, -mx)}{1 - 2u(m-1) - 3u(m) - 3u(-m) - u(2m) - u(2m-1)}$$

for all $x \in G$.

Proof. Replacing (x, y, z) by $(mx, (m-1)x, -mx)$ in (3.3), we get

$$\begin{aligned}\|2f((m-1)x) + 3f(mx) + 3f(-mx) - f(2mx) - f((2m-1)x) - f(x)\| \\ \le \varphi(mx, (m-1)x, -mx) := \varepsilon_m(x)\end{aligned} \tag{3.5}$$

for all $x \in G$ and $m \in \mathbb{Z}^*$. Further let us define a mapping $\mathcal{T} : E^G \to E^G$ by

$$\mathcal{T}\xi(x) := 2\xi((m-1)x) + 3\xi(mx) + 3\xi(-mx) - \xi(2mx) - \xi((2m-1)x), \;\; \forall x \in G, \; \xi \in E^G, \; m \in \mathbb{Z}^*.$$

Then the inequality (3.5) takes the form

$$\|\mathcal{T}f(x) - f(x)\| \le \varepsilon_m(x), \;\; x \in G.$$

Now, we define an operator $\Lambda : \mathbb{R}_+^G \to \mathbb{R}_+^G$ for $m \in \mathbb{Z}^*$ by

$$\Lambda\delta(x) := 2\delta((m-1)x) + 3\delta(mx) + 3\delta(-mx) + \delta(2mx) + \delta((2m-1)x), \; x \in G, \;\; \delta \in \mathbb{R}_+^G.$$

This operator has the form described in (H3) with $k = 5$ and $f_1(x) = (m-1)x$, $f_2(x) = mx = -f_3(x)$, $f_4(x) = 2mx$, $f_5(x) = (2m-1)x$, $L_1(x) = 2$, $L_2(x) = 3 = L_3(x)$, and $L_4(x) = L_5(x) = 1$ for $x \in G$. Moreover, for every $\xi, \mu \in E^G$ and $x \in G$, we obtain

$$\begin{aligned}&\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \\ &\quad = \|2(\xi-\mu)(f_1(x)) + 3(\xi-\mu)(f_2(x)) + 3(\xi-\mu)(f_3(x)) - (\xi-\mu)(f_4(x)) - (\xi-\mu)(f_5(x))\| \\ &\quad \le 2\,\|(\xi-\mu)(f_1(x))\| + 3\,\|(\xi-\mu)(f_2(x))\| + 3\|(\xi-\mu)(f_3(x))\| \\ &\qquad + \|(\xi-\mu)(f_4(x))\| + \|(\xi-\mu)(f_5(x))\| \\ &\quad = \sum_{i=1}^{5} L_i(x)\,\|(\xi-\mu)(f_i(x))\|,\end{aligned}$$

where $(\xi-\mu)(y) = \xi(y) - \mu(y)$ for all $y \in G$. So, (H2) is valid. It is easy to check that, in view of (3.2)

$$\begin{aligned}\Lambda\varepsilon_k(x) &= 2\varepsilon_k((m-1)x) + 3\varepsilon_k(mx) + 3\varepsilon_k(-mx) + \varepsilon_k(2mx) + \varepsilon_k((2m-1)x) \\ &\le 2u(m-1)\varepsilon_k(x) + 3u(m)\varepsilon_k(x) + 3u(-m)\varepsilon_k(x) + u(2m)\varepsilon_k(x) + u(2m-1)\varepsilon_k(x) \\ &= [2u(m-1) + 3u(m) + 3u(-m) + u(2m) + u(2m-1)]\varepsilon_k(x)\end{aligned} \tag{3.6}$$

for all $x \in G$ and $k, m \in \mathbb{Z}^*$. Therefore, since the operator Λ is linear, we have

$$\begin{aligned}\varepsilon^*(x) : &= \sum_{n=0}^{\infty} \Lambda^n \varepsilon_m(x) \\ &\le \sum_{n=0}^{\infty} (2u(m-1) + 3u(m) + 3u(-m) + u(2m) + u(2m-1))^n \varepsilon_m(x) \\ &= \frac{\varepsilon_m(x)}{1 - 2u(m-1) - 3u(m) - 3u(-m) - u(2m) - u(2m-1)} < \infty\end{aligned}$$

for all $x \in G$ and $m \in \mathbb{Z}^*$. Thus, according to Theorem 2.1, for each $m \in \mathcal{M}$ there exists a unique mapping $Q_m : G \to E$ such that

$$Q_m(x) = 2Q_m((m-1)x) + 3Q_m(mx) + 3Q_m(-mx) - Q_m(2mx) - Q_m((2m-1)x), \;\; x \in G,$$

$$\|f(x) - Q_m(x)\| \le \frac{\varepsilon_m(x)}{1 - 2u(m-1) - 3u(m) - 3u(-m) - u(2m) - u(2m-1)} \tag{3.7}$$

for all $x \in G$ and $m \in \mathbb{Z}^*$. Moreover,

$$Q_m(x) = \lim_{n\to\infty} \mathcal{T}^n f(x), \quad x \in G, \quad m \in \mathcal{M}.$$

Next, we show that

$$\begin{aligned} &\|\mathcal{T}^n f(x+y+z) + \mathcal{T}^n f(x-y) + \mathcal{T}^n f(x-z) + \mathcal{T}^n f(y-z) - 3\mathcal{T}^n f(x) - 3\mathcal{T}^n f(y) - 3\mathcal{T}^n f(z)\| \\ &\quad \le (2u(m-1) + 3u(m) + 3u(-m) + u(2m) + u(2m-1))^n \varphi(x,y,z). \end{aligned} \tag{3.8}$$

Fix $m \in \mathcal{M}$. Indeed, if $n = 0$, then (3.8) is simply (3.3). So, fix $n \in \mathbb{N}$ and suppose that (3.8) holds for n. Then

$$\begin{aligned} &\|\mathcal{T}^{n+1} f(x+y+z) + \mathcal{T}^{n+1} f(x-y) + \mathcal{T}^{n+1} f(x-z) + \mathcal{T}^{n+1} f(y-z) - 3\mathcal{T}^{n+1} f(x) \\ &\quad - 3\mathcal{T}^{n+1} f(y) - 3\mathcal{T}^{n+1} f(z)\| \\ &= \|2\mathcal{T}^n f((m-1)(x+y+z)) + 3\mathcal{T}^n f(m(x+y+z)) + 3\mathcal{T}^n f(-m(x+y+z)) \\ &\quad - \mathcal{T}^n f(2m(x+y+z)) - \mathcal{T}^n f((2m-1)(x+y+z)) + 2\mathcal{T}^n f((m-1)(x-y)) \\ &\quad + 3\mathcal{T}^n f(m(x-y)) + 3\mathcal{T}^n f(-m(x-y)) - \mathcal{T}^n f(2m(x-y)) - \mathcal{T}^n f((2m-1)(x-y)) \\ &\quad + 2\mathcal{T}^n f((m-1)(x-z)) + 3\mathcal{T}^n f(m(x-z)) + 3\mathcal{T}^n f(-m(x-z)) \\ &\quad - \mathcal{T}^n f(2m(x-z)) - \mathcal{T}^n f((2m-1)(x-z)) + 2\mathcal{T}^n f((m-1)(y-z)) \\ &\quad + 3\mathcal{T}^n f(m(y-z)) + 3\mathcal{T}^n f(-m(y-z)) - \mathcal{T}^n f(2m(y-z)) - \mathcal{T}^n f((2m-1)(y-z)) \\ &\quad - 3[2\mathcal{T}^n f((m-1)x) + 3\mathcal{T}^n f(mx) + 3\mathcal{T}^n f(-mx) - \mathcal{T}^n f(2mx) - \mathcal{T}^n f((2m-1)x)] \\ &\quad - 3[2\mathcal{T}^n f((m-1)y) + 3\mathcal{T}^n f(my) + 3\mathcal{T}^n f(-my) - \mathcal{T}^n f(2my) - \mathcal{T}^n f((2m-1)y)] \\ &\quad - 3[2\mathcal{T}^n f((m-1)z) + 3\mathcal{T}^n f(mz) + 3\mathcal{T}^n f(-mz) - \mathcal{T}^n f(2mz) - \mathcal{T}^n f((2m-1)z)]\| \\ &\le 2(\lambda(m))^n \varphi((m-1)x, (m-1)y, (m-1)z) + 3(\lambda(m))^n \varphi(mx, my, mz) \\ &\quad + 3(\lambda(m))^n \varphi(-mx, -my, -mz) + (\lambda(m))^n \varphi(2mx, 2my, 2mz) \\ &\quad + (\lambda(m))^n \varphi((2m-1)x, (2m-1)y, (2m-1)z) \\ &\le (\lambda(m))^{n+1} \varphi(x,y,z) \end{aligned}$$

for all $x, y, z \in G$, where $\lambda(m) := 2u(m-1)+3u(m)+3u(-m)+u(2m)+u(2m-1)$. Thus, by the induction, we have shown that (3.8) holds for all $x, y, z \in G$ and for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (3.8), we obtain that

$$Q_m(x+y+z) + Q_m(x-y) + Q_m(x-z) + Q_m(y-z) = 3[Q_m(x) + Q_m(y) + Q_m(z)] \tag{3.9}$$

for all $x, y, z \in G$ and $m \in \mathcal{M}$ such that

$$\|f(x) - Q_m(x)\| \le \frac{\varepsilon_m(x)}{1-\lambda(m)}.$$

Now, we prove that $Q_m = Q_k$ for all $m, k \in \mathcal{M}$. Let us fix $m, k \in \mathcal{M}$ and note that Q_k satisfies (3.7) with m replaced by k. Hence, by replacing (x, y, z) by $(mx, (m-1)x, -mx)$ in (3.9), we get $\mathcal{T}Q_j = Q_j$ for $j = m, k$ and

$$\|Q_m(x) - Q_k(x)\| \le \frac{\varepsilon_m(x)}{1-\lambda(m)} + \frac{\varepsilon_k(x)}{1-\lambda(k)}$$

for all $x \in G$. It follows from the linearity of Λ and (3.6) that

$$\begin{aligned} \|Q_m(x) - Q_k(x)\| &= \|\mathcal{T}^n Q_m(x) - \mathcal{T}^n Q_k(x)\| \\ &\le \frac{\Lambda^n \varepsilon_m(x)}{1-\lambda(m)} + \frac{\Lambda^n \varepsilon_k(x)}{1-\lambda(k)} \end{aligned}$$

$$\leq (\lambda(m))^n [A_m(x) + A_k(x)],$$

where

$$A_m(x) := \frac{\varepsilon_m(x)}{1 - \lambda(m)}$$

for all $x \in G$ and $n \in \mathbb{N}$. Letting $n \to \infty$ we get $Q_m = Q_k =: Q$. Thus, we have

$$\|f(x) - Q(x)\| \leq A_m(x), \quad x \in G, \quad m \in \mathcal{M},$$

thus, we derive (3.4). Due to (3.9), it is easy to notice that Q is a solution of (1.1).

To prove the uniqueness of the mapping Q, let us assume that there exists a mapping $Q' : G \to E$ which satisfies (1.1) and the inequality

$$\left\|f(x) - Q'(x)\right\| \leq \phi(x), \quad x \in G.$$

Then

$$\left\|Q(x) - Q'(x)\right\| \leq 2\phi(x), \quad x \in G.$$

Further, $\mathcal{T}Q'(x) = Q'(x)$ for all $x \in G$. Consequently, with a fixed $m \in \mathcal{M}$

$$\|Q(x) - Q'(x)\| = \|\mathcal{T}^n Q(x) - \mathcal{T}^n Q'(x)\| \leq 2\Lambda^n \phi(x) \leq \frac{2\Lambda^n \varepsilon_m(x)}{1 - \lambda(m)} \leq \frac{2[\lambda(m)]^n \varepsilon_m(x)}{1 - \lambda(m)}$$

for all $x \in G$ and $n \in \mathbb{N}$. Letting $n \to \infty$ we get $Q = Q'$. The proof of the theorem is complete. □

Theorem 3.2. *Let $(G, +)$ be an abelian group and E be a Banach space. Let $f : G \to E$, $\varphi : G^3 \to [0, \infty)$ and $u : \mathbb{Z}^* = \mathbb{Z}\backslash\{0\} \to [0, \infty)$ be functions, and the conditions* (3.1), (3.2) *and* (3.3) *be valid. Assume that*

$$\inf\{\varphi(mx, (m-1)x, -mx) : m \in \mathcal{M}\} = 0 \tag{3.10}$$

for all $x \in G$. Then f satisfies (1.1) *on G.*

Proof. Suppose that

$$\inf\{\varphi(mx, (m-1)x, -mx) : m \in \mathcal{M}\} = 0$$

for all $x \in G$. Hence from Theorem 3.1 we have $\phi(x) = 0$ for all $x \in G$. Then f satisfies (1.1) on G. □

Remark 3.3. In Theorem 3.1, if

$$\inf\{2u(m-1) + 3u(m) + 3u(-m) + u(2m) + u(2m-1) : m \in \mathcal{M}\} = 0,$$

(this is the case when, i.e., $\lim_{|m| \to \infty} u(m) = 0$), then (3.1) holds and

$$\phi(x) = \inf_{m \in \mathcal{M}} \varphi(mx, (m-1)x, -mx)$$

for all $x \in G$.

In a similar way we can prove that Theorem 3.1 holds if the inequality (3.3) is defined on $G\backslash\{0\} := G_0$.

Theorem 3.4. *Let $(G, +)$ be an abelian group and E be a Banach space. Let $f : G \to E$, $\varphi : G_0^3 \to [0, \infty)$, and $u : \mathbb{Z}^* = \mathbb{Z}\backslash\{0\} \to [0, \infty)$ be functions satisfying the following three conditions*

$$\mathcal{M} := \{m \in \mathbb{Z}^* : 2u(m-1) + 3u(m) + 3u(-m) + u(2m) + u(2m-1) < 1\} \neq \emptyset, \tag{3.11}$$

$$\varphi(tx, ty, tz) \leq u(t)\varphi(x, y, z), \tag{3.12}$$

$$\|f(x+y+z) + f(x-y) + f(x-z) + f(y-z) - 3[f(x) + f(y) + f(z)]\| \leq \varphi(x, y, z) \tag{3.13}$$

for all $x, y, z \in G_0$, $t \in \{m-1, m, -m, 2m, 2m-1\}$, and $m \in \mathcal{M}$ with $x+y+z, x-y, x-z, y-z \neq 0$. Then there exists a unique function $Q : G \to E$ satisfying (1.1) *and*

$$\|f(x) - Q(x)\| \leq \phi(x),$$

where

$$\phi(x) := \inf_{m \in \mathcal{M}} \frac{\varphi(mx, (m-1)x, -mx)}{1 - 2u(m-1) - 3u(m) - 3u(-m) - u(2m) - u(2m-1)}$$

for all $x \in G_0$.

4. Applications

In this section we give some applications of Theorem 3.4, in the two cases:

$$\varphi_1(x,y,z) = \theta\|x\|^p.\|y\|^q.\|z\|^r, \ \ p+r<0 \text{ and } q<0,$$

and

$$\varphi_2(x,y,z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p), \ \ p<0,$$

where $\varphi(x,y,z) = \varphi_j(x,y,z)$ for $j \in \{1,2\}$, $\theta \in \mathbb{R}_+$, $p,q,r \in \mathbb{R}$ and $x,y \neq 0$.

Corollary 4.1. *Let E_1 and E_2 be a normed space and a Banach space, respectively. Assume $S := (S,+)$ be a subgroup of the group $(E_1,+)$, $p,q,r \in \mathbb{R}$, $p+r<0$, $q<0$ and $\theta \geq 0$. If $f : S \to E_2$ satisfies*

$$\|f(x+y+z)+f(x-y)+f(x-z)+f(y-z)-3[f(x)+f(y)+f(z)]\| \leq \theta\|x\|^p\|y\|^q\|y\|^r$$

for all $x,y,z \in S\backslash\{0\}$ with $x+y+z, x-y, x-z, y-z \neq 0$, then f is a solution of (1.1) on $S\backslash\{0\}$ such that $x+y+z, x-y, x-z, y-z \neq 0$.

Proof. Let $\varphi_1(x,y,z) = \theta\|x\|^p.\|y\|^q.\|z\|^r$ and $u(t) = |t|^{p+q+r}$ in Theorem 3.4 where $p,q,r \in \mathbb{R}$, $p+r<0$, $q<0$, and $t \in \mathbb{Z}^*$, then we get that the condition (3.12) is valid. Obviously, (3.10) holds, and there exists $m_0 \in \mathbb{N}$, $m_0 > 1$ such that

$$2|m-1|^{p+q+r} + (6+2^{p+q+r})|m|^{p+q+r} + |2m-1|^{p+q+r} < 1, \ \ m \geq m_0.$$

So we obtain (3.11), as well. Consequently, by Theorem 3.2, every function $f : S \to E_2$ fulfilling the inequality (3.13), satisfies (1.1) on $S\backslash\{0\}$. □

Corollary 4.2. *Let E_1 and E_2 be a normed space and a Banach space, respectively. Assume $S := (S,+)$ is a subgroup of the group $(E_1,+)$, $p \in \mathbb{R}$, $p<0$ and $\theta \geq 0$. If $f : S \to E_2$ satisfies*

$$\|f(x+y+z)+f(x-y)+f(x-z)+f(y-z)-3[f(x)+f(y)+f(z)]\| \leq \theta(\|x\|^p+\|y\|^p+\|y\|^p)$$

for all $x,y,z \in S\backslash\{0\}$ with $x+y+z, x-y, x-z, y-z \neq 0$, then f is a solution of (1.1) on $S\backslash\{0\}$ such that $x+y+z, x-y, x-z, y-z \neq 0$.

Proof. Let $\varphi_2(x,y,z) = \theta(\|x\|^p+\|y\|^p+\|z\|^p)$ and $u(t) = |t|^p$ in Theorem 3.4 where $p \in \mathbb{R}$, $p<0$ and $t \in \mathbb{Z}^*$, then we get the condition (3.12) is valid. Obviously, (3.10) holds, and there exists $m_0 \in \mathbb{N}$, $m_0 > 1$ such that

$$2|m-1|^p + (6+2^p)|m|^p + |2m-1|^p < 1, \ \ m \geq m_0.$$

So we obtain (3.11), as well. Consequently, by Theorem 3.2, every mapping $f : S \to E_2$ fulfilling the inequality (3.13), satisfies (1.1) on $S\backslash\{0\}$ such that $x+y+z, x-y, x-z, y-z \neq 0$. □

In this part, we show that Corollaries 4.1 and 4.2 yield a characterization of the inner product spaces.

Corollary 4.3. *Let X be a normed space and $X_0 = X\backslash\{0\}$. Write*

$$\Delta(x,y,z) = \big|\|x+y+z\|^2 + \|x-y\|^2 + \|x-z\|^2 + \|y-z\|^2 - 3[\|x\|^2+\|y\|^2+\|z\|^2]\big|$$

for all $x,y \in X$. Assume that one of the following two hypotheses is valid

(i) $\sup_{x,y,z\in X_0} \frac{\Delta(x,y,z)}{\varphi_1(x,y,z)} < \infty$;

(ii) $\sup_{x,y,z\in X_0} \frac{\Delta(x,y,z)}{\varphi_2(x,y,z)} < \infty$,

where $x+y+z, x-y, x-z, y-z \neq 0$. Then X is an inner product space.

Proof. Write $f(x) = \|x\|^2$. Then from Corollaries 4.1 and 4.2, we easily derive f is a solution of the functional equation (1.1). That implies $\Delta(x, y) = 0$. Thus, the norm $\|.\|$ on X satisfies the parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in X.$$

Therefore, X is an inner product space. □

Corollary 4.4. *Let G be an abelian group and E be a Banach space. Let $\varphi : G^3 \to [0, \infty)$ and $u : \mathbb{Z}^* = \mathbb{Z}\backslash\{0\} \to [0, \infty)$ be functions and the conditions (3.1), (3.2), and (3.10) be valid. If $F : G^3 \to E$ is a mapping such that $F(x_0, y_0, z_0) \neq 0$ for some $x_0, y_0, z_0 \in G$ and*

$$\|F(x, y, z)\| \leq \varphi(x, y, z)$$

for all $x, y, z \in G$, then the functional equation

$$g(x + y + z) + g(x - y) + g(x - z) + g(y - z) = F(x, y, z) + 3[g(x) + g(y) + g(z)], \quad x, y, z \in G, \tag{4.1}$$

has no solution in the class of functions $g : G \to E$.

Proof. Suppose that $g : G \to E$ is a solution of (4.1). Then (3.3) holds, and consequently, according to Theorem 3.2, g satisfies (1.1) on G, which means that $F(x_0, y_0, z_0) = 0$. This is a contradiction. □

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References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64–66. 1, 1

[2] J.-H. Bae, *On the stability of 3-dimensional quadratic functional equation*, Bull. Korean Math. Soc., **37** (2000), 477–486. 1, 1.3, 2, 2

[3] J.-H. Bae, K.-W. Jun, *On the generalized Hyers-Ulam-Rassias stability of a quadratic functional equation*, Bull. Korean Math. Soc., **38** (2001), 325–336. 1

[4] J.-H. Bae, Y.-S. Jung, *The Hyers-Ulam stability of the quadratic functional equations on abelian groups*, Bull. Korean Math. Soc., **39** (2002), 199–209. 1, 2, 2

[5] A. Bahyrycz, M. Piszczek, *Hyperstability of the Jensen functional equation*, Acta Math. Hungar., **142** (2014), 353–365. 1

[6] D. G. Bourgin, *Approximately isometric and multiplicative transformations on continuous function rings*, Duke Math. J., **16** (1949), 385–397. 1, 1, 1

[7] J. Brzdęk, *Hyperstability of the Cauchy equation on restricted domains*, Acta Math. Hungar., **141** (2013), 58–67. 1

[8] J. Brzdęk, *Remarks on hyperstability of the Cauchy functional equation*, Aequationes Math., **86** (2013), 255–267.

[9] J. Brzdęk, *A hyperstability result for the Cauchy equation*, Bull. Aust. Math. Soc., **89** (2014), 33–40. 1

[10] J. Brzdęk, J. Chudziak, Z. Páles, *A fixed point approach to stability of functional equations*, Nonlinear Anal., **74** (2011), 6728–6732. 2

[11] J. Brzdęk, K. Ciepliński, *Hyperstability and superstability*, Abstr. Appl. Anal., **2013** (2013), 13 pages. 1

[12] St. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg, **62** (1992), 59–64. 1

[13] Iz. EL-Fassi, S. Kabbaj, *On the hyperstability of a Cauchy-Jensen type functional equation in Banach spaces*, Proyecciones, **34** (2015), 359–375. 1

[14] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci., **14** (1991), 431–434. 1, 1

[15] P. Găvruţa, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431–436. 1

[16] E. Gselmann, *Hyperstability of a functional equation*, Acta Math. Hungar., **124** (2009), 179–188. 1

[17] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A., **27** (1941), 222–224. 1.1, 1

[18] S.-M. Jung, *On the Hyers-Ulam-Rassias stability of a quadratic functional equation*, J. Math. Anal. Appl., **232** (1999), 384–393. 1, 2

[19] P. Kannappan, *Quadratic functional equation and inner product spaces*, Results Math., **27** (1995), 368–372. 1

[20] G. Maksa, Z. Páles, *Hyperstability of a class of linear functional equations*, Acta Math. Acad. Paedagog. Nyhzi. (N.S.), **17** (2001), 107–112. 1

[21] M. Mirzavaziri, M. S. Moslehian, *fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc. (N.S.), **37** (2006), 361–376. 1, 2

[22] M. Piszczek, *Remark on hyperstability of the general linear equation*, Aequationes Math., **88** (2013), 163–168. 1

[23] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300. 1, 1

[24] Th. M. Rassias, *On a modified Hyers-Ulam sequence*, J. Math. Anal. Appl., **158** (1991), 106–113. 1

[25] J. M. Rassias, *On the stability of the Euler-Lagrange functional equation*, Chinese J. Math., **20** (1992), 185–190. 1

[26] Th. M. Rassias, *On the stability of the quadratic functional equation and its applications*, Studia Univ. Babe-Bolyai Math., **43** (1998), 89–124.

[27] Th. M. Rassis, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl., **251** (2000), 264–284.

[28] Th. M. Rassias, P. Šemrl, *On the behavior of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc., **114** (1992), 989–993. 1

[29] S. M. Ulam, *Problems in modern mathematics*, Science Editions John Wiley & Sons, Inc., New York, (1960). 1