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# The perturbed Riemann problem for the chromatography system of Langmuir isotherm with one inert component

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**Abstract**

The solutions of the perturbed Riemann problem for the chromatography system of Langmuir isotherm with one inert component are constructed in completely explicit forms when the initial data are taken as three piecewise constant states. The wave interaction problem is investigated in detail by using the method of characteristics. In addition, the generalized Riemann problem with the delta-type initial data is considered and the delta contact discontinuity is discovered. Moreover, the strength of delta contact discontinuity decreases linearly at a constant rate and then the delta contact discontinuity degenerates to be the contact discontinuity when across the critical point. 

*Keywords:* Chromatography system, Riemann problem, wave interaction, Temple class, hyperbolic conservation law.
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## 1. Introduction

Chromatography is the terminology adopted by engineers and chemists to describe a process of separating two chemical components in a fluid phase [16]. Various mathematical models are used to understand and analyze dynamic composition front in chromatographic columns and thus it is necessary to develop the theory of nonlinear chromatography system in order to investigate the separation process of chromatography. It is well-known that the Langmuir model [12] is very effective to describe a variety of real systems in the

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local equilibrium theory of chromatography [16]. The process of Langmuir isotherm can be described by the system of conservation laws to account for convection and exchange between the adsorbed phases and the fluid at the thermodynamic equilibrium [13]. Thus, it is very necessary to look for the exact solutions of these models associated with suitable initial and boundary value conditions to describe different chromatographic behaviors. Fortunately, it is amenable to give an appropriate treatment in the theory of hyperbolic conservation laws. Recently, several generalizations of the Langmuir model within the time scale calculus have been proposed in [2]. In addition, the Langmuir model was also modified in [3, 4] by taking into account some nonlinear effects such as diffusion and condensation.

In this paper, we are concerned with the chromatography separation of two chemical species through a Langmuir isotherm reactor [15, 16] when one component is inert and the other component is active, which can be described by the following hyperbolic system of conservation laws

$$\begin{cases} u_t = 0, \\ v_t + (\dfrac{kv}{1+u+v})_x = 0, \end{cases} \tag{1.1}$$

where $u$, $v$ are the non-negative functions of the variables $(x, t) \in R \times R_+$, which stand for the concentrations of the species. The system (1.1) can be derived from the more general two-component chromatography system [16, 25]

$$\begin{cases} u_t + (\dfrac{k_1 u}{1+u+v})_x = 0, \\ v_t + (\dfrac{k_2 v}{1+u+v})_x = 0, \end{cases} \tag{1.2}$$

by letting $k_1 = 0$ and $k_2 = k$, where $k_1$, $k_2 \in [0, 1]$ are all known constants dependent on the nature of the Langmuir isotherm. If $k_1 = k_2$ is taken in (1.2), then it is called as the simplified chromatography system, which has been widely studied such as in [1, 19, 23] recently. It is well-known that a component with concentration $u$ (or $v$) is called as inert if $k_1 = 0$ (or $k_2 = 0$) is taken [5]. Thus, the mass balances of the two chemical components in the process of chromatography separation governed by the Langmuir isotherm can be described by the system (1.1) when the component $u$ is inert and the component $v$ is active.

It is easy to see that the chromatography system (1.1) is strictly hyperbolic and the first characteristic field is linearly degenerated and while the second characteristic field is genuinely nonlinear. It is noteworthy that the system (1.1) belongs to the Temple class [25] for the shock curve coincides with the rarefaction one in the $(u, v)$ phase plane. One of the main purposes in this paper is to construct the global solutions of the particular Cauchy problem for the system (1.1) when the initial data are taken to be three piecewise constant states as

$$(u, v)(x, 0) = \begin{cases} (u_-, v_-), & -\infty < x < -\varepsilon, \\ (u_m, v_m), & -\varepsilon < x < \varepsilon, \\ (u_+, v_+), & \varepsilon < x < +\infty, \end{cases} \tag{1.3}$$

in which $\varepsilon > 0$ is arbitrarily small. It is worthwhile to notice that the initial data (1.3) may be viewed as the perturbation of the corresponding Riemann initial data

$$(u, v)(x, 0) = \begin{cases} (u_-, v_-), & -\infty < x < 0, \\ (u_+, v_+), & 0 < x < +\infty. \end{cases} \tag{1.4}$$

Thus, the particular Cauchy problem (1.1) and (1.3) is often called as the perturbed Riemann problem (or the double Riemann problems) below. The Riemann problem is of great importance in the theory of chromatography and needs to be considered when a stream of a given composition is fed to a column initially saturated at a different composition or the saturation of an initially clean column with a feedstream has constant concentrations of the two solutions [16]. It can be seen that the solutions of Riemann problem (1.1) and (1.4) consist of constant states separated by elementary waves including rarefaction wave, shock wave and contact discontinuity. To study the perturbed Riemann problem (1.1) and (1.3), it is essential to study the wave interaction problem for the system (1.1). The method adopted in this paper is to first construct the solutions of the perturbed Riemann problem (1.1) and (1.3) in the $(u, v)$ phase plane and consequently map them onto the $(x, t)$ physical plane. In fact, this type of initial data (1.3) has been widely used to study all kinds of chromatography systems such as in [10, 20, 24, 27] for the reason that the wave interaction problem is one of the most basic problems [16] in the study of chromatography separation process. More precisely, the three piecewise constant initial states (1.3) should be taken when we deal with the problem of multi-component separation by the chromatographic cycle [16]. The perturbed Riemann problem (1.1) and (1.3) is one of the most important questions in the theory of chromatography and the fundamental features of wave interactions for the system (1.1) can be examined thoroughly by studying the perturbed Riemann problem (1.1) and (1.3).

In this paper, the wave interaction problem for the system (1.1) has been investigated in detail by using the method of characteristics and then the global solutions of the perturbed Riemann problem (1.1) and (1.3) have been constructed in completely explicit forms for all the possible situations. During the process of wave interaction, the propagation speeds of shock and rarefaction waves are delivered and then the explicit expressions of shock curves are given. It is interesting to observe that the propagation speeds of shock and rarefaction waves increase or decrease when across the contact discontinuity which depends on the choice of initial data. Furthermore, the stability of solutions to the Riemann problem (1.1) and (1.4) can also be analyzed under the particular small perturbation (1.3) of the Riemann initial data (1.4) which is summarized in the theorem below.

**Theorem 1.1.** *The limits of the global solutions to the perturbed Riemann problem* (1.1) *and* (1.3) *are identical with the corresponding Riemann solutions of* (1.1) *and* (1.4) *when the limit* $\varepsilon \to 0$ *is taken. Thus, the Riemann solutions are stable with respect to the particular small perturbations* (1.3) *of the Riemann initial data* (1.4).

Recently, the delta shock wave has been observed experimentally in [11, 14] for the local equilibrium model of two-component nonlinear chromatography attributed to a mixed competitive-cooperative generalized Langmuir isotherm. In fact, the delta shock wave is a nonclassical and singular transition front between two constant composition states that may occur in the theory of chromatography due to the competitive-cooperative interaction between two chemical components. The delta shock wave may be regarded as a traveling spike superposed on a discontinuity to separate the initial and feed states [13]. Thus, the delta shock wave can also be seen as a reasonable supplement of classical waves involving the rarefaction wave, shock wave and contact discontinuity in the theory of chromatography. Consequently, the study of delta shock wave for all kinds of chromatography systems has attracted extensive attention such as in [8, 18, 20, 23, 26]. Thus, it is natural to study the generalized Riemann problem for the chromatography system (1.1) with the delta-type initial data

$$(u, v)(x, 0) = \begin{cases} (u_-, v_-), & -\infty < x < 0, \\ (u_0, m\delta(x)), & x = 0, \\ (u_+, v_+), & 0 < x < +\infty, \end{cases} \tag{1.5}$$

where the symbol $\delta$ indicates the standard Dirac delta function. In other words, it may be assumed that the occurrence of a spike with infinitely high concentration [22] appears initially without loss of generality and then the spike propagation can be observed. During the construction of solutions to the generalized Riemann problem (1.1) and (1.5), the delta contact discontinuity is captured which is the Dirac delta function supported on the line $x = 0$ of contact discontinuity. It is interesting to discover that the strength of delta contact discontinuity decreases linearly at a constant rate and then becomes zero at the critical point, such that the delta contact discontinuity will degenerate to be the contact discontinuity when across the critical point. That is to say, the position of spike keeps invariant and the height of spike decreases linearly with respect to the time $t$ such that the spike disappears in finite time for the generalized Riemann problem (1.1) and (1.5).

The paper is organized as follows: In Section 2, we obtain the solutions of the Riemann problem (1.1) and (1.4). In Section 3, we mainly discuss all kinds of wave interactions when the initial data are taken to be three piecewise constant states and then the global solutions of the perturbed Riemann problem (1.1) and (1.3) are constructed completely. In Section 4, the generalized Riemann problem with delta-type initial data is considered and the solutions are constructed. At the end, the conclusion is drawn in Section 5.

## 2. The Riemann problem

In this section, we need to solve the Riemann problem (1.1) and (1.4) by using the standard technique such as in the classical books [6, 7, 9, 17, 21]. The eigenvalues of (1.1) are

$$\lambda_1 = 0, \qquad \lambda_2 = \frac{k(1+u)}{(1+u+v)^2},$$

such that $\lambda_1 < \lambda_2$ holds for arbitrary $u$ and $v$. Thus, the system (1.1) is strictly hyperbolic in the quarter of $(u, v)$ phase plane. The corresponding right eigenvectors for the system (1.1) are $\overrightarrow{r_1} = (u+1, v)^T$ and $\overrightarrow{r_2} = (0, 1)^T$, respectively. Let us use the notation $\nabla = (\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ to stand for the gradient operator. We have $\nabla\lambda_1 \cdot \overrightarrow{r_1} = 0$ for the first characteristic field $\lambda_1$, therefore the first characteristic field $\lambda_1$ is linearly degenerated. Thus, the wave associated with $\lambda_1$ is the contact discontinuity denoted by $J$. On the other hand, we have $\nabla\lambda_2 \cdot \overrightarrow{r_2} = \frac{-2k(u+1)}{(1+u+v)^3} \neq 0$ for the second characteristic field $\lambda_2$, such that the second characteristic field $\lambda_2$ is genuinely nonlinear. Thus, the wave associated with $\lambda_2$ will be either the shock wave (denoted by $S$) or the rarefaction wave (denoted by $R$).

Let us first consider the smooth solutions. If $u(x, t)$ is a solution of the Riemann problem (1.1) and (1.4), then $u(\alpha x, \alpha t)$ is also a solution of the Riemann problem (1.1) and (1.4) for any $\alpha > 0$. Thus, it is natural to consider the solution of the Riemann problem (1.1) and (1.4) which only depends on $\xi = \frac{x}{t}$. Therefore, the Riemann problem (1.1) and (1.4) is reduced to the following boundary value problem for the ordinary differential equations

$$\begin{cases} -\xi u_\xi = 0, \\ -\xi v_\xi + \Big(\dfrac{kv}{1+u+v}\Big)_\xi = 0, \end{cases} \tag{2.1}$$

associated with the boundary condition $(u, v)(\pm\infty) = (u_\pm, v_\pm)$. Let us rewrite (2.1) into the following form

$$\begin{pmatrix} \xi & 0 \\ \dfrac{-kv}{(1+u+v)^2} & \dfrac{k(1+u)}{(1+u+v)^2} - \xi \end{pmatrix} \begin{pmatrix} u_\xi \\ v_\xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, for the given left state $(u_-, v_-)$, the contact discontinuity curve which is a wave of the first characteristic family can be expressed by

$$J(u_-, v_-) : \begin{cases} \xi = \lambda_1 = 0, \\ \dfrac{v}{1+u} = \dfrac{v_-}{1+u_-}, \end{cases}$$

and the rarefaction wave curve which is a wave of second characteristic family can be expressed by

$$R(u_-, v_-) : \begin{cases} \xi = \lambda_2 = \dfrac{k(1+u)}{(1+u+v)^2}, \\ u = u_-, \qquad v < v_-. \end{cases}$$

Let us turn to the discontinuous solutions. For a discontinuous curve $x = x(t)$, the Rankine-Hugoniot relation

$$\begin{cases} \sigma[u] = 0, \\ \sigma[v] = \Big[\dfrac{kv}{1+u+v}\Big], \end{cases}$$

should hold, where $\sigma = \frac{dx}{dt}$ is the propagation speed of the discontinuity and $[u] = u_r - u_l$ is the jump across the discontinuity with $u_l = u(x(t) - 0, t)$ and $u_r = u(x(t) + 0, t)$, etc. By a simply calculation, we can also obtain the contact discontinuity which is a wave of the first characteristic family

$$J(u_-, v_-) : \begin{cases} \tau = 0, \\ \dfrac{v}{1+u} = \dfrac{v_-}{1+u_-}, \end{cases}$$

and the shock wave which is a wave of the second characteristic family

$$S(u_-, v_-) : \begin{cases} \sigma = \dfrac{k(1+u_-)}{(1+u_-+v_-)(1+u+v)}, \\ u = u_-, \qquad v > v_-. \end{cases}$$

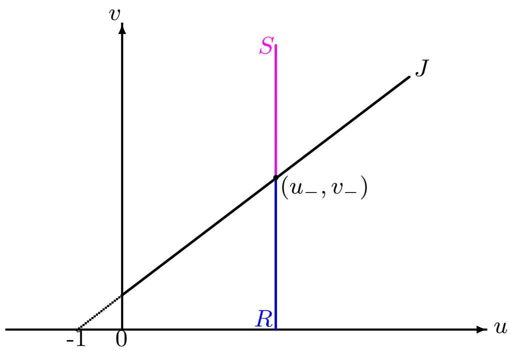


Figure 1: For the given left state $(u_-, v_-)$, the $(u, v)$ phase plane is shown for the Riemann problem (1.1) and (1.4).

Clearly, the system (1.1) is attributed to the so-called Temple class [25] for the reason that the shock curve coincides with the rarefaction one in the $(u, v)$ phase plane. Let us draw Figure 1 to illustrate this situation. In summary, for the given left state $(u_-, v_-)$, there exist two kinds of Riemann solutions to (1.1) and (1.4) described below.

(1) When $\frac{v_+}{u_++1} > \frac{v_-}{u_-+1}$, the Riemann solution is a contact discontinuity $J$ followed by a shock wave $S$

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & -\infty < x < 0, \\ \Big(u_+, \dfrac{(u_++1)v_-}{u_-+1}\Big), & 0 < x < \sigma t, \\ (u_+, v_+), & \sigma t < x < +\infty, \end{cases}$$

in which $\sigma = \frac{k(1+u_-)}{(1+u_-+v_-)(1+u_++v_+)}$ is the propagation speed of the shock wave.

(2) When $\frac{v_+}{u_++1} < \frac{v_-}{u_-+1}$, the Riemann solution is a contact discontinuity $J$ followed by a rarefaction wave $R$

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & -\infty < x < 0, \\ \Big(u_+, \dfrac{(u_++1)v_-}{u_-+1}\Big), & 0 < x < \lambda_2\Big(u_+, \dfrac{(u_++1)v_-}{u_-+1}\Big)t, \\ (u_+, v), & \lambda_2\Big(u_+, \dfrac{(u_++1)v_-}{u_-+1}\Big)t < x < \lambda_2\Big(u_+, v_+\Big)t, \\ (u_+, v_+), & \lambda_2(u_+, v_+)t < x < +\infty, \end{cases}$$

in which the state variable $v$ in $R$ varies from $\frac{(u_++1)v_-}{u_-+1}$ to $v_+$.

## 3. Construction of global solutions to the perturbed Riemann problem (1.1) and (1.3)

In this section, we are planing to construct the global solutions of the perturbed Riemann problem (1.1) and (1.3) for all kinds of situations. In other words, we need to study all the possible wave interactions for the system (1.1) by employing the method of characteristics.

**Case 1.** $J+S$ and $J+S$.

First of all, we need to consider the case that both a contact discontinuity followed by a shock wave emitting from the initial points $(-\varepsilon,0)$ and $(\varepsilon,0)$ (see Figure 2). Obviously, the occurrence of this case depends on the condition

$$\frac{v_+}{u_+ + 1} > \frac{v_m}{u_m + 1} > \frac{v_-}{u_- + 1}.$$

For the sufficiently small time $t$, the solution may be represented succinctly as:

$$(u_-,v_-) + J_1 + (u_1,v_1) + S_1 + (u_m,v_m) + J_2 + (u_2,v_2) + S_2 + (u_+,v_+),$$

in which the states $(u_1,v_1)$ and $(u_2,v_2)$ are given respectively by

$$(u_1,v_1) = \Big(u_m, \frac{(u_m+1)v_-}{u_- + 1}\Big), \qquad (u_2,v_2) = \Big(u_+, \frac{(u_+ + 1)v_m}{u_m + 1}\Big). \tag{3.1}$$

The propagation speed of $S_1$ is $\sigma_1 = \frac{k(1+u_1)}{(1+u_m+v_m)(1+u_1+v_1)} > 0$ and that of $J_2$ is $\tau_2 = 0$, such that $S_1$ will interact with $J_2$ at a finite time $t_1$ and the interaction point is given by

$$\begin{cases} x_1 = \varepsilon, \\ x_1 + \varepsilon = \sigma_1 t_1, \end{cases} \tag{3.2}$$

which means that

$$(x_1,t_1) = \Big(\varepsilon, \frac{2\varepsilon(1+u_1+v_1)(1+u_m+v_m)}{k(1+u_1)}\Big). \tag{3.3}$$

At the point $(x_1,t_1)$, a new local Riemann problem will be formulated where the initial data are taken to be

$$(u,v)(x,0) = \begin{cases} (u_1,v_1), & x<\varepsilon, \\ (u_2,v_2), & x>\varepsilon. \end{cases}$$

Furthermore, the solution of the new local Riemann problem at the point $(x_1,t_1)$ is a contact discontinuity $J_2$ followed by a shock wave $S_3$. Analogously, the intermediate state $(u_3,v_3)$ between $J_2$ and $S_3$ can also be obtained by

$$(u_3,v_3) = \Big(u_2, \frac{(u_2+1)v_1}{u_1+1}\Big),$$

in which $(u_1,v_1)$ and $(u_2,v_2)$ are given by (3.1). Then, we use the following lemma to describe the interaction between $S_2$ and $S_3$ (see Figure 2).

**Lemma 3.1.** *If $u_+ > u_m$, then we have $\sigma_1 > \sigma_3$, namely the shock wave $S_1$ decelerates when it passes through $J_2$. Otherwise if $u_+ < u_m$, then we have $\sigma_1 < \sigma_3$, namely the shock wave $S_1$ accelerates when it passes through $J_2$.*

*Proof.* The propagation speeds of $S_1$ and $S_3$ can be computed respectively by

$$\sigma_1 = \frac{k(1+u_1)}{(1+u_m+v_m)(1+u_1+v_1)}, \qquad \sigma_3 = \frac{k(1+u_3)}{(1+u_2+v_2)(1+u_3+v_3)}. \tag{3.4}$$

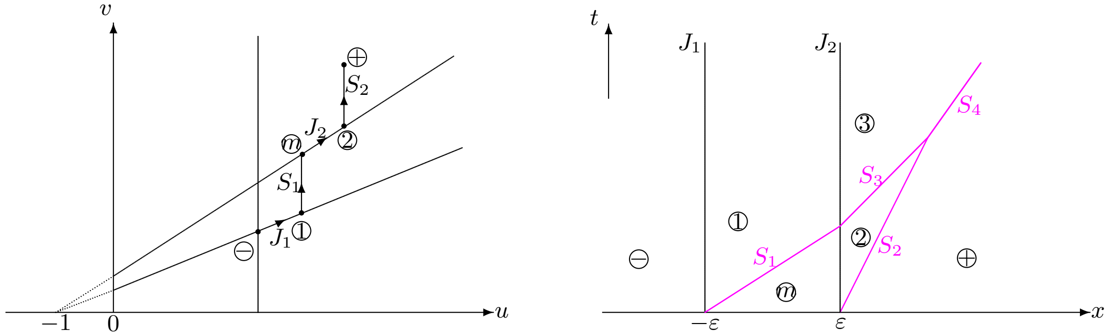


Figure 2: The interaction between $J+S$ and $J+S$ is displayed when $\frac{v_+}{u_++1} > \frac{v_m}{u_m+1} > \frac{v_-}{u_-+1}$ and $u_+ > u_m$.

Then, we have

$$\begin{aligned}\sigma_1-\sigma_3&=k\frac{(1+u_1)(1+u_3+v_3)(1+u_2+v_2)-(1+u_3)(1+u_1+v_1)(1+u_m+v_m)}{(1+u_m+v_m)(1+u_1+v_1)(1+u_2+v_2)(1+u_3+v_3)}\\&=k\frac{[(1+u_1)(1+u_3)+v_3(1+u_1)](1+u_2+v_2)-[(1+u_3)(1+u_1)+v_1(1+u_3)](1+u_m+v_m)}{(1+u_m+v_m)(1+u_1+v_1)(1+u_2+v_2)(1+u_3+v_3)}\\&=\frac{k[(1+u_1)(1+u_3)+v_3(1+u_1)](u_2+v_2-u_m-v_m)}{(1+u_m+v_m)(1+u_1+v_1)(1+u_2+v_2)(1+u_3+v_3)}\\&=\frac{k(1+u_1)(u_2+v_2-u_m-v_m)}{(1+u_m+v_m)(1+u_1+v_1)(1+u_2+v_2)},\end{aligned}$$

in which $v_3(1+u_1)=v_1(1+u_3)$ has been used. If $u_+ > u_m$, then we have $v_2 > v_m$ and $u_2 > u_m$, such that $\sigma_1 > \sigma_3$. Otherwise if $u_+ < u_m$, then $\sigma_1 < \sigma_3$ can be achieved similarly. The proof is completed. □

Finally, we consider the coalescence of two shock waves belonging to the same family shown below.

**Lemma 3.2.** *The two shock waves $S_3$ and $S_2$ belonging to the second family coalesce into a new shock wave of the second family.*

*Proof.* The propagation speed of $S_2$ is

$$\sigma_2=\frac{k(1+u_2)}{(1+u_2+v_2)(1+u_++v_+)},$$

which, together with (3.4), yields

$$\sigma_3-\sigma_2=\frac{k(1+u_2)(v_+-v_3)}{(1+u_3+v_3)(1+u_2+v_2)(1+u_++v_+)}>0,$$

in which $u_+=u_2=u_3$ and $v_+>v_2>v_3$ have been used. Hence, $S_3$ catches up with $S_2$ in finite time and the intersection $(x_2,t_2)$ is determined by

$$\begin{cases} x_2-\varepsilon=\sigma_2t_2,\\ x_2-\varepsilon=\sigma_3(t_2-t_1),\end{cases}$$

which yields

$$(x_2,t_2)=\Big(\varepsilon+\frac{2\varepsilon(1+u_2)(1+u_1+v_1)(1+u_m+v_m)}{(1+u_1)(1+u_2+v_2)(v_+-v_3)},\frac{2\varepsilon(1+u_1+v_1)(1+u_m+v_m)(1+u_++v_+)}{k(1+u_1)(v_+-v_3)}\Big).$$

It can be seen from $u_3=u_+$ that the two states $(u_+,v_+)$ and $(u_3,v_3)$ can also be connected by a shock wave directly. Thus, after $t_2$, $S_2$ and $S_3$ coalesce into a new shock wave $S_4$ whose propagation speed is

$$\sigma_4=\frac{k(1+u_3)}{(1+u_3+v_3)(1+u_++v_+)}.$$

It is easy to get $\sigma_3>\sigma_4>\sigma_2$. The proof is completed. □

**Case 2.** $J+S$ and $J+R$.

For this case, we need to cope with the situation that the Riemann solution at $(-\varepsilon,0)$ is $J_1+S_1$ and at $(\varepsilon,0)$ is $J_2+R_2$. On this occasion, the initial data (1.3) should satisfy the condition

$$\frac{v_m}{u_m+1}>\max\Big(\frac{v_-}{u_-+1},\frac{v_+}{u_++1}\Big).$$

The solution of (1.1) and (1.3) for sufficiently small $t$ may be indicated as

$$(u_-,v_-)+J_1+(u_1,v_1)+S_1+(u_m,v_m)+J_2+(u_2,v_2)+R_2+(u_+,v_+).$$

Here and below the states $(u_1,v_1)$, $(u_2,v_2)$, and $(u_3,v_3)$ have the same presentations as those in Case 1.

As in Case 1, $S_1$ collides with $J_2$ at the point $(x_1,t_1)$ which has the same expression with (3.3). After the time $t_1$, the new local Riemann problem whose left state is $(u_1,v_1)$ and right state is $(u_2,v_2)$ can also be solved by a contact discontinuity $J_2$ and a shock wave denoted by $S_3$. The result of Lemma 3.1 can also be obtained here in contrast with the two propagation speeds of $S_1$ and $S_3$. Therefore, we are now in a position to consider the situation that the shock wave $S_3$ penetrates the rarefaction wave $R_2$ which can be summarized below.

**Lemma 3.3.** *The shock wave $S_3$ catches up with the wave back of the rarefaction wave $R_2$ in finite time and consequently begins to penetrate $R_2$. More precisely, if $\frac{v_+}{u_++1}>\frac{v_-}{u_-+1}$, then $S_3$ is able to cancel the whole $R_2$ thoroughly. Otherwise, if $\frac{v_+}{u_++1}<\frac{v_-}{u_-+1}$, then $S_3$ penetrates $R_2$ partially and finally has the line $x=\varepsilon+\frac{k(1+u_3)t}{(1+u_3+v_3)^2}$ in $R_2$ as its asymptotic line.*

*Proof.* One can see that the propagation speed of $S_3$ is given by (3.4) and that of the wave back of $R_2$ is calculated by

$$\xi_2(u_2,v_2)=\frac{k(1+u_2)}{(1+u_2+v_2)^2}.$$

By virtue of $u_2=u_3$ and $v_2>v_3$, we have

$$\sigma_3-\xi_2(u_2,v_2)=\frac{k(1+u_2)(v_2-v_3)}{(1+u_3+v_3)(1+u_2+v_2)^2}>0,$$

thus $S_3$ keeps up with the wave back of $R_2$ at the point $(x_2,t_2)$ which is computed by

$$\begin{cases} x_2-\varepsilon=\sigma_3(t_2-t_1),\\ x_2-\varepsilon=\xi_2(u_2,v_2)\cdot t_2, \end{cases} \tag{3.5}$$

in which $t_1$ is given by (3.3). Thus, we have

$$(x_2,t_2)=\Big(\varepsilon+\frac{2\varepsilon(1+u_2)(1+u_1+v_1)(1+u_m+v_m)}{(1+u_1)(1+u_2+v_2)(v_2-v_3)},\frac{2\varepsilon(1+u_1+v_1)(1+u_m+v_m)(1+u_2+v_2)}{k(1+u_1)(v_2-v_3)}\Big). \tag{3.6}$$

Consequently, $S_3$ begins to penetrate $R_2$ after the time $t_2$. During the process of penetration, we denote it with $S_4$ whose propagation speed is determined by

$$\begin{cases} \dfrac{dx}{dt}=\dfrac{k(1+u_3)}{(1+u_3+v_3)(1+u+v)},\\ x-\varepsilon=\dfrac{k(1+u)t}{(1+u+v)^2},\\ x(t_2)=x_2, \end{cases} \tag{3.7}$$

where $(u,v)$ changes from $(u_2,v_2)$ to $(u_+,v_+)$. Taking into account $u=u_2=u_3=u_+$, differentiating (3.7) with respect to $t$ leads to

$$\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{k(1+u_3)}{(1+u_3+v_3)(1+u_3+v)^2}\cdot\frac{dv}{dt},\\ \frac{dx}{dt} &= \frac{k(1+u_3)}{(1+u_3+v)^2} - \frac{2k(1+u_3)t}{(1+u_3+v)^3}\cdot\frac{dv}{dt}.\end{aligned} \tag{3.8}$$

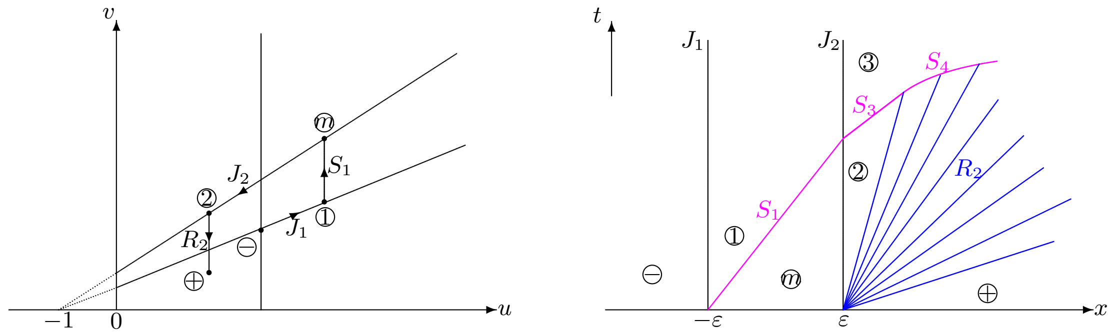


Figure 3: The interaction between $J+S$ and $J+R$ is shown when $u_+ < u_m$ and $\frac{v_+}{u_+ + 1} < \frac{v_-}{u_- + 1} < \frac{v_m}{u_m+1}$.

It yields $\frac{dv}{dt} < 0$ by substituting the first equation of (3.7) into (3.8). Furthermore, we have $\frac{d^2x}{dt^2} > 0$, which means that $S_3$ accelerates during the process of penetration. On the other hand, it follows from (3.6) and (3.7) that the curve of the shock wave $S_4$ is determined by

$$\sqrt{x-\varepsilon} = \frac{\sqrt{k(1+u_3)t}}{1+u_3+v_3} - \frac{1}{1+u_3+v_3}\sqrt{\frac{2\varepsilon(1+u_2)(1+u_1+v_1)(1+u_m+v_m)(v_2-v_3)}{(1+u_1)(1+u_2+v_2)}}.$$

Therefore, there exist two possible solutions as follows:

(a) If $\frac{v_+}{u_+ + 1} > \frac{v_-}{u_- + 1}$, then $S_4$ is able to cross the whole $R_2$ completely and terminates at the point $(x_3, t_3)$ which is given by

$$\begin{cases} x_3 - \varepsilon = \xi_2(u_+, v_+)t_3, \\ \sqrt{x_3-\varepsilon} = \dfrac{\sqrt{k(1+u_3)t_3}}{1+u_3+v_3} - \dfrac{1}{1+u_3+v_3}\sqrt{\dfrac{2\varepsilon(1+u_2)(1+u_1+v_1)(1+u_m+v_m)(v_2-v_3)}{(1+u_1)(1+u_2+v_2)}}, \end{cases}$$

such that we have

$$\begin{aligned}(x_3, t_3) = \Big(\varepsilon + &\frac{2\varepsilon(1+u_+)(1+u_1+v_1)(1+u_m+v_m)(v_2-v_3)}{(1+u_1)(1+u_2+v_2)(v_3-v_+)^2}\\ &, \frac{2\varepsilon(1+u_1+v_1)(1+u_m+v_m)(v_2-v_3)(1+u_+ + v_+)^2}{k(1+u_1)(1+u_2+v_2)(v_3-v_+)^2}\Big).\end{aligned} \tag{3.9}$$

After the penetration, we denote the shock wave with $S_5$ whose propagation speed is

$$\sigma_5 = \frac{k(1+u_3)}{(1+u_3+v_3)(1+u_+ + v_+)}. \tag{3.10}$$

(b) If $\frac{v_+}{u_+ + 1} < \frac{v_-}{u_- + 1}$, then $S_4$ cannot cancel the entire $R_2$ thoroughly and ultimately has the characteristic line $x = \varepsilon + \frac{k(1+u_3)t}{(1+u_3+v_3)^2}$ in $R_2$ as the asymptotic line (see Figure 3). □

**Case 3.** $J + R$ and $J + S$.

In this case, we consider that the Riemann solution at $(-\varepsilon, 0)$ is $J_1 + R_1$ and at $(\varepsilon, 0)$ is $J_2 + S_2$. This case happens if and only if

$$\frac{v_m}{u_m + 1} < \min\Big(\frac{v_-}{u_- + 1}, \frac{v_+}{u_+ + 1}\Big)$$

is satisfied. When $t$ is small enough, the solution of (1.1) and (1.3) is

$$(u_-, v_-) + J_1 + (u_1, v_1) + R_1 + (u_m, v_m) + J_2 + (u_2, v_2) + S_2 + (u_+, v_+).$$

Let us first consider the interaction between $R_1$ and $J_2$ and use the following lemma to depict it.

**Lemma 3.4.** *The rarefaction wave $R_1$ passes through $J_2$ and then a transmitted rarefaction wave denoted by $R_2$ is generated during the process of penetration. If $u_+ > u_m$, then the rarefaction wave slows down across $J_2$. On the contrary, if $u_+ < u_m$, then it speeds up across $J_2$.*

*Proof.* The propagation speed of $J_2$ is $\tau_2 = 0$ and those of the characteristic lines in $R_1$ are

$$\xi_1(u^-, v^-) = \frac{k(1 + u^-)}{(1 + u^- + v^-)^2} > 0,$$

where the states $(u^-, v^-)$ in $R_1$ vary from $(u_1, v_1)$ to $(u_m, v_m)$. It is obvious that the rarefaction wave $R_1$ can across $J_2$ absolutely. In addition, $J_2$ interacts with the wave front of $R_1$ at the point which is determined by

$$\begin{cases} x_1 = \varepsilon, \\ x_1 + \varepsilon = \xi_1(u_m, v_m)t_1, \end{cases}$$

which yields

$$(x_1, t_1) = \Big(\varepsilon, \frac{2\varepsilon(1 + u_m + v_m)^2}{k(1 + u_m)}\Big). \tag{3.11}$$

On the other hand, the intersection of $J_2$ and the wave back of $R_1$ can also be calculated by

$$(x_2, t_2) = \Big(\varepsilon, \frac{2\varepsilon(1 + u_1 + v_1)^2}{k(1 + u_1)}\Big).$$

Now, let us compare the propagation speeds of rarefaction waves before and after penetration when across $J_2$. The state $(u^-, v^-)$ in $R_1$ becomes the matched one $(u^+, v^+)$ in $R_2$ when acrosses $J_2$, which should satisfy

$$\frac{v^+}{u^+ + 1} = \frac{v^-}{u^- + 1}. \tag{3.12}$$

The propagation speeds of the matched characteristic lines in $R_1$ and $R_2$ can be calculated respectively by

$$\xi_1(u^-, v^-) = \frac{k(1 + u^-)}{(1 + u^- + v^-)^2}, \qquad \xi_2(u^+, v^+) = \frac{k(1 + u^+)}{(1 + u^+ + v^+)^2}.$$

Then, we have

$$\begin{aligned}
&\xi_1(u^-, v^-) - \xi_2(u^+, v^+) \\
&= k\frac{(1 + u^-)(1 + u^+ + v^+)^2 - (1 + u^+)(1 + u^- + v^-)^2}{(1 + u^- + v^-)^2(1 + u^+ + v^+)^2} \\
&= k\frac{[(1 + u^-)(1 + u^+) + v^+(1 + u^-)](1 + u^+ + v^+) - [(1 + u^+)(1 + u^-) + v^-(1 + u^+)](1 + u^- + v^-)}{(1 + u^- + v^-)^2(1 + u^+ + v^+)^2} \\
&= \frac{k[(1 + u^-)(1 + u^+) + v^+(1 + u^-)](u^+ + v^+ - u^- - v^-)}{(1 + u^- + v^-)^2(1 + u^+ + v^+)^2}
\end{aligned}$$

$$= \frac{k(1+u^-)(u^+ + v^+ - u^- - v^-)}{(1+u^- + v^-)^2(1+u^+ + v^+)},$$

in which (3.12) has been used. If $u_+ > u_m$, then we have $u^+ > u^-$ and $v^+ > v^-$, such that $\xi_1(u^-, v^-) > \xi_2(u^+, v^+)$, which means that $R_1$ decelerates when it passes through $J_2$. Otherwise, if $u_+ < u_m$, then we have $u^+ < u^-$ and $v^+ < v^-$, such that $\xi_1(u^-, v^-) < \xi_2(u^+, v^+)$, which means that $R_1$ accelerates when it passes through $J_2$. Thus, the conclusion of the lemma can be drawn. □

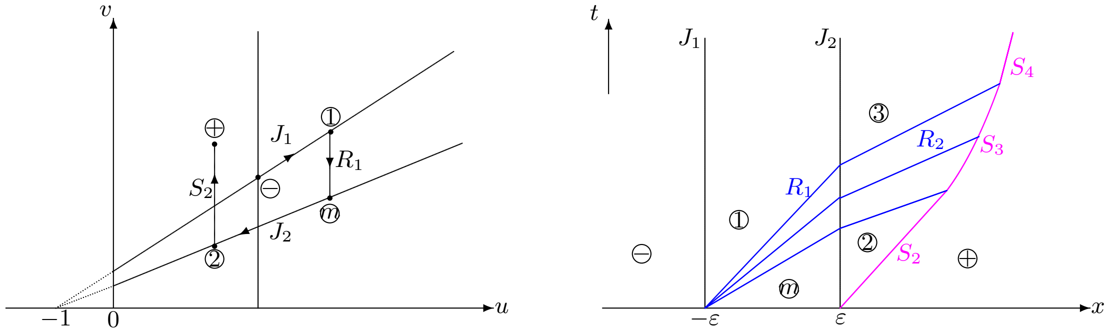


Figure 4: The interaction between $J + R$ and $J + S$ is shown when $u_+ < u_m$ and $\frac{v_+}{u_+ + 1} > \frac{v_-}{u_- + 1} > \frac{v_m}{u_m + 1}$.

On the other hand, the rarefaction wave $R_2$ continues to move forwards and penetrates the shock wave $S_2$ which can be summarized in the following lemma.

**Lemma 3.5.** *If $\frac{v_+}{u_+ + 1} > \frac{v_-}{u_- + 1}$, then $S_2$ has the ability to cancel the whole $R_2$ thoroughly. Otherwise, if $\frac{v_+}{u_+ + 1} < \frac{v_-}{u_- + 1}$, then $S_2$ penetrates $R_2$ partially and eventually takes the characteristic line $x = \varepsilon + \frac{k(1+u_+)t}{(1+u_+ + v_+)^2}$ in $R_2$ as its asymptotic line.*

*Proof.* The propagation speed of $S_2$ and that of the wave front in $R_2$ are computed respectively by

$$\sigma_2 = \frac{k(1+u_2)}{(1+u_2+v_2)(1+u_+ + v_+)}, \qquad \xi_2(u_2, v_2) = \frac{k(1+u_2)}{(1+u_2+v_2)^2}.$$

Noticing that $u_2 = u_+$ and $v_+ > v_2$, it is easy to know

$$\xi_2(u_2, v_2) - \sigma_2 = \frac{k(1+u_2)(v_+ - v_2)}{(1+u_2+v_2)^2(1+u_+ + v_+)} > 0.$$

Equivalently, the wave back of $R_2$ catches up with the shock wave $S_2$ in finite time. In fact, the intersection is determined by

$$\begin{cases} x_3 - \varepsilon = \xi_2(u_2, v_2)(t_3 - t_1), \\ x_3 - \varepsilon = \sigma_2 t_3, \end{cases}$$

in which $(x_1, t_1)$ is given by (3.11), which enables us to have

$$(x_3, t_3) = \Big(\varepsilon + \frac{2\varepsilon(1+u_m+v_m)}{v_+ - v_2}, \frac{2\varepsilon(1+u_+ + v_+)(1+u_m+v_m)^2}{k(1+u_m)(v_+ - v_2)}\Big).$$

After the time $t_3$, the shock wave starts to penetrate $R_2$ with varying propagation speeds and is labeled by $S_3$ during the process of penetration. The curve of $S_3$ may be determined by

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{k(1+u^+)}{(1+u^+ + v^+)(1+u_+ + v_+)}, \\[2ex] x - \varepsilon = \dfrac{k(1+u^+)}{(1+u^+ + v^+)^2}(t - \bar{t}), \\[2ex] x(t_3) = x_3, \end{cases}$$

in which $\bar{t}$ changes from $t_1$ to $t_2$ and $(u^+, v^+)$ is the corresponding state that the characteristic line starting from the point $(\varepsilon, \bar{t})$ in $R_2$ arrives at $S_3$. For our knowledge, it is impossible to calculate the explicit form for the curve of $S_3$ due to the fact that $R_2$ is a non-centered rarefaction wave. Depending on the relation between $\frac{v_+}{u_+ +1}$ and $\frac{v_-}{u_- +1}$, there are two possible situations as follows:

(a) If $\frac{v_+}{u_+ +1} > \frac{v_-}{u_- +1}$, then $S_3$ is able to cancel the entire $R_2$ thoroughly (see Figure 4). The shock wave is denoted with $S_4$ after penetration whose propagation speed is given by

$$\sigma_4 = \frac{k(1+u_-)}{(1+u_- + v_-)(1+u_+ + v_+)}.$$

(b) If $\frac{v_+}{u_+ +1} < \frac{v_-}{u_- +1}$, then $S_2$ cannot penetrate the whole $R_2$ completely and at last has the characteristic line $x = \varepsilon + \frac{k(1+u_+)t}{(1+u_+ + v_+)^2}$ associated with the state $(u_3, v_3)$ in $R_2$ as its asymptotic line. □

**Case 4.** $J + R$ and $J + R$.

In the end, we consider the situation that there are both $J + R$ originating from $(-\varepsilon, 0)$ and $(\varepsilon, 0)$. This case arises when

$$\frac{v_+}{u_+ + 1} < \frac{v_m}{u_m + 1} < \frac{v_-}{u_- + 1}$$

is satisfied. The solution of (1.1) and (1.3) for sufficiently small $t$ may be symbolized as

$$(u_-, v_-) + J_1 + (u_1, v_1) + R_1 + (u_m, v_m) + J_2 + (u_2, v_2) + R_2 + (u_+, v_+).$$

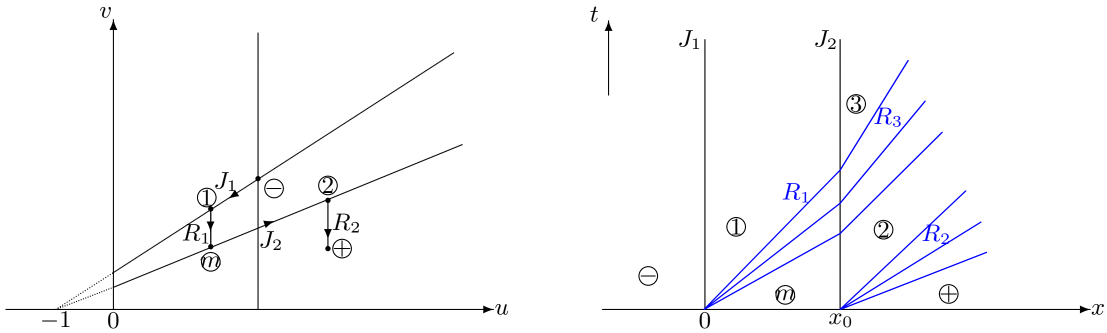


Figure 5: The interaction between $J + R$ and $J + R$ is shown when $\frac{v_+}{u_+ +1} < \frac{v_m}{u_m +1} < \frac{v_-}{u_- +1}$ and $u_+ > u_m$.

Similar to that in Case 3, the forward rarefaction wave $R_1$ penetrates $J_2$ at a time. This penetration also gives rise to a transmitted rarefaction wave $R_3$. In addition, the propagation speed will change and obey the same rule in Lemma 3.4 when the rarefaction wave $R_1$ crosses the contact discontinuity $J_2$. On the other hand, the wave front of $R_3$ is parallel to the wave back of $R_2$, such that these two waves $R_2$ and $R_3$ cannot interact with each other (see Figure 5).

## 4. The generalized Riemann problem with delta-type initial data

In this section, we draw our attention on the generalized Riemann problem for the system (1.1) with the delta-type initial data (1.5). In order to construct the solution of the generalized Riemann problem (1.1) and (1.5), we should also consider the particular Cauchy problem for the system (1.1) with the following perturbed Riemann initial data

$$(u, v)(x, 0) = \begin{cases} (u_-, v_-), & -\infty < x < -\varepsilon, \\ (u_0, \dfrac{m}{2\varepsilon}), & -\varepsilon < x < \varepsilon, \\ (u_+, v_+), & \varepsilon < x < +\infty, \end{cases} \tag{4.1}$$

in which $\varepsilon(>0)$ is sufficiently small. Then, the solutions of (1.1) and (1.5) can be obtained by letting $\varepsilon \to 0$ in the solutions of (1.1) and (4.1).

Obviously, $\frac{m}{2\varepsilon(1+u_0)}$ is much bigger than $\frac{v_-}{u_-+1}$ as well as $\frac{v_+}{u_++1}$ for $\varepsilon$ sufficiently small. Provided that $\varepsilon$ is a sufficiently small positive number, the Riemann solution emitting from $(-\varepsilon,0)$ is always a contact discontinuity $J_1$ followed by a shock wave $S_1$ and the Riemann solution emitting from $(\varepsilon,0)$ is always a contact discontinuity $J_2$ followed by a rarefaction wave $R_2$, respectively. As in Case 2, $S_1$ interacts with $J_2$ at the point determined by (3.2), which yields

$$(x_1,t_1)=\Big(\varepsilon,\frac{2\varepsilon(1+u_-+v_-)(1+u_0+\frac{m}{2\varepsilon})}{k(1+u_-)}\Big). \tag{4.2}$$

At the point $(x_1,t_1)$, a new Riemann problem with the initial data

$$(u_1,v_1)=\Big(u_0,\frac{(1+u_0)v_-}{1+u_-}\Big),\qquad (u_2,v_2)=\Big(u_+,\frac{m(1+u_+)}{2\varepsilon(1+u_0)}\Big), \tag{4.3}$$

is formed. Analogously, the Riemann solution is also a contact discontinuity $J_2$ followed by a shock wave $S_3$, in which the intermediate state $(u_3,v_3)$ is given by

$$(u_3,v_3)=\Big(u_+,\frac{(1+u_+)v_-}{1+u_-}\Big). \tag{4.4}$$

After that, $S_3$ begins to penetrate the rarefaction wave $R_2$ at the point which is determined by (3.5), which together with (4.2) gives

$$(x_2,t_2)=\Big(\varepsilon+\frac{k(1+u_2)t_1}{(1+u_2+v_2)(v_2-v_3)},\frac{(1+u_2+v_2)t_1}{v_2-v_3}\Big). \tag{4.5}$$

Making use of the relation between $\frac{v_-}{u_-+1}$ and $\frac{v_+}{u_++1}$, there are also two possibilities which are similar as that in Lemma 3.3. Besides, when $\frac{v_-}{u_-+1}<\frac{v_+}{u_++1}$, $S_4$ is able to cancel $R_2$ entirely and the process ends at the point given by (3.9), where the intermediate states are given by (4.3) and (4.4). It follows from (4.2) and (4.5) that

$$\lim_{\varepsilon\to 0}(x_1,t_1)=\Big(0,\frac{m(1+u_-+v_-)}{k(1+u_-)}\Big),\qquad \lim_{\varepsilon\to 0}x_2=0. \tag{4.6}$$

By making use of (3.5), we have

$$\lim_{\varepsilon\to 0}\frac{t_1}{t_2}=\lim_{\varepsilon\to 0}\frac{\sigma_3-\xi_2(u_2,v_2)}{\sigma_3}=\lim_{\varepsilon\to 0}\Big(1-\frac{(1+u_0)(1+u_-+v_-)}{(1+u_-)(1+u_0+\frac{m}{2\varepsilon})}\Big)=1. \tag{4.7}$$

On the other hand, taking into account (4.3) and (4.4), it follows from (3.4) that

$$\lim_{\varepsilon\to 0}\sigma_3=\lim_{\varepsilon\to 0}\frac{k(1+u_-)(1+u_0)}{(1+u_+)(1+u_-+v_-)(1+u_0+\frac{m}{2\varepsilon})}=0. \tag{4.8}$$

For convenience, let us denote $\bar t=\frac{m(1+u_-+v_-)}{k(1+u_-)}$. It is clear to see from (4.6) and (4.7) that both the points $(x_1,t_1)$ and $(x_2,t_2)$ will tend to the same point $(0,\bar t)$ in the limit $\varepsilon\to 0$ situation. Furthermore, it is observed from (4.8) that the shock wave $S_3$ is also compressed at the point $(0,\bar t)$ in the limit $\varepsilon\to 0$ situation. Thus, the shock wave $S_4$ starts to propagate from the point $(0,\bar t)$ in the limit $\varepsilon\to 0$ situation. On the other hand, it follows from (3.9) that

$$\lim_{\varepsilon\to 0}(x_3,t_3)=\Big(\frac{m(1+u_+)(1+u_-)(1+u_-+v_-)}{(v_--v_++u_+v_--u_-v_+)^2},\frac{m(1+u_-)(1+u_-+v_-)(1+u_++v_+)^2}{k(v_--v_++u_+v_--u_-v_+)^2}\Big).$$

It follows from (3.7) that the tangent slope of $S_4$ at the point $(x_2,t_2)$ can be calculated by

$$\frac{dx}{dt}\Big|_{(x_2,t_2)} = \Big(\frac{\sqrt{k(1+u_3)}}{1+u_3+v_3}\cdot\sqrt{\frac{x-\varepsilon}{t}}\Big)\Big|_{(x_2,t_2)} = \frac{k(1+u_-)(1+u_0)}{(1+u_+)(1+u_-+v_-)(1+u_0+\frac{m}{2\varepsilon})},$$

such that we have $\lim\limits_{\varepsilon\to 0}\frac{dx}{dt}|_{(x_2,t_2)}=0$. That is to say, the shock curve $S_4$ is tangent to the $t-$axis at the point $(0,\bar{t})$ in the limit $\varepsilon\to 0$ situation. Finally, it can be seen from (3.10) that the propagation speed of $S_5$ is unchanged in the limit $\varepsilon\to 0$ situation.

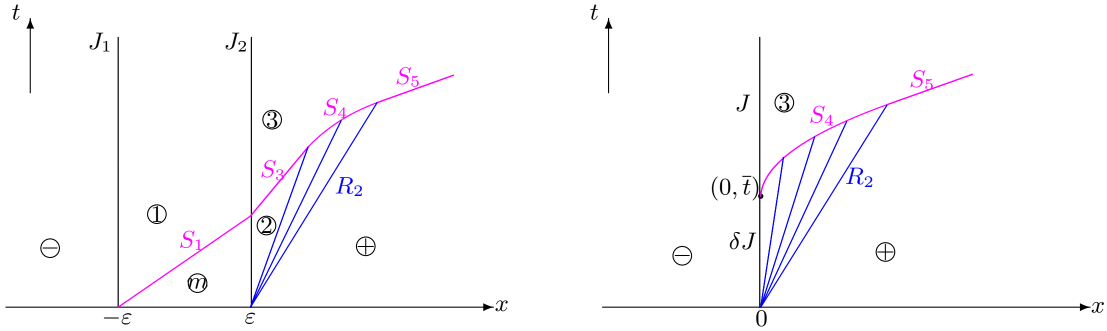


Figure 6: When $\frac{v_-}{u_-+1}<\frac{v_+}{u_++1}$, the solution of the particular Cauchy problem (1.1) and (4.1) is shown for the given sufficiently small $\varepsilon$ on the left-hand side and the solution of the generalized Riemann problem (1.1) and (1.5) is shown which is the limit $\varepsilon\to 0$ of the solution of (1.1) and (4.1) on the right-hand side.

Now, let us consider the limit $\varepsilon\to 0$ situation for the rarefaction wave $R_2$. First of all, the propagation speed of the wave front of $R_2$ is also $\xi_2(u_+,v_+)=\frac{k(1+u_+)}{(1+u_++v_+)^2}$ which remains unchanged after taking the limit $\varepsilon\to 0$. On the other hand, about the wave back of $R_2$, we have

$$\lim_{\varepsilon\to 0}\xi_2(u_2,v_2)=\lim_{\varepsilon\to 0}\frac{k(1+u_0)^2}{(1+u_+)(1+u_0+\frac{m}{2\varepsilon})^2}=0,$$

which means that the wave back of $R_2$ coincides with the $t-$axis.

In the end, let us turn our attention on the mass accumulation on the $t-$axis in the limit $\varepsilon\to 0$ situation. Let us use $x_1(t)$ and $x_2(t)$ to denote the curves of $S_1$ and the wave back of $R_2$, which are expressed respectively by

$$x_1(t)=\frac{k(1+u_-)t}{(1+u_-+v_-)(1+u_0+\frac{m}{2\varepsilon})}-\varepsilon,\qquad x_2(t)=\frac{k(1+u_0)^2t}{(1+u_+)(1+u_0+\frac{m}{2\varepsilon})}+\varepsilon.$$

In what follows, let us calculate the mass of $v$ in the region $(-\varepsilon,\varepsilon)$ as below

$$\begin{aligned}
\beta_\varepsilon(t) &= \int_{-\varepsilon}^{x_1(t)} v_1dx+\int_{x_1(t)}^{\varepsilon} v_mdx+\int_{\varepsilon}^{x_2(t)} v_2dx\\
&= \int_{-\varepsilon}^{x_1(t)} \frac{(1+u_0)v_-}{1+u_-}dx+\int_{x_1(t)}^{\varepsilon}\frac{m}{2\varepsilon}dx+\int_{\varepsilon}^{x_2(t)}\frac{1+u_+}{1+u_0}\cdot\frac{m}{2\varepsilon}dx\\
&= \frac{(1+u_0)v_-}{1+u_-}\cdot(x_1(t)+\varepsilon)+\frac{m}{2\varepsilon}\cdot(\varepsilon-x_1(t))+\frac{1+u_+}{1+u_0}\cdot\frac{m}{2\varepsilon}\cdot(x_2(t)-\varepsilon)\\
&= \frac{(1+u_0)v_-}{1+u_-}\cdot\Big(\frac{k(1+u_-)t}{(1+u_-+v_-)(1+u_0+\frac{m}{2\varepsilon})}\Big)+\frac{m}{2\varepsilon}\cdot\Big(2\varepsilon-\frac{k(1+u_-)t}{(1+u_-+v_-)(1+u_0+\frac{m}{2\varepsilon})}\Big)\\
&\quad+\frac{1+u_+}{1+u_0}\cdot\frac{m}{2\varepsilon}\cdot\Big(\frac{k(1+u_0)^2t}{(1+u_+)(1+u_0+\frac{m}{2\varepsilon})}\Big),
\end{aligned}$$

which enables us to have

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(t)=m-\frac{k(1+u_-)t}{1+u_-+v_-}.$$

Thus, we can see that the two contact discontinuities $J_1$ and $J_2$ coalesce into the delta contact discontinuity $\delta J$ on the $t-$axis before the time $\bar{t}$ in the limit $\varepsilon \to 0$ situation. But the strength of the delta contact discontinuity $\delta J$ decreases linearly at the rate $\frac{k(1+u_-)}{1+u_-+v_-}$ and becomes zero at the point $(0, \bar{t})$. Thus, the delta contact discontinuity $\delta J$ degenerates to be the contact discontinuity $J$ after the time $\bar{t}$.

If $\frac{v_-}{1+u_-} < \frac{v_+}{1+u_+}$, then the shock wave $S_4$ is able to penetrate $R_2$ completely in the limit $\varepsilon \to 0$ situation (see Figure 6). Otherwise, if $\frac{v_-}{1+u_-} > \frac{v_+}{1+u_+}$, then the shock wave $S_4$ cannot penetrate $R_2$ completely and eventually takes the characteristic line $x = \frac{k(1+u_+)t}{(1+u_++v_+)^2}$ in $R_2$ as its asymptotic line in the limit $\varepsilon \to 0$ situation. Thus, we can obtain the solutions of the generalized Riemann problem (1.1) and (1.5) by taking the limit $\varepsilon \to 0$ of the solutions to the particular Cauchy problem (1.1) and (4.1). Furthermore, it is easily seen that the solutions of the generalized Riemann problem (1.1) and (1.5) also converge to the corresponding solutions of the Riemann problem (1.1) and (1.4) when the limit $m \to 0$ is taken.

## 5. Conclusion

So far, we have finished the discussion for all kinds of wave interactions and the global solutions of the perturbed Riemann problem (1.1) and (1.3) have been constructed completely. It is clear to see that the large-time asymptotic states of the global solutions to the perturbed Riemann problem (1.1) and (1.3) are exactly the corresponding Riemann solutions of (1.1) and (1.4) and the asymptotic behaviors of the solutions to the perturbed Riemann problem (1.1) and (1.3) are governed completely by the Riemann initial data $(u_\pm, v_\pm)$. Thus, the Riemann solutions are stable with respect to the particular small perturbations (1.3) of the Riemann initial data (1.4) and the proof of Theorem 1.1 has been finished in view of the above analysis. In addition, the generalized Riemann problem for the system (1.1) with the delta-type initial data (1.5) can also be considered by virtue of the solutions of the perturbed Riemann problem (1.1) and (4.1) and then the delta contact discontinuity is captured and observed clearly.

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