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Solving variational inequality and split equality common fixed-point problem without prior knowledge of operator norms

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Abstract

In this paper, we introduce a viscosity iterative algorithm for finding common solution of variational inequality for Lipschitzian and strongly monotone operators and the split equality common fixed-point problem for firmly quasi-nonexpansive operators. We prove the strong convergence of the proposed algorithm which does not need any prior information about the bounded linear operator norms.

Keywords: Split equality problem, firmly quasi-nonexpansive operators, strong convergence, viscosity iterative algorithm, Hilbert space.
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1. Introduction and preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let I denote the identity operator on H. Let $T : H \to H$ be a mapping. A point $x \in H$ is said to be a fixed point of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set of T.

Let $F : H \to H$ be a nonlinear operator. Let C be a nonempty closed convex subset of H. The classical variational inequality, denoted by $VI(F, C)$, is to find $u \in C$ such that

$$\langle Fu, v - u \rangle \geq 0, \quad \forall v \in C.$$

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The theory of variational inequalities has played an important role in the study of a wide class of problems arising in pure and applied sciences including mechanics, optimization and optimal control, partial differential equation, operations research and engineering sciences. During the last decades, this problem has been studied by many authors (see [6, 9, 17, 20, 22]).

In [25], Yamada introduced the following hybrid iterative method

$$x_{k+1} = Tx_k - \mu\lambda_k F(Tx_k), \quad k \geq 0,$$

for solving variational inequality

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad x \in F(T),$$

where T is a nonexpansive operator and F is a κ-Lipschitzian and η-strongly monotone operator on H with $\kappa > 0$, $\eta > 0$, $0 < \mu < 2\eta/k^2$.

Tian [21] introduced the following general iterative method

$$x_{k+1} = \alpha_k \sigma f(x_k) + (I - \mu\alpha_k F)Tx_k, \quad k \geq 0,$$

for solving variational inequality

$$\langle (\sigma f - \mu F)x^*, x - x^* \rangle \leq 0, \quad x \in F(T),$$

where T is a nonexpansive operator, f is ρ-contraction, F is a κ-Lipschitzian and η-strongly monotone operator on H with $\kappa > 0$, $\eta > 0$, $0 < \mu < 2\eta/k^2$ and $0 < \sigma < \mu(\eta - \frac{\mu\kappa^2}{2})/\rho$.

Let C and Q be nonempty closed convex subset of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is to find a point

$$x \in C, \quad \text{such that} \quad Ax \in Q, \tag{1.1}$$

where $A: H_1 \to H_2$ is a bounded linear operator. The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [10] for the modeling inverse problems which arise from the phase retrievals and in the medical image reconstruction [4].

Note that if the split feasibility problem (1.1) is consistent (i.e., (1.1) has a solution) then (1.1) can be formulated as a fixed point equation by using the fact

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*, \tag{1.2}$$

where P_C and P_Q are the (orthogonal) projection onto C and Q, respectively, $\gamma > 0$ is any positive constant and A^* denotes the adjoint of A. That is, x^* solves the SFP (1.1) if and only if x^* solves the fixed point equation (1.2) (see [24] for the details). This implies that we can use the fixed point algorithms (see [2, 23, 24]) to solve SFP. To solve (1.2), Byrne [4] proposed his CQ algorithm which generates a sequence $\{x_k\}$ by

$$x_{k+1} = P_C(I - \gamma A^*(I - P_Q)A)x_k, \quad k \in N,$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A.

Censor and Segal [12] introduced the following split common fixed-point problem (SCFP):

$$\text{find} \quad x^* \in F(U), \quad \text{such that} \quad Ax^* \in F(T), \tag{1.3}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two nonexpansive operators with nonempty fixed-point sets $F(U) = C$ and $F(T) = Q$. SCFP is in itself at the core of the modeling of many inverse problems in various areas of mathematics and physical sciences and has been used to model significant real-world inverse problems in many areas (see [11]).

To solve (1.3), Censor and Segal [12] proposed and proved, in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{k+1} = U(x_k + \gamma A^t(T - I)Ax_k), \quad k \in N,$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the largest eigenvalue of the matrix A^tA (A^t stands for matrix transposition).

Let H_1, H_2, H_3 be real Hilbert spaces, let $C \subset H_1$, $Q \subset H_2$ be two nonempty closed convex sets, let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be two firmly quasi-nonexpansive operators. In [18], Moudafi introduced the following split equality problem (SEP):

$$\text{find } \ x \in C, y \in Q, \ \text{ such that } \ Ax = By, \tag{1.4}$$

and split equality common fixed-point problem (SECFP):

$$\text{find } \ x \in F(U), y \in F(T), \ \text{ such that } \ Ax = By, \tag{1.5}$$

which allows asymmetric and partial relations between the variables x and y. The interest is to cover many situation, for instance in decomposition methods for PDE's, applications in game theory and in intensity-modulated radiation therapy (IMRT). In decision sciences, this allows to consider agents who interplay only via some components of their decision variables (see [1]). In (IMRT), this amounts to envisage a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity (see [5]).

If $H_2 = H_3$ and $B = I$, then the SECFP (1.5) reduces to the SCFP (1.3). For solving the SECFP (1.5), Moudafi [18] introduced the following alternating algorithm

$$\begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_{k+1} - By_k)), \end{cases} \tag{1.6}$$

for firmly quasi-nonexpansive operators U and T, where non-decreasing sequence $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, λ_A, λ_B stand for the spectral radius of A^*A and B^*B, respectively.

Very recently, Moudafi [19] introduced the following simultaneous iterative method to solve SECFP (1.5):

$$\begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_k - By_k)), \end{cases} \tag{1.7}$$

for firmly quasi-nonexpansive operators U and T, where $\gamma_k \in (\varepsilon, \frac{2}{\lambda_A + \lambda_B} - \varepsilon)$, λ_A, λ_B stand for the spectral radius of A^*A and B^*B, respectively.

Note that in the algorithms (1.6) and (1.7) mentioned above, the determination of the stepsize $\{\gamma_k\}$ depends on the operator (matrix) norms $\|A\|$ and $\|B\|$ (or the largest eigenvalues of A^*A and B^*B). In order to implement the above algorithms for solving SECFP (1.5), one has first to compute (or, at least, estimate) operator norms of A and B, which is in general not an easy work in practice. To overcome this difficulty, López et al. [16] and Zhao and Yang [28] presented a helpful method for estimating the stepsizes which do not need prior knowledge of the operator norms for solving the split feasibility problems and multiple-set split feasibility problems, respectively.

Some algorithms have been invented to solve SECFP (1.5) (see [13, 14, 27] and references therein). In this paper, inspired and motivated by the works mentioned above, to get the strong convergence of the algorithm, we introduce the viscosity iterative algorithm without prior knowledge of operators norms for finding common solution of variational inequality for Lipschitzian and strongly monotone operators and the split equality common fixed-point problem for firmly quasi-nonexpansive operators. The organization of this paper is as follows. Some useful definitions and results are listed for the convergence analysis of the iterative algorithm in the Section 2. In Section 3, the strong convergence theorem of the proposed general iterative algorithm is obtained.

2. Preliminaries

In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. We use $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ stand for the weak ω-limit set of $\{x_k\}$ and use Γ stand for the solution set of the SECFP (1.5).

Definition 2.1. An operator $T : H \to H$ is said to be

(i) nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in H$.
(ii) quasi-nonexpansive if $F(T) \ne \emptyset$ and $\|Tx - q\| \le \|x - q\|$, for all $x \in H$ and $q \in F(T)$.
(iii) firmly nonexpansive if $\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2$, for all $x, y \in H$.
(iv) firmly quasi-nonexpansive if $F(T) \ne \emptyset$ and $\|Tx - q\|^2 \le \|x - q\|^2 - \|x - Tx\|^2$, for all $x \in H$ and $q \in F(T)$.

Remark 2.2. A firmly quasi-nonexpansive operator is also called a separating operator [7], cutter operator [8], directed operators [12, 26], or class-$\mathfrak{T}$ operator which was introduced by Bauschke and Combettes [3]. Firmly quasi-nonexpansive operators are important because they include many types of nonlinear operators arising in applied mathematics such as approximation and convex optimization. For instance, the subgradient projection T of a continuous convex function $f : H \to \mathbb{R}$ is a firmly quasi-nonexpansive operator. Recall that the subgradient projection T is defined by assuming the level set $\{x \in H : f(x) \le 0\} \ne \emptyset$,

$$Tx := \begin{cases} x - \frac{f(x)}{\|g(x)\|^2} g(x), & f(x) > 0, \\ x, & f(x) \le 0, \end{cases}$$

where g is a selection of the subdifferential ∂f(i.e., $g(x) \in \partial f(x)$ for all $x \in H$).

Particularly, projections are firmly quasi-nonexpansive operators. Recall that, given a closed convex subset C of a Hilbert space H, the projection $P_C : H \to C$ assigns each $x \in H$ to its closest point from C, defined by

$$P_C x = argmin_{z\in C} \|x - z\|.$$

It is well-known that $P_C x$ is characterized by the inequality:

$$P_C x \in C, \quad \langle x - P_C x, z - P_C x\rangle \le 0, \quad z \in C.$$

Lemma 2.3 ([3, 8])**.** *The fixed point set of a firmly quasi-nonexpansive operator is closed and convex.*

We also need other classes of operators.

Definition 2.4. An operator $T : H \to H$ is called demiclosed at the origin, if for any sequence $\{x_n\}$ which weakly converges to x, and if the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx = 0$.

Definition 2.5. An operator $T : H \to H$ is called contraction with constant $0 < \rho < 1$, if for any $x, y \in H$,

$$\|Tx - Ty\| \le \rho\|x - y\|.$$

Definition 2.6. An operator $T : H \to H$ is called κ-Lipschitzian operater with constant $\kappa > 0$, if for any $x, y \in H$,

$$\|Tx - Ty\| \le \kappa\|x - y\|.$$

Definition 2.7. An operator $F : H \to H$ is called η-strongly monotone with constant $\eta > 0$, if for any $x, y \in H$,

$$\langle x - y, Fx - Fy\rangle \ge \eta\|x - y\|^2.$$

Let F be a κ-Lipschitzian and η-strongly monotone operator on H with $\kappa > 0$, $\eta > 0$. Assume that $\mu > 0$, $\alpha \in [0, 1]$, we have, for any x, $y \in H$,

$$\begin{aligned} \|(I - \mu\alpha F)x - (I - \mu\alpha F)y\|^2 =&\|(x - y) - \mu\alpha(Fx - Fy)\|^2 \\ =&\|x - y\|^2 - 2\mu\alpha\langle x - y, Fx - Fy\rangle + \mu^2\alpha^2\|Fx - Fy\|^2 \\ \le&\|x - y\|^2 - 2\mu\alpha\eta\|x - y\|^2 + \mu^2\alpha^2\kappa^2\|x - y\|^2 \\ \le&[1 - \alpha(2\mu\eta - \mu^2\kappa^2)]\|x - y\|^2, \end{aligned} \tag{2.1}$$

which implies that $I - \mu\alpha F$ is $\sqrt{1 - \alpha(2\mu\eta - \mu^2\kappa^2)}$-Lipschitzian. It is easy to see that $I - \mu\alpha F$ is contraction if $0 < \mu < \frac{2\eta}{k^2}$, $\alpha \in [0, 1]$. The following lemma is easy to prove.

Lemma 2.8. *Let H be a Hilbert space, f be a contraction with constant $0 < \rho < 1$ and F be a η-strongly monotone operator on H with $\eta > 0$. Then, for $\mu > 0$ and $0 < \sigma < \frac{\mu\eta}{\rho}$,*

$$\langle x - y,\ (\mu F - \sigma f)x - (\mu F - \sigma f)y\rangle \geq (\mu\eta - \sigma\rho)\|x - y\|^2, \quad x, y \in H.$$

That is, $\mu F - \sigma f$ is $\mu\eta - \sigma\rho$-strongly monotone operator.

In real Hilbert space, we easily get the following results:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H; \tag{2.2}$$

$$2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2, \quad \forall x, y \in H. \tag{2.3}$$

We end this section by the following lemma, which are important in convergence analysis for our iterative algorithm.

Lemma 2.9 ([15]). *Assume $\{s_k\}$ is a sequence of nonnegative real numbers such that*

$$\begin{cases} s_{k+1} \leq (1 - \lambda_k)s_k + \lambda_k\delta_k, & k \geq 0, \\ s_{k+1} \leq s_k - \eta_k + \mu_k, & k \geq 0, \end{cases}$$

where $\{\lambda_k\}$ is a sequence in $(0, 1)$, $\{\eta_k\}$ is a sequence of nonnegative real numbers and $\{\delta_k\}$ and $\{\mu_k\}$ are two sequences in $\mathbb{R}$ such that

(i) $\Sigma_{k=1}^{\infty}\lambda_k = \infty$;
(ii) $\lim_{k\to\infty} \mu_k = 0$;
(iii) $\lim_{l\to\infty} \eta_{k_l} = 0$, *implies* $\limsup_{l\to\infty} \delta_{k_l} \leq 0$, *for any subsequence* $\{k_l\} \subset \{k\}$.

Then $\lim_{k\to\infty} s_k = 0$.

3. Strong convergence result of viscosity iterative algorithm for SECFP (1.5)

In this section we introduce a viscosity iterative algorithm for finding common solution of variational inequality for Lipschitzian and strongly monotone operators and the split equality common fixed-point problem for firmly quasi-nonexpansive operators. Here the stepsizes $\{\gamma_k\}$ do not depend on the operator norms $\|A\|$ and $\|B\|$ and we prove the strong convergence of the proposed algorithm.

Algorithmm 3.1. *Let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be two contractions with constants ρ_1, $\rho_2 \in [0, 1)$, $\alpha_k \in [0, 1]$ and $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone with $\kappa > 0$, $\eta > 0$. Choose an initial guess $x_0 \in H_1$, $y_0 \in H_2$ arbitrarily. Assume that the k-th iterate $x_k \in H_1$, $y_k \in H_2$ has been constructed, then we calculate the (k + 1)-th iterate (x_{k+1}, y_{k+1}) via the formula:*

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \sigma\alpha_k f_1(x_k) + (I - \mu\alpha_k F)U(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \sigma\alpha_k f_2(x_k) + (I - \mu\alpha_k F)T(v_k). \end{cases}$$

The stepsize γ_k is chosen in such a way that

$$\gamma_k \in \big(\epsilon, \min\{\tau, \frac{2\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2}\} - \epsilon\big), \quad k \in \Omega \tag{3.1}$$

for small enough ϵ, otherwise, $\gamma_k = \gamma$ (γ being any nonnegative value), where the set of indices $\Omega = \{k : Ax_k - By_k \neq 0\}$,

$$\tau = \frac{2\|Ax_N - By_N\|^2}{\|A^*(Ax_N - By_N)\|^2 + \|B^*(Ax_N - By_N)\|^2},$$

and $N = \min\{k : k \in \Omega\}$.

Lemma 3.2. *Assume the solution set* Γ *of* (1.5) *is nonempty. Then* γ_k *defined by* (3.1) *is well-defined.*

Proof. Take $(x, y) \in \Gamma$, i.e., $x \in F(U)$, $y \in F(T)$ and $Ax = By$. We have

$$\langle A^*(Ax_k - By_k), x_k - x\rangle = \langle Ax_k - By_k, Ax_k - Ax\rangle,$$

and

$$\langle B^*(Ax_k - By_k), y - y_k\rangle = \langle Ax_k - By_k, By - By_k\rangle.$$

By adding the two above equalities and by taking into account the fact that $Ax = By$, we obtain

$$\begin{aligned}\|Ax_k - By_k\|^2 &= \langle A^*(Ax_k - By_k), x_k - x\rangle + \langle B^*(Ax_k - By_k), y - y_k\rangle \\ &\le \|A^*(Ax_k - By_k)\| \cdot \|x_k - x\| + \|B^*(Ax_k - By_k)\| \cdot \|y - y_k\|.\end{aligned}$$

Consequently, for $k \in \Omega$, that is, $\|Ax_k-By_k\| > 0$, we have $\|A^*(Ax_k-By_k)\| \neq 0$ or $\|B^*(Ax_k-By_k)\| \neq 0$. This leads that γ_k is well-defined. □

Theorem 3.3. *Let* H_1, H_2, H_3 *be real Hilbert spaces. Given two bounded linear operators* $A: H_1 \to H_3$, $B: H_2 \to H_3$, *let* $U : H_1 \to H_1$ *and* $T : H_2 \to H_2$ *be firmly quasi-nonexpansive operators with the solution set* Γ *of* (1.5) *is nonempty. Let* $f_1 : H_1 \to H_1$ *and* $f_2 : H_2 \to H_2$ *be two contractions with constants* ρ_1, $\rho_2 \in [0, 1)$ *and* $F : H \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone. Assume that we choose* μ, β *and* σ *such that* $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \beta < \frac{\tau}{2}$, $0 < \sigma < \min\{\frac{\sqrt{\beta(\tau-2\beta)}}{\rho}, \frac{\mu\eta}{\rho}\}$, *where* $\tau = 2\mu\eta - \mu^2\kappa^2$ *and* $\rho = \max\{\rho_1, \rho_2\}$. *Let the sequence* $\{(x_k, y_k)\}$ *be generated by Algorithm* 3.1. *Assume that the following conditions are satisfied:*

(1) $\lim_{k\to\infty} \alpha_k = 0$ *and* $\sum_{k=0}^{\infty} \alpha_k = \infty$;
(2) $U - I$ *and* $T - I$ *are demiclosed at origin.*

Then sequence $\{(x_k, y_k)\}$ *strongly converges to a solution* (x^*, y^*) *of* (1.5) *which solves the variational inequality problem:*

$$\begin{cases}\langle(\sigma f_1 - \mu F)x^*, x - x^*\rangle \le 0, \\ \langle(\sigma f_2 - \mu F)y^*, y - y^*\rangle \le 0,\end{cases} \qquad (x, y) \in \Gamma. \tag{3.2}$$

Proof. Since f_1, f_2 are two contractions and F is Lipschitzian, we have $\mu F - \sigma f_1$ and $\mu F - \sigma f_2$ are Lipschitzian. By Lemma 2.8, $\mu F - \sigma f_1$ and $\mu F - \sigma f_2$ are strongly monotone, so the variational inequality (3.2) has only one solution. From assumption on μ we have $\tau > 0$. Let $(x^*, y^*) \in \Gamma$ be the solution of the variational inequality problem (3.2). Then $x^* \in F(U)$, $y^* \in F(T)$ and $Ax^* = By^*$. We have

$$\begin{aligned}\|u_k - x^*\|^2 &= \|x_k - \gamma_k A^*(Ax_k - By_k) - x^*\|^2 \\ &= \|x_k - x^*\|^2 - 2\gamma_k\langle x_k - x^*, A^*(Ax_k - By_k)\rangle + \gamma_k^2\|A^*(Ax_k - By_k)\|^2.\end{aligned} \tag{3.3}$$

By using the equality (2.3), we have

$$\begin{aligned}-2\langle x_k - x^*, A^*(Ax_k - By_k)\rangle &= -2\langle Ax_k - Ax^*, Ax_k - By_k\rangle \\ &= -\|Ax_k - Ax^*\|^2 - \|Ax_k - By_k\|^2 + \|By_k - Ax^*\|^2.\end{aligned} \tag{3.4}$$

By (3.3) and (3.4), we obtain

$$\begin{aligned}\|u_k - x^*\|^2 \le& \|x_k - x^*\|^2 - \gamma_k\|Ax_k - Ax^*\|^2 - \gamma_k\|Ax_k - By_k\|^2 \\ &+ \gamma_k\|By_k - Ax^*\|^2 + \gamma_k^2\|A^*(Ax_k - By_k)\|^2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|v_k - y^*\|^2 \le& \|y_k - y^*\|^2 - \gamma_k\|By_k - By^*\|^2 - \gamma_k\|Ax_k - By_k\|^2 \\ &+ \gamma_k\|Ax_k - By^*\|^2 + \gamma_k^2\|B^*(Ax_k - By_k)\|^2.\end{aligned}$$

By adding the two last inequalities and by taking into account the fact that $Ax^* = By^*$, we obtain

$$\begin{aligned}\|u_k - x^*\|^2 + \|v_k - y^*\|^2 \leq &\|x_k - x^*\|^2 + \|y_k - y^*\|^2 \\ &- \gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 \\ &+ \|B^*(Ax_k - By_k)\|^2)].\end{aligned} \tag{3.5}$$

With assumption on γ_k we obtain

$$\|u_k - x^*\|^2 + \|v_k - y^*\|^2 \leq \|x_k - x^*\|^2 + \|y_k - y^*\|^2. \tag{3.6}$$

It follows from (2.1) that

$$\|(I - \mu\alpha_k F)U(u_k) - (I - \mu\alpha_k F)x^*\|^2 \leq (1 - \alpha_k\tau)\|U(u_k) - x^*\|^2.$$

By the fact that U is a firmly quasi-nonexpansive operator, it follows from (2.2) that

$$\begin{aligned}\|x_{k+1} - x^*\|^2 =&\|\sigma\alpha_k f_1(x_k) + (I - \mu\alpha_k F)U(u_k) - x^*\|^2 \\ =&\|\sigma\alpha_k f_1(x_k) + (I - \mu\alpha_k F)U(u_k) - (I - \mu\alpha_k F)x^* - \mu\alpha_k F x^*\|^2 \\ \leq&(1 - \alpha_k\tau)\|U(u_k) - x^*\|^2 + 2\alpha_k\langle\sigma f_1(x_k) - \mu F x^*, x_{k+1} - x^*\rangle \\ \leq&(1 - \alpha_k\tau)\|u_k - x^*\|^2 + 2\alpha_k\|\sigma f_1(x_k) - \mu F x^*\| \cdot \|x_{k+1} - x^*\|.\end{aligned}$$

Obviously, we have

$$\|\sigma f_1(x_k) - \mu F x^*\| \cdot \|x_{k+1} - x^*\| \leq \beta\|x_{k+1} - x^*\|^2 + \frac{1}{4\beta}\|\sigma f_1(x_k) - \mu F x^*\|^2.$$

So, we can obtain

$$\|x_{k+1} - x^*\|^2 \leq (1 - \alpha_k\tau)\|u_k - x^*\|^2 + 2\alpha_k\beta\|x_{k+1} - x^*\|^2 + \frac{\alpha_k}{2\beta}\|\sigma f_1(x_k) - \mu F x^*\|^2,$$

which implies that

$$\begin{aligned}(1 - 2\alpha_k\beta)\|x_{k+1} - x^*\|^2 \leq&(1 - \alpha_k\tau)\|u_k - x^*\|^2 + \frac{\alpha_k}{2\beta}(\|\sigma f_1(x_k) - \sigma f_1(x^*) + \sigma f_1(x^*) - \mu F x^*\|^2) \\ \leq&(1 - \alpha_k\tau)\|u_k - x^*\|^2 + \frac{\alpha_k\sigma^2\rho_1^2}{\beta}\|x_k - x^*\|^2 + \frac{\alpha_k}{\beta}\|\sigma f_1(x^*) - \mu F x^*\|^2.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|x_{k+1} - x^*\|^2 \leq&\frac{1 - \alpha_k\tau}{1 - 2\alpha_k\beta}\|u_k - x^*\|^2 + \frac{\alpha_k\sigma^2\rho_1^2}{\beta(1 - 2\alpha_k\beta)}\|x_k - x^*\|^2 \\ &+ \frac{\alpha_k}{\beta(1 - 2\alpha_k\beta)}\|\sigma f_1(x^*) - \mu F x^*\|^2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|y_{k+1} - y^*\|^2 \leq&\frac{1 - \alpha_k\tau}{1 - 2\alpha_k\beta}\|v_k - y^*\|^2 + \frac{\alpha_k\sigma^2\rho_2^2}{\beta(1 - 2\alpha_k\beta)}\|y_k - y^*\|^2 \\ &+ \frac{\alpha_k}{\beta(1 - 2\alpha_k\beta)}\|\sigma f_2(y^*) - \mu F y^*\|^2.\end{aligned}$$

By adding up the last two inequalities and by using (3.6) and setting $s_k = \|x_k - x^*\|^2 + \|y_k - y^*\|^2$, we get

$$s_{k+1} \leq \frac{1 - \alpha_k\tau + \frac{\alpha_k\sigma^2\rho^2}{\beta}}{1 - 2\alpha_k\beta}s_k + \frac{\alpha_k}{\beta(1 - 2\alpha_k\beta)}(\|\sigma f_1(x^*) - \mu F x^*\|^2 + \|\sigma f_2(y^*) - \mu F y^*\|^2),$$

where $\rho = \max\{\rho_1, \rho_2\}$. So,

$$\begin{aligned} s_{k+1} \leq & \frac{1-\alpha_k(\tau - \frac{\sigma^2\rho^2}{\beta})}{1-2\alpha_k\beta} s_k \\ & + \frac{\alpha_k(\tau - 2\beta - \frac{\sigma^2\rho^2}{\beta})}{1-2\alpha_k\beta} \frac{1}{\beta(\tau - 2\beta - \frac{\sigma^2\rho^2}{\beta})} (\|\sigma f_1(x^*) - \mu F x^*\|^2 + \|\sigma f_2(y^*) - \mu F y^*\|^2). \end{aligned}$$

From $0 < \beta < \frac{\tau}{2}$ and $0 < \sigma < \frac{\sqrt{\beta(\tau-2\beta)}}{\rho}$, we have $\tau - 2\beta - \frac{\sigma^2\rho^2}{\beta} > 0$. Since $\alpha_k \to 0$, we have $0 \leq \frac{\alpha_k(\tau-2\beta-\frac{\sigma^2\rho^2}{\beta})}{1-2\alpha_k\beta} \leq 1$, for large enough $k \in N$. Since

$$\frac{\alpha_k(\tau - 2\beta - \frac{\sigma^2\rho^2}{\beta})}{1-2\alpha_k\beta} + \frac{1-\alpha_k(\tau - \frac{\sigma^2\rho^2}{\beta})}{1-2\alpha_k\beta} = 1,$$

without loss of generality, it follows from the induction that

$$s_k \leq \max\left\{s_0, \frac{1}{\beta(\tau - 2\beta - \frac{\sigma^2\rho^2}{\beta})}(\|\sigma f_1(x^*) - \mu F x^*\|^2 + \|\sigma f_2(y^*) - \mu F y^*\|^2)\right\}$$

for $k \geq 0$. Then we have $\{x_k\}$ and $\{y_k\}$ are bounded. It follows that $\{u_k\}, \{v_k\}, \{f_1(x_k)\}$ and $\{f_2(y_k)\}$ are bounded. Note that U is a firmly quasi-nonexpansive operator, we have

$$\begin{aligned} \|x_{k+1} - x^*\|^2 =& \|\alpha_k(\sigma f_1(x_k) - \mu F x^*) + (I - \mu\alpha_k F)U(u_k) - (I - \mu\alpha_k F)Ux^*\|^2 \\ =& \alpha_k^2\|\sigma f_1(x_k) - \mu F x^*\|^2 + \|(I - \mu\alpha_k F)U(u_k) - (I - \mu\alpha_k F)Ux^*\|^2 \\ & + 2\alpha_k\langle (I - \mu\alpha_k F)U(u_k) - (I - \mu\alpha_k F)Ux^*, \sigma f_1(x_k) - \mu F x^*\rangle \\ \leq& (1 - \alpha_k\tau)\|u_k - x^*\|^2 + \alpha_k^2\|\sigma f_1(x_k) - \mu F x^*\|^2 \\ & + 2\alpha_k[\langle U(u_k) - Ux^*, \sigma f_1(x_k) - \mu F x^*\rangle \\ & - \mu\alpha_k\langle FU(u_k) - FUx^*, \sigma f_1(x_k) - \mu F x^*\rangle] \\ \leq& (1 - \alpha_k\tau)\|u_k - x^*\|^2 + \alpha_k^2\|\sigma f_1(x_k) - \mu F x^*\|^2 \\ & + 2\alpha_k\langle U(u_k) - Ux^*, \sigma f_1(x_k) - \sigma f_1(x^*)\rangle \\ & + 2\alpha_k\langle U(u_k) - Ux^*, \sigma f_1(x^*) - \mu F x^*\rangle \\ & + 2\mu\alpha_k^2\|FU(u_k) - FUx^*\| \cdot \|\sigma f_1(x_k) - \mu F x^*\| \\ \leq& (1 - \alpha_k\tau)\|u_k - x^*\|^2 + \alpha_k^2\|\sigma f_1(x_k) - \mu F x^*\|^2 \\ & + 2\alpha_k(\beta\|U(u_k) - Ux^*\|^2 + \frac{1}{4\beta}\|\sigma f_1(x_k) - \sigma f_1(x^*)\|^2) \\ & + 2\alpha_k\langle U(u_k) - Ux^*, \sigma f_1(x^*) - \mu F x^*\rangle \\ & + 2\mu\alpha_k^2\kappa\|U(u_k) - x^*\| \cdot \|\sigma f_1(x_k) - \mu F x^*\| \\ \leq& (1 - \alpha_k\tau)\|u_k - x^*\|^2 + \alpha_k^2\|\sigma f_1(x_k) - \mu F x^*\|^2 + 2\alpha_k\beta\|u_k - x^*\|^2 \\ & + \frac{\alpha_k\sigma^2\rho_1^2}{2\beta}\|x_k - x^*\|^2 + 2\alpha_k\langle U(u_k) - Ux^*, \sigma f_1(x^*) - \mu F x^*\rangle \\ & + 2\mu\alpha_k^2\kappa\|u_k - x^*\| \cdot \|\sigma f_1(x_k) - \mu F x^*\| \\ =& (1 - \alpha_k\tau + 2\alpha_k\beta)\|u_k - x^*\|^2 + \alpha_k^2\|\sigma f_1(x_k) - \mu F x^*\|^2 \\ & + \frac{\alpha_k\sigma^2\rho^2}{2\beta}\|x_k - x^*\|^2 + 2\alpha_k\langle U(u_k) - Ux^*, \sigma f_1(x^*) - \mu F x^*\rangle \\ & + 2\mu\alpha_k^2\kappa\|u_k - x^*\| \cdot \|\sigma f_1(x_k) - \mu F x^*\|. \end{aligned} \tag{3.7}$$

Similarly, we have

$$\begin{aligned}\|y_{k+1}-y^*\|^2 \leq &(1-\alpha_k\tau+2\alpha_k\beta)\|v_k-y^*\|^2+\alpha_k^2\|\sigma f_2(y_k)-\mu Fy^*\|^2\\&+\frac{\alpha_k\sigma^2\rho^2}{2\beta}\|y_k-y^*\|^2+2\alpha_k\langle T(v_k)-Ty^*,\sigma f_2(y^*)-\mu Fy^*\rangle\\&+2\mu\alpha_k^2\kappa\|v_k-y^*\|\cdot\|\sigma f_2(y_k)-\mu Fy^*\|.\end{aligned}\tag{3.8}$$

So, by (3.6), (3.7) and (3.8) we obtain

$$\begin{aligned}s_{k+1}\leq&[1-\alpha_k(\tau-2\beta-\frac{\sigma^2\rho^2}{2\beta})]s_k\\&+\alpha_k(\tau-2\beta-\frac{\sigma^2\rho^2}{2\beta})\{\frac{\alpha_k(\|\sigma f_1(x_k)-\mu Fx^*\|^2+\|\sigma f_2(y_k)-\mu Fy^*\|^2)}{\tau-2\beta-\frac{\sigma^2\rho^2}{2\beta}}\\&+\frac{2(\langle U(u_k)-x^*,\sigma f_1(x^*)-\mu Fx^*\rangle+\langle T(v_k)-y^*,\sigma f_2(y^*)-\mu Fy^*\rangle)}{\tau-2\beta-\frac{\sigma^2\rho^2}{2\beta}}\\&+\frac{2\mu\alpha_k\kappa(\|u_k-x^*\|\cdot\|\sigma f_1(x_k)-\mu Fx^*\|+\|v_k-y^*\|\cdot\|\sigma f_2(y_k)-\mu Fy^*\|)}{\tau-2\beta-\frac{\sigma^2\rho^2}{2\beta}}\}\\=&(1-\lambda_k)s_k+\lambda_k\delta_k,\end{aligned}$$

where

$$\begin{aligned}\lambda_k=&\alpha_k(\tau-2\beta-\frac{\sigma^2\rho^2}{2\beta}),\\\delta_k=&\frac{1}{\tau-2\beta-\frac{\sigma^2\rho^2}{2\beta}}\{\alpha_k(\|\sigma f_1(x_k)-\mu Fx^*\|^2+\|\sigma f_2(y_k)-\mu Fy^*\|^2)\\&+2(\langle U(u_k)-x^*,\sigma f_1(x^*)-\mu Fx^*\rangle+\langle T(v_k)-y^*,\sigma f_2(y^*)-\mu Fy^*\rangle)\\&+2\mu\alpha_k\kappa(\|u_k-x^*\|\cdot\|\sigma f_1(x_k)-\mu Fx^*\|+\|v_k-y^*\|\cdot\|\sigma f_2(y_k)-\mu Fy^*\|)\}.\end{aligned}$$

On the other hand, since U is firmly quasi-nonexpansive we have

$$\begin{aligned}\|x_{k+1}-x^*\|^2=&\|\alpha_k\sigma f_1(x_k)+(I-\alpha_k\mu F)U(u_k)-x^*\|^2\\=&\|\alpha_k\sigma f_1(x_k)+(I-\alpha_k\mu F)U(u_k)-(I-\alpha_k\mu F)x^*-\alpha_k\mu Fx^*\|^2\\\leq&(1-\alpha_k\tau)\|U(u_k)-x^*\|^2+2\alpha_k\langle\sigma f_1(x_k)-\mu Fx^*,x_{k+1}-x_k\rangle\\\leq&(1-\alpha_k\tau)\|u_k-x^*\|^2-(1-\alpha_k\tau)\|U(u_k)-u_k\|^2\\&+2\alpha_k\|\sigma f_1(x_k)-\mu Fx^*\|\cdot\|x_{k+1}-x^*\|\\\leq&(1-\alpha_k\tau)\|u_k-x^*\|^2-(1-\alpha_k\tau)\|U(u_k)-u_k\|^2\\&+2\alpha_k(\frac{1}{4\beta}\|\sigma f_1(x_k)-\mu Fx^*\|^2+\beta\|x_{k+1}-x^*\|^2).\end{aligned}$$

Hence, we can obtain

$$\begin{aligned}\|x_{k+1}-x^*\|^2\leq&\frac{1-\alpha_k\tau}{1-2\alpha_k\beta}\|u_k-x^*\|^2-\frac{1-\alpha_k\tau}{1-2\alpha_k\beta}\|U(u_k)-u_k\|^2\\&+\frac{\alpha_k}{2\beta(1-2\alpha_k\beta)}\|\sigma f_1(x_k)-\mu Fx^*\|^2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|y_{k+1}-y^*\|^2\leq&\frac{1-\alpha_k\tau}{1-2\alpha_k\beta}\|v_k-y^*\|^2-\frac{1-\alpha_k\tau}{1-2\alpha_k\beta}\|T(v_k)-v_k\|^2\\&+\frac{\alpha_k}{2\beta(1-2\alpha_k\beta)}\|\sigma f_2(y_k)-\mu Fy^*\|^2.\end{aligned}$$

By adding up the last two inequalities and by using (3.5), it follows from $\tau > 2\beta$ that

$$\begin{aligned}
s_{k+1} \leq & \frac{1-\alpha_k\tau}{1-2\alpha_k\beta}(\|u_k - x^*\|^2 + \|v_k - y^*\|^2) \\
& - \frac{1-\alpha_k\tau}{1-2\alpha_k\beta}(\|U(u_k) - u_k\|^2 + \|T(v_k) - v_k\|^2) \\
& + \frac{\alpha_k}{2\beta(1-2\alpha_k\beta)}(\|\sigma f_1(x_k) - \mu F x^*\|^2 + \|\sigma f_2(y_k) - \mu F y^*\|^2) \\
\leq & \frac{1-\alpha_k\tau}{1-2\alpha_k\beta} s_k - \frac{1-\alpha_k\tau}{1-2\alpha_k\beta}\{\gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 \\
& + \|B^*(Ax_k - By_k)\|^2)] + (\|U(u_k) - u_k\|^2 + \|T(v_k) - v_k\|^2)\} \\
& + \frac{\alpha_k}{2\beta(1-2\alpha_k\beta)}(\|\sigma f_1(x_k) - \mu F x^*\|^2 + \|\sigma f_2(y_k) - \mu F y^*\|^2) \\
\leq & s_k + \frac{\alpha_k}{2\beta(1-2\alpha_k\beta)}(\|\sigma f_1(x_k) - \mu F x^*\|^2 + \|\sigma f_2(y_k) - \mu F y^*\|^2) \\
& - \frac{1-\alpha_k\tau}{1-2\alpha_k\beta}\{\gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)] \\
& + (\|U(u_k) - u_k\|^2 + \|T(v_k) - v_k\|^2)\}.
\end{aligned} \tag{3.9}$$

Now, by setting

$$\begin{aligned}
\mu_k = & \frac{\alpha_k}{2\beta(1-2\alpha_k\beta)}(\|\sigma f_1(x_k) - \mu F x^*\|^2 + \|\sigma f_2(y_k) - \mu F y^*\|^2), \\
\eta_k = & \frac{1-\alpha_k\tau}{1-2\alpha_k\beta}\{\gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)] \\
& + (\|U(u_k) - u_k\|^2 + \|T(v_k) - v_k\|^2)\},
\end{aligned}$$

Eq. (3.9) can be rewritten as the following form:

$$s_{k+1} \leq s_k - \eta_k + \mu_k, \quad k \geq 0.$$

By the assumption on α_k, we get $\sum_{k=0}^{\infty} \lambda_k = \infty$ and $\lim_{k\to\infty} \mu_k = 0$ which thanks to the boundedness of $\{x_k\}$ and $\{y_k\}$.

To use Lemma 2.9, we need to prove that, for any subsequence $\{k_l\} \subset \{k\}$, $\lim_{l\to\infty} \eta_{k_l} = 0$ implies

$$\limsup_{l\to\infty} \delta_{k_l} \leq 0. \tag{3.10}$$

It follows from $\lim_{k\to\infty} \eta_{k_l} = 0$ that

$$\lim_{l\to\infty} \gamma_{k_l}[2\|Ax_{k_l} - By_{k_l}\|^2 - \gamma_{k_l}(\|A^*(Ax_{k_l} - By_{k_l})\|^2 + \|B^*(Ax_{k_l} - By_{k_l})\|^2)] = 0,$$

and

$$\lim_{l\to\infty} \|u_{k_l} - U(u_{k_l})\| = \lim_{l\to\infty} \|v_{k_l} - T(v_{k_l})\| = 0. \tag{3.11}$$

From the assumption on γ_k, we can obtain

$$\lim_{l\to\infty} \|Ax_{k_l} - By_{k_l}\| = 0.$$

So, we have

$$\lim_{l\to\infty} \|u_{k_l} - x_{k_l}\| = \lim_{l\to\infty} \gamma_{k_l}\|A^*(Ax_{k_l} - By_{k_l})\| = 0, \tag{3.12}$$

and

$$\lim_{l\to\infty} \|v_{k_l} - y_{k_l}\| = \lim_{l\to\infty} \gamma_{k_l}\|B^*(Ax_{k_l} - By_{k_l})\| = 0. \tag{3.13}$$

By taking $(\tilde{x}, \tilde{y}) \in \omega_w(x_{k_l}, y_{k_l})$, from (3.12) and (3.13) we have $(\tilde{x}, \tilde{y}) \in \omega_w(u_{k_l}, v_{k_l})$. Combined with the demiclosednesses of $U - I$ and $T - I$ at 0, (3.11) yields $U\tilde{x} = \tilde{x}$ and $T\tilde{y} = \tilde{y}$. So $\tilde{x} \in F(U)$ and $\tilde{y} \in F(T)$. On the other hand, $A\tilde{x} - B\tilde{y} \in \omega_w(Ax_{k_l} - By_{k_l})$ and weakly lower semicontinuity of the norm imply

$$\|A\tilde{x} - B\tilde{y}\| \leq \liminf_{l\to\infty} \|Ax_{k_l} - By_{k_l}\| = 0,$$

hence $(\tilde{x}, \tilde{y}) \in \Gamma$. So $\omega_w(x_{k_l}, y_{k_l}) \subset \Gamma$. Since

$$\begin{aligned}\lim_{k\to\infty} \{&\alpha_k(\|\sigma f_1(x_k) - \mu F x^*\|^2 + \|\sigma f_2(y_k) - \mu F y^*\|^2) \\ &+ 2\mu\alpha_k\kappa(\|u_k - x^*\|\|\sigma f_1(x_k) - \mu F x^*\| + \|v_k - y^*\|\|\sigma f_2(y_k) - \mu F y^*\|)\} = 0,\end{aligned}$$

to get (3.10), we only need to verify

$$\limsup_{l\to\infty}(\langle\sigma f_1(x^*) - \mu F x^*, U(u_{k_l}) - x^*\rangle + \langle\sigma f_2(y^*) - \mu F y^*, T(v_{k_l}) - y^*\rangle) \leq 0.$$

Indeed, from (3.11), (3.12) and (3.13) we have

$$\begin{aligned}&\limsup_{l\to\infty} \big(\langle\sigma f_1(x^*) - \mu F x^*, U(u_{k_l}) - x^*\rangle + \langle\sigma f_2(y^*) - \mu F y^*, T(v_{k_l}) - y^*\rangle\big) \\ &\quad= \limsup_{l\to\infty} \big(\langle\sigma f_1(x^*) - \mu F x^*, u_{k_l} - x^*\rangle + \langle\sigma f_2(y^*) - \mu F y^*, v_{k_l} - y^*\rangle\big) \\ &\quad= \limsup_{l\to\infty} \big(\langle\sigma f_1(x^*) - \mu F x^*, x_{k_l} - x^*\rangle + \langle f_2(\sigma y^*) - \mu F y^*, y_{k_l} - y^*\rangle\big) \\ &\quad= -\liminf_{l\to\infty} \big(\langle(\mu F - \sigma f_1)x^*, x_{k_l} - x^*\rangle + \langle(\mu F - \sigma f_2)y^*, y_{k_l} - y^*\rangle\big).\end{aligned} \tag{3.14}$$

We can take subsequence $\{(x_{k_{l_j}}, y_{k_{l_j}})\}$ of $\{(x_{k_l}, y_{k_l})\}$ such that $(x_{k_{l_j}}, y_{k_{l_j}}) \rightharpoonup (\tilde{x}, \tilde{y})$ as $j \to \infty$ and

$$\begin{aligned}&-\liminf_{l\to\infty} \big(\langle(\mu F - \sigma f_1)x^*, x_{k_l} - x^*\rangle + \langle(\mu F - \lambda f_2)y^*, y_{k_l} - y^*\rangle\big) \\ &\quad= -\lim_{j\to\infty} \big(\langle(\mu F - \sigma f_1)x^*, x_{k_{l_j}} - x^*\rangle + \langle(\mu F - \sigma f_2)y^*, y_{k_{l_j}} - y^*\rangle\big) \\ &\quad= -\big(\langle(\mu F - \sigma f_1)x^*, \tilde{x} - x^*\rangle + \langle(\mu F - \sigma f_2)y^*, \tilde{y} - y^*\rangle\big).\end{aligned} \tag{3.15}$$

Since $\omega_w(x_{k_l}, y_{k_l}) \subset \Gamma$ and (x^*, y^*) is the solution of the variational inequality problem (3.2), from (3.14) and (3.15) we obtain

$$\limsup_{l\to\infty}(\langle\sigma f_1(x^*) - \mu F x^*, U(u_{k_l}) - x^*\rangle + \langle\sigma f_2(y^*) - \mu F y^*, T(v_{k_l}) - y^*\rangle) \leq 0.$$

From Lemma 2.9, it follows

$$\lim_{k\to\infty} (\|x_k - x^*\|^2 + \|y_k - y^*\|^2) = 0,$$

which implies that $x_k \to x^*$ and $y_k \to y^*$. □

Remark 3.4. Our main result generalized the main results of Moudafi [18, 19] from weak convergence to strong convergence which is more desirable. Our algorithm does not require the operator norms.

By taking $U = P_C$ and $T = P_Q$, we have the following viscosity iterative algorithm without prior knowledge of operators norms for finding common solution of variational inequality for Lipschitzian and strongly monotone operators and the SEP (1.4).

Algorithmm 3.5. *Let $x_0 \in H_1$, $y_0 \in H_2$ be arbitrary. Assume that the k-th iterate $x_k \in H_1$, $y_k \in H_2$ has been constructed, then we calculate the (k + 1)-th iterate (x_{k+1}, y_{k+1}) via the formula:*

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \sigma\alpha_k f_1(x_k) + (I - \mu\alpha_k F)P_C(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \sigma\alpha_k f_2(x_k) + (I - \mu\alpha_k F)P_Q(v_k), \end{cases}$$

where γ_k is chosen by (3.1).

At last, noting that for a maximal monotone operator $M : H_1 \to 2^{H_1}$, its associated resolvent mapping, $J_\mu^M(x) := (I + \mu M)^{-1}(x)$, is firmly quasi-nonexpansive and $0 \in M(x)$ if and only if $x = J_\mu^M(x)$. In other words, zeroes of M are exactly fixed-points of its resolvent mapping. Let $S : H_2 \to 2^{H_2}$ be another maximal monotone operator, the problem under consideration is nothing but

$$\text{find} \quad x^* \in M^{-1}(0), y^* \in S^{-1}(0), \quad \text{such that} \quad Ax^* = By^*. \tag{3.16}$$

For finding common solution of variational inequality for Lipschitzian and strongly monotone operators and the problem (3.16), by taking $U = J_\mu^M$, $T = J_\nu^S$, Algorithm 3.1 takes the following equivalent form

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \sigma\alpha_k f_1(x_k) + (I - \mu\alpha_k F)J_\mu^M(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \sigma\alpha_k f_2(x_k) + (I - \mu\alpha_k F)J_\nu^S(v_k), \end{cases}$$

where γ_k is chosen by (3.1).

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