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On the approximate solution of nonlinear time-fractional KdV equation via modified homotopy analysis Laplace transform method

Chong Li[a], Amit Kumar[b], Sunil Kumar[b], Xiao-Jun Yang[c,*]

[a]*School of Mines, Key Laboratory of Deep Coal Resource Mining of Ministry of Education, China University of Mining and Technology, Xuzhou 221116, China.*

[b]*Department of Mathematics, National Institute of Technology, Jamshedpur 831014, India.*

[c]*School of Mechanics and Civil Engineering, China University of Mining and Technology, Xuzhou 221116, China.*

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Abstract

The approximate solution of the time-fractional KdV equation (KdV) by using modified homotopy analysis Laplace transform method, which is a combined form of the Laplace transform and homotopy analysis methods, is investigated for the first time in this article. Comparison of series solutions between under a rapid convergence and the optimal values of convergence parameter $\hbar$ is made. The results through the L_2 and L_∞ error norms are also analyzed. The validity, flexibility, and accuracy of the proposed method is conformed through the numerical computations as well as graphical presentations of the results.

Keywords: Time-fractional KdV, homotopy analysis Laplace transform method, homotopy polynomial, approximate solution, optimal value.
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1. Introduction

In the past about forty years, the theory and applications of the fractional-order partial differential equations (FPDEs) have become an increasing interest for the researchers to generalize the integer-order

*Corresponding author
Email address: dyangxiaojun@163.com (Xiao-Jun Yang)

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differential equations [3, 10]. The FPDEs were adopted to model the thermal science, fluid dynamics, electrical network, chemical physics, optics and so on (see [4, 5, 13]). Conventionally various technologies, e.g., the Adomian decomposition method (ADM) [6], variation iteration method (VIM) [18], homotopy perturbation method (HPM) [1, 15], homotopy decomposition method (HDM) [2], homotopy analysis method (HAM) [11, 12, 17], residual power series method (RPSM) [8], traveling wave method (TWM) [16], and homotopy analysis Laplace transform method (MHALTM) [7] were used for the solutions of such type of the FPDEs.

In the present paper, we apply the MHALTM to study the approximate solution of time-fractional KdV equation [14]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + au\frac{\partial u}{\partial x} + b\frac{\partial^3 u}{\partial x^3} = 0,\ t > 0, 0 < \alpha \leq 1,$$

subject to the initial condition:

$$u(x,0) = f(x),$$

where a and b are two constants, and the fractional derivative is considered in sense of Caputo type [3–5, 10, 13]. The above model plays an important role in modeling of the complicated physical phenomena, such as the particle vibrations in lattices, thermal science and current flow in electrical flow. Recently, it was studied by the HPM [14]. But as far the possible information of the authors, this technology is for the first time attempted for finding the approximate solution of the model by using the MHALTM.

The rest of the present paper is organized as follows. The basic idea of the MHALTM is presented in Section 2. A new application to the KdV is discussed in Section 3. The numerical simulations are given in Section 4. In Section 5, the optimal values of $\hbar$ in the MHALTM are given. Finally, the conclusions are drawn in Section 6.

2. Analysis of the method

We consider the following general FPDE of Caputo type (see [3–5, 10, 13]):

$$D_t^\alpha u(x,t) + R[x]u(x,t) + N[x]u(x,t) = g(x,t), \quad t > 0, \quad x \in \mathbb{R}, \quad 0 < \alpha \leq 1, \tag{2.1}$$

where $R[x]$ is a general linear operator in x, $N[x]$ is a general nonlinear operator in x, $g(x,t)$ is a continuous function, $u(x,t)$ is an unknown function, and the fractional derivative is considered in sense of Caputo type [3–5, 10, 13]. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way.

Due to the methodology discussed in [7] and by applying to (2.1) the m-th order deformation equation can be written in the form:

$$\begin{aligned} u_m(x,t) = (\chi_m + \hbar)u_{m-1} - \hbar(1-\chi_m)\sum_{i=0}^{j-1} t^i u^{(i-1)}(0) + \hbar L^{-1}\Big(\frac{1}{s^\alpha} L\Big(R_{m-1}[t]u_{m-1}(t) \\ + \sum_{k=0}^{m-1} P_k(u_0, u_1, \ldots, u_m) - g(x,t)\Big)\Big), \end{aligned} \tag{2.2}$$

where P_k are the homotopy polynomials, and the Laplace transform of the Caputo fractional derivative, $D_t^\alpha u(x,t)$, with respect to the variable t is given by (see [3, 10]):

$$L[D_t^\alpha u(x,t)] = s^\alpha L[u(x,t)] - s^{(\alpha-1)}u(x,0), \quad 0 < \alpha \leq 1.$$

For the sake of convenience, the expression in nonlinear operator can be written by using the HANLTM, i.e., the nonlinear term $N[x,t]u(x,t)$ is expanded in terms of homotopy polynomials as:

$$N[u(x,t)] = N\Big(\sum_{k=0}^{m-1} u_m(x,t)\Big) = \sum_{m=0}^{\infty} P_m u^m.$$

The novelty of our proposed algorithm is that a new correction functional (2.2) is constructed and expanding the nonlinear term as a series of homotopy polynomials in the equation (2.2). Now from the equation (2.2), we calculate the various $u_m(x,t)$ for $m \geq 1$ and the series solution of equation (2.1) is thus entirely determined by:

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t).$$

3. Solving the time-fractional KdV

We now consider time-fractional KdV as follows [14]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + au\frac{\partial u}{\partial x} + b\frac{\partial^3 u}{\partial x^3} = 0, \ t > 0, 0 < \alpha \leq 1, \tag{3.1}$$

subject to the initial condition:

$$u(x,0) = 12\frac{k^2 b}{a} \operatorname{sech}^2(kx).$$

For $\alpha = 1$, the exact solution of (3.1) is given by [14],

$$u(x,t) = 12\frac{k^2 b}{a} \operatorname{sech}^2(k(x - 4k^2 bt)).$$

Adopting the Laplace transform on both sides in (3.1) and after using the differentiation property of Laplace transform for fractional derivative, we have

$$L[u(x,t)] - s^{\alpha-1}u(x,0) + L[auu_x + bu_{xxx}] = 0.$$

We choose the linear operator as

$$\mathcal{L}[u(x,t;q)] = L[u(x,t;q)],$$

with property $\mathcal{L}[c] = 0$, where c is a constant.

Now we define a nonlinear operator as

$$N[\phi(x,t;q)] = L[\phi(x,t;q)] - \frac{1}{s}\Big(12\frac{k^2 b}{a} \operatorname{sech}^2(kx)\Big) + s^{-\alpha}L[a\phi(x,t;q)\phi_x(x,t;q) + b\phi_{xxx}(x,t;q).$$

With assumption $H(x,t) = 1$ and with the help of the above definitions, we construct the so-called zeroth-order deformation equation

$$(1-q)\mathcal{L}[\phi(x,t;q) - w_0(x,t)] = q\hbar N[\phi_j(x,t;q)].$$

Obviously, when $q = 0$ and $q = 1$, we have that

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t).$$

Thus, we obtain the m-th order deformation equation

$$L[u_m(x,t) - \xi_m u_{m-1}(x,t)] = \hbar R_m(\vec{u}_{m-1}, x, t). \tag{3.2}$$

Operating the inverse Laplace transform on both sides in (3.2) gives

$$u_m(x,t) = \xi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1}, x, t)],$$

where

$$R_m(\vec{u}_{m-1}, x, t) = L[u_{m-1}] - \frac{1-\xi_m}{s}\Big(12\frac{k^2 b}{a} \operatorname{sech}^2(kx)\Big) + s^{-\alpha}L[aP_m^1 + b(u_{m-1})_{xxx}], \ m \geq 1.$$

Now the solution of m-th order deformation equation (3.2) is

$$u_m(x,t) = (\xi_m + \hbar)u_{m-1}(x,t) - \hbar(1-\xi_m)\left(\left(12\frac{k^2 b}{a}\ \mathrm{sech}^2(kx)\right)\right) - \hbar L^{-1}[s^{-\alpha} L(aP_m^1 + b(u_{m-1})_{xxx})], \quad (3.3)$$

where P_m^1 is the homotopy polynomial given by:

$$P_m^1 = \frac{1}{\Gamma(m+1)}\left[\frac{\partial^m}{\partial q^m} Nu[(qu(x,t;q))(qu(x,t;q))_x u]\right]_{q=0},$$

where $u(x,t;q)$ is given by

$$u(x,t;q) = u_0 + qu_1 + q^2 u_2 + q^3 u_3 + \cdots.$$

Finally, we have

$$u(x,t) = u_0(x,t) + \sum_{m=0}^{\infty} u_m(x,t).$$

In view of the initial approximation, $u_0(x,t) = u(x,0) = -\sqrt{c}\ \tanh(\sqrt{c}\ x)$, and the iterative scheme (3.3), we obtain the various iterates

$$u_1(x,t) = \frac{-96\hbar\ b^2\ k^5\ \mathrm{sech}^2(kx)\ \tanh^2(kx)\ t^\alpha}{a\ \Gamma(\alpha+1)},$$

$$u_2(x,t) = \frac{-96\hbar(1+\hbar)\ b^2\ k^5\ \mathrm{sech}^2(k\ x)\ \tanh^2(kx)t^{2\alpha}}{a\ \Gamma(\alpha+1)} - \frac{768\ \hbar^2\ b^3\ k^8\ \mathrm{sech}^4(kx)}{a\ \Gamma(2\alpha+1)} + \frac{384\ \hbar^2\ b^3\ k^8\ t^{2\alpha}\ \cosh(2kx)\ \mathrm{sech}^4\ (kx)}{a\ \Gamma(2\alpha+1)},$$

and so on.

Similarly, the rest terms of $u_m(x,t)$ for $m \geq 3$ can be completely obtained.

Hence, the solution of equation (3.1) is given as

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t). \quad (3.4)$$

4. Numerical simulations

In this section, the results obtained by the proposed method are being discussed one by one. The different graphical representations with tabulated data are taken into account for the verification of the MHALTM.

Figs. 1, 2, 3, and 4 show the comparison between the 4th order approximate solution obtained by the MHALTM and the exact solution in the different values of α. Next, in Fig. 5 we present the absolute error curve $E_4(x,t) = |u_h(x,t) - u(x,t)|$, where $u_h(x,t)$ is the exact solution.

The analytical behavior of the approximate solution of (3.1) obtained by the MHALTM for the different fractional Brownian motions $\alpha = 0.7, \alpha = 0.8$ and $\alpha = 0.9$, and standard motions, i.e., $\alpha = 1$ is shown in Fig. 6. It is seen from Fig. 6 that the solution obtained by the MHALTM increases very rapidly with the increases in t at the value of $x = 1$. Fig. 7 demonstrates the $\hbar$- curve obtained by the MHALTM. It is obvious from Fig. 7, for the convergence of series solution (3.4) we can choose any value of $\hbar$, where $\hbar \in (\hbar_1, \hbar_2)$, $\hbar_1 \approx -1.3$, and $\hbar_2 \approx -0.3$. In particular, if we take $\hbar = -1$ the rate of convergence is optimum.

The comparative results among the approximate and exact solutions for the time-fractional KdV and the absolute error are presented in Table 1 and Table 2, respectively. The tabulated data shows that our approximate solution is very nearer to the exact solution.

Table 1: The absolute error in the solution of time-fractional KdV using MHALTM at different points of x and t for $\alpha = 1$.

(x,t)	Exact Solution	Approximation Solution	Absolute Error
(0.1,0.1)	0.00498777	0.00498777	2.03442×10^{-16}
(0.1,0.2)	0.00498801	0.00498801	3.26508×10^{-15}
(0.1,0.3)	0.00498826	0.00498826	1.6523×10^{-14}
(0.2,0.1)	0.00495082	0.00495082	1.89671×10^{-16}
(0.2,0.2)	0.00495131	0.00495131	3.05813×10^{-15}
(0.2,0.3)	0.0049518	0.0049518	1.54813×10^{-14}
(0.3,0.1)	0.00488989	0.00488989	1.54813×10^{-16}
(0.3,0.2)	0.00489062	0.00489062	2.72966×10^{-15}
(0.3,0.3)	0.00489134	0.00489134	1.38293×10^{-14}

Table 2: The L_2 and L_∞ error norms for the fractional KdV using MHALTM at various points x for $\alpha = 1$.

x	L_2 error norm	L_∞ error norm
0.1	1.43210×10^{-15}	2.03442×10^{-16}
0.2	1.65243×10^{-15}	1.89671×10^{-16}
0.3	1.98623×10^{-15}	1.54813×10^{-16}

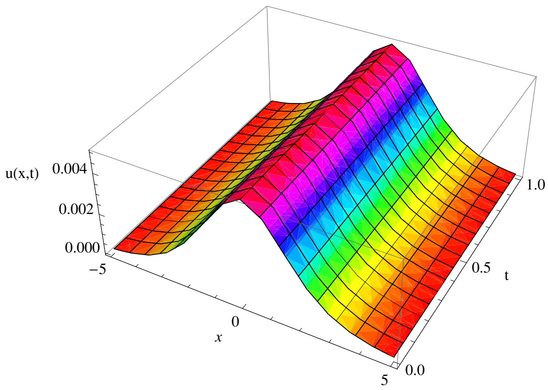

Figure 1: The 4th order approximate solution of the KdV equation: (a) $u(x, t)$ when $\alpha = 1$.

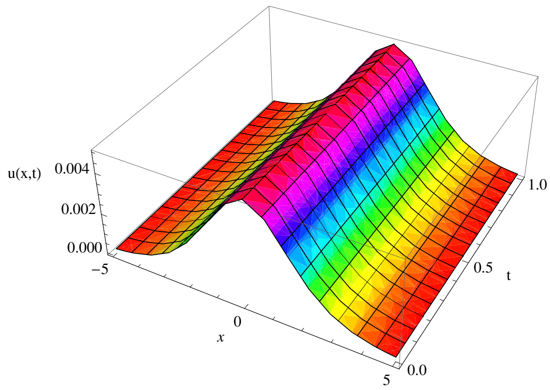

Figure 2: The 4th order approximate solution of the KdV equation: (b) $u(x, t)$ when $\alpha = 0.75$.

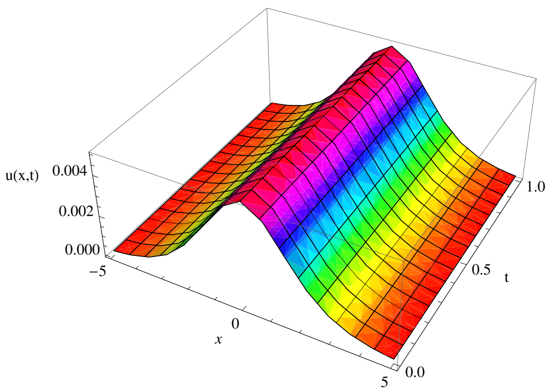

Figure 3: The 4th order approximate solution of the KdV equation: (c) $u(x, t)$ when $\alpha = 0.5$.

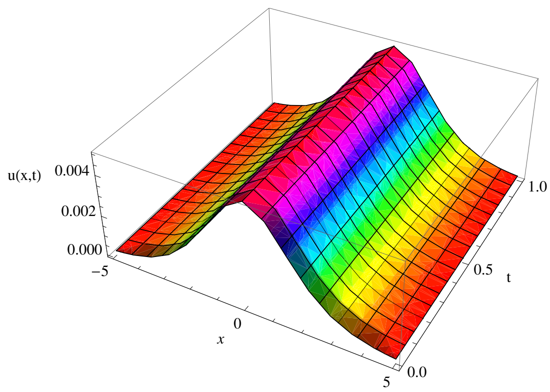

Figure 4: The 4th order approximate solution of the KdV equation: (d) $u(x, t)$ when $\alpha = 0.25$.

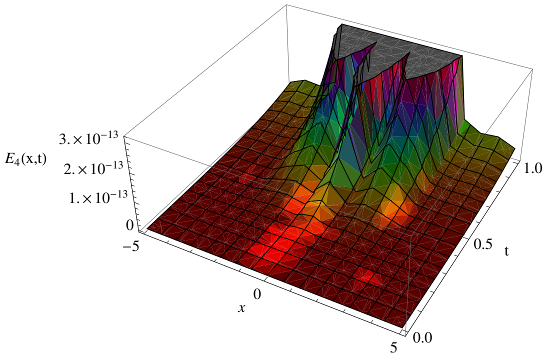

Figure 5: Plot of absolute error $E_4(x,t) = |u_h(x,t) - u(x,t)|$.

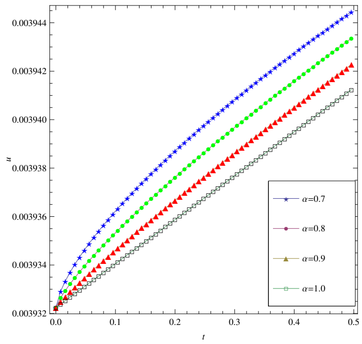

Figure 6: Plot of $u(x,t)$ vs. time t at $x = 1$ and different values of α.

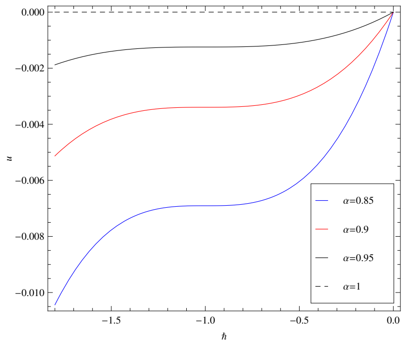

Figure 7: Plot of $\hbar$- curve for different values of α.

5. Optimal values of $\hbar$ in MHALTM

At the m-th order of the approximation, the exact square residual error is defined by:

$$\Delta_m^u = \int_0^1 \int_0^1 \left(N \left[\sum_{i=0}^{m} u_i(x,t) \right] \right)^2 dx\, dt,$$

where $N[u(x,t)] = \frac{\partial^\beta u}{\partial t^\alpha} + au\frac{\partial u}{\partial x} + b\frac{\partial^3 u}{\partial x^3}$.

Even if the order of the approximation is not very high, the exact square residual error needs too much CPU time to calculate. In order to overcome this disadvantage, we introduced here the so-called averaged residual error defined by [9]:

$$E_m^u = \frac{1}{k_1^2} \sum_{j=1}^{k_1} \sum_{l=1}^{k_1} \left(N \left[\sum_{i=0}^{m} u_i(j\Delta x, l\Delta t) \right] \right)^2,$$

where $\Delta x = \frac{1}{40k_1}, \Delta t = \frac{1}{40k_2}, k_1 = 5$, and $k_2 = 5$. The optimal value of $\hbar$ can be obtained by means of minimizing the so-called averaged residual error.

Thus, the nonlinear algebraic equations are $\frac{\partial E_m^u}{\partial \hbar} = 0$.

Table 3 shows the selection of the values of $\hbar$ as well as the averaged residual error for the different orders of the approximations. Here we see that there is a great freedom to choose the auxiliary parameters $\hbar$.

Table 3: Optimal value of $\hbar$.

Order of approx.	Optimal value of $\hbar$ for $\alpha = 1$	Optimal value of $\hbar$ for $\alpha = 0.9$	value of E_m^u for $\alpha = 1$	value of E_m^u for $\alpha = 0.9$
1	-0.83090	-0.721927	1.45691×10^{-4}	3.56713×10^{-4}
2	-0.82271	-0.75123	2.36789×10^{-5}	1.23478×10^{-4}
3	-0.76235	-0.65467	4.56732×10^{-6}	2.87612×10^{-5}

6. Conclusion

In the present work, an effective and innovative method called the MHALTM was adopted for finding approximate solution of the time-fractional KdV. The approximate solution obtained by the present method was verified through the different graphical representations as well as tabulated data. We found that there exists a very good agreement between our solution and the exact solution. From the above discussion we concluded that the present method is reliable. The more realistic series solutions converge very rapidly in the physical problems.

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