APPLICATION OF THE DOUBLE LAPLACE ADOMIAN DECOMPOSITION METHOD FOR SOLVING LINEAR SINGULAR ONE DIMENSIONAL THERMO-ELASTICITY COUPLED SYSTEM

HASSAN ELTAYEB, ADEM KILIÇMAN AND SAID MESLOUB

Abstract. In the present work, the Adomain decomposition and double Laplace transform methods are combined to solve linear singular one dimensional hyperbolic equation and linear singular one dimensional thermo-elasticity coupled system. Also we address the convergence of double Laplace transform decomposition method. Moreover, some examples are given to establish our method.

1. Introduction
In general, it is reported in the literature that finding the exact solutions for partial differential equations are a complicated task. Therefore, some recent approximate methods to over come this task have been improved, such as homotopy perturbation method [1, 2], combined Laplace transforms and decomposition method [3] to solve first order differential equation, An auxiliary parameter method using A domain polynomials and Laplace transformation have been powerfully combined [5] to study the nonlinear differential equation. The one-dimensional nonlinear hyperbolic equation with Bessel operator is one of the fundamental nonlinear wave equations having many applications in science. The energy-integral method is used to handle nonlinear singular one dimensional hyperbolic equation [4]. In [6] authors studied the initial boundary value problem for a nonlinear singular system of thermo-elasticity by using a functional analysis approach and an iteration method. The aim of the present paper is to investigate the application of the modified double Laplace transform decomposition method for solving the linear singular one dimensional hyperbolic equation and linear singular one dimensional thermo-elasticity coupled system. The convergence of Adomian’s method has been studied by several authors [11, 12, 13, 14, 16]. Now, we recall the following definitions which are given by [10, 7, 8, 9]. The double Laplace transform is defined as

\[ L_x L_t [f(x, s)] = F(q, s) = \int_0^\infty e^{-qx} \int_0^\infty e^{-st} f(x, t) dt dx, \]  

(1.1)
where \( x, t > 0 \) and \( q, s \) are complex values, and further double Laplace transform of the first order partial derivatives is given by

\[
L_x L_t \left[ \frac{\partial v(x,t)}{\partial x} \right] = qV(q,s) - V(0,s).
\] (1.2)

Similarly the double Laplace transform for second partial derivative with respect to \( x \) and \( t \) are defined as follows

\[
L_x L_t \left[ \frac{\partial^2 u(x,t)}{\partial^2 x} \right] = q^2 U(q,s) - qU(q,0) - \frac{\partial U(q,0)}{\partial t},
\] (1.3)

\[
L_x L_t \left[ \frac{\partial^2 u(x,t)}{\partial^2 t} \right] = s^2 U(q,s) - sU(q,0) - \frac{\partial U(q,0)}{\partial t}.
\] (1.4)

First of all we need the following lemma for future use in this paper.

**Lemma 1.** Double Laplace transform of the non constant coefficient second order partial derivative \( x^n \frac{\partial^2 u}{\partial t^2} \) and the function \( x^n f(x,t) \) given by

\[
(-1)^n \frac{d^n}{dq^n} \left[ L_x L_t \left( \frac{\partial^2 u}{\partial t^2} \right) \right] = L_x L_t \left( x^n \frac{\partial^2 u}{\partial t^2} \right)
\] (1.4)

and

\[
L_x L_t (x^n f(x,t)) = (-1)^n \frac{d^n}{dq^n} \left[ L_x L_t (f(x,t)) \right] = (-1)^n \frac{d^n F(q,s)}{dq^n}.
\] (1.5)

One can prove this lemma by using the definition of double Laplace transform in Eq.(1.1), Eq.(1.3) and Eq.(1.2).

2. SINGULAR ONE DIMENSIONAL HYPERBOLIC EQUATION

In this part of the paper we discuss how to obtain the solution of the singular one dimensional hyperbolic equation:

\[
\frac{\partial^2 u}{\partial t^2} = \frac{1}{x} \left( x \frac{\partial u}{\partial x} \right)_x + f(x,t),
\] (2.1)

subject to

\[
u(x, 0) = f_1(x), \quad \frac{\partial u(x, 0)}{\partial t} = f_2(x), \tag{2.2}
\]

where \( \frac{1}{x} \left( x \frac{\partial u}{\partial x} \right)_x \) is called Bessel operator and the functions \( f(x,t), f_1(x) \) and \( f_2(x) \) are known. In the following theorem we apply modified double Laplace decomposition methods.

**Theorem 1.** The solution of the singular one dimensional hyperbolic equation given in Eq. (2.1) exists and is given by

\[
u(x, t) = f_1(x) + tf_2(x) - L_q^{-1} L_s^{-1} \left[ \frac{1}{s^2} L_x L_t \left[ \int_0^q x f(x,t) \, dq \right] \right] - L_q^{-1} L_s^{-1} \left[ \frac{1}{s^2} L_x L_t \left[ \int_0^q \left( x \frac{\partial}{\partial x} \sum_{n=0}^{\infty} u_n \right)_x \, dq \right] \right]
\]

(2.3)

where

\[
u(x, t) = \sum_{n=0}^{\infty} u_n(x, t),
\]
HYPERBOLIC EQUATION WITH BESSEL OPERATOR

and \( L_x, L_t \) double Laplace transform with respect to \( x, t \) and \( L_q^{-1} L_s^{-1} \) double inverse Laplace transform with respect to \( q, s \) further \( A_n \) represents the linear terms. Here we provided double inverse Laplace transform with respect to \( p, s \) exist for each terms in the right hand side of Eq. (2.3).

**Proof.** The method consists of first applying the double Laplace transform to both sides of equations in Eq.(2.1) and then by using initial conditions Eq.(2.2), and the differentiation property of double Laplace transform together with lemma 1, we obtain:

\[
\frac{dU(q,s)}{dp} = \frac{1}{s} \frac{dF_1(q)}{dq} + \frac{1}{s^2} \frac{dF_2(q)}{dq} - \frac{1}{s^2} L_x L_t \left[ \left( x \frac{\partial u}{\partial x} \right)_x \right] - \frac{1}{s^2} L_x L_t \left[ x f(x,t) \right] \tag{2.4}
\]

applying the integral for both sides of Eq.(2.4) from 0 to \( q \), we get

\[
U(q,s) = \frac{F_1(q)}{s} + \frac{1}{s^2} \int_0^q L_x L_t \left[ x f(x,t) \right] dq - \frac{1}{s^2} \int_0^q L_x L_t \left[ \left( x \frac{\partial u}{\partial x} \right)_x \right] dq. \tag{2.5}
\]

The double Laplace Adomain decomposition methods (DLADM) defines the solution of linear singular one dimensional hyperbolic equation as \( u(x,t) \) by the infinite series

\[
u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{2.6}\]

By applying double inverse Laplace transform for Eq.(2.5) we obtain

\[
u(x,t) = f_1(x) + tf_2(x) - L_q^{-1} L_s^{-1} \left[ \frac{1}{s^2} \int_0^q L_x L_t \left[ x f(x,t) \right] dq \right]
- L_q^{-1} L_s^{-1} \left[ \frac{1}{s^2} \int_0^q L_x L_t \left[ \left( x \frac{\partial u}{\partial x} \right)_x \right] dq \right], \tag{2.7}\]

then the general decomposition formula for the (2.7) is given by

\[
u(x,t) = f_1(x) + tf_2(x) - L_q^{-1} L_s^{-1} \left[ \frac{1}{s^2} \int_0^q L_x L_t \left[ \int_0^q x f(x,t) dq \right] \right]
- L_q^{-1} L_s^{-1} \left[ \frac{1}{s^2} \int_0^q L_x L_t \left[ \int_0^q \left( x \frac{\partial u}{\partial x} \sum_{n=0}^{\infty} u_n \right)_x dq \right] \right].
\]

In particular, we have

\[
u_0(x,t) = f_1(x) + tf_2(x) - L_q^{-1} L_s^{-1} \left[ \frac{1}{s^2} \int_0^q L_x L_t \left[ \int_0^q (x f(x,t)) dq \right] \right]
\]

and

\[
u_{n+1}(x,t) = -L_q^{-1} L_s^{-1} \left[ \frac{1}{s^2} \int_0^q L_x L_t \left[ \int_0^q \left( x \frac{\partial u}{\partial x} \sum_{n=0}^{\infty} u_n \right)_x dq \right] \right], \tag{2.8}\]
where \( n = 0, 1, 2, \ldots \), and by calculating the terms \( u_0, u_1, u_2, \ldots \) we obtain the solution as follows

\[
 u(x, t) = u_0 + u_1 + u_2 + \ldots
\]

\( \square \)

In order to establish our method for solving the singular one dimensional hyperbolic equation, we consider the following example:

**Example 1.** Consider the following one dimensional hyperbolic equation

\[
\frac{\partial^2 u}{\partial t^2} - \frac{1}{x} \left( \frac{\partial u}{\partial x} \right)_x = -x^2 \sin t - 4 \sin t, \tag{2.9}
\]

subject to

\[
u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = x^2. \tag{2.10}\]

By taking the double and single Laplace transform for Eq. (2.9) and Eq. (2.10) respectively, we obtain

\[
\frac{dU(q, s)}{dq} = -\frac{6}{q^4 s^2} + \frac{6}{q^4 s^2 (s^2 + 1)} + \frac{4}{q^2 s^2 (s^2 + 1)} - \frac{1}{s^2} L_x L_t \left[ \left( \frac{\partial u}{\partial x} \right)_x \right], \tag{2.11}\]

the integral for both sides of Eq.(2.11) from 0 to \( q \), we have

\[
U(q, s) = \frac{2}{q^4 s^2} - \frac{2}{q^4 s^2 (s^2 + 1)} - \frac{4}{qs^2 (s^2 + 1)} - \frac{1}{s^2} \int_0^q L_x L_t \left[ \left( \frac{\partial u}{\partial x} \right)_x \right] dq. \tag{2.12}\]

On using double inverse Laplace transform, we have

\[
u(x, t) = x^2 \sin t + 4 \sin t - L_q^{-1} L_s^{-1} \left( \frac{1}{s^2} \int_0^q L_x L_t \left[ \left( \frac{\partial u}{\partial x} \right)_x \right] dq \right), \tag{2.13}\]

by using equation Eq.(2.3), we obtain

\[
\sum_{n=0}^\infty u_n(x, t) = x^2 \sin t + 4 \sin t - 4t - L_q^{-1} L_s^{-1} \left( \frac{1}{s^2} \int_0^q L_x L_t \left[ \left( \frac{\partial}{\partial x} \sum_{n=0}^\infty u_n \right)_x \right] dq \right), \tag{2.14}\]

The other components given by

\[
u_{n+1} = -L_q^{-1} L_s^{-1} \left( \frac{1}{s^2} \int_0^q L_x L_t \left[ \left( \frac{\partial}{\partial x} \sum_{n=0}^\infty u_n \right)_x \right] dq \right). \tag{2.15}\]

therefore

\[
u_1 = -L_q^{-1} L_s^{-1} \left( \frac{1}{s^2} \int_0^q L_x L_t \left[ \left( \frac{\partial}{\partial x} u_0 \right)_x \right] dq \right)
\]

\[
u_1 = 4t - 4 \sin t,
\]

and

\[
u_2 = 0.
\]

It is obvious that the rest coming terms all zeros, we have

\[
u(x, t) = u_0 + u_1 + \ldots
\]

Therefore, the exact solution is given by

\[
u(x, t) = x^2 \sin t.
\]
3. Linear singular one dimensional thermo-elasticity coupled system

In this section of the paper, we apply our technique to solve the linear singular one dimensional thermo-elasticity coupled system given below

\[
\frac{\partial^2 u}{\partial t^2} - \frac{1}{x^n} \left( x^n \frac{\partial u}{\partial x} \right)_x + x \frac{\partial v}{\partial x} = f(x, t), \quad x \in \Omega \\
\frac{\partial v}{\partial t} - \frac{1}{x^n} \left( x^n \frac{\partial v}{\partial x} \right)_x + x \frac{\partial^2 u}{\partial x \partial t} = g(x, t), \quad t > 0 \tag{3.1}
\]

subject to

\[
u(x, 0) = f_1(x), \quad \frac{\partial u(x, 0)}{\partial t} = f_2(x), \quad v(x, 0) = g_1(x) \tag{3.2}
\]

where \( \frac{1}{x^n} \left( x^n \frac{\partial u}{\partial x} \right)_x \) and \( \frac{1}{x^n} \left( x^n \frac{\partial v}{\partial x} \right)_x \) are called Bessel’s operators, \( f(x, t) \), \( g(x, t) \), \( f_1(x) \), \( f_2(x) \) and \( g_1(x) \) are known functions and \( n = 1, 2, 3, \ldots \). To obtain the solution of Linear singular one dimensional thermo-elasticity coupled system of Eq.(3.1) we apply our method as follows. On using the definition of partial derivatives of the double Laplace transform and single Laplace transform for Eq.(3.1) and Eq.(3.2) respectively and lemma 1, we get

\[
\frac{d^n U(q, s)}{d q^n} = \frac{d^n F_1(q)}{s d q^n} + \frac{d^n F_2(q)}{s^2 d q^n} + \frac{d^n F(q, s)}{s^3 d q^n} + \left( \frac{-1}{s} \right)^n L_x L_t \left[ \left( x^n \frac{\partial u}{\partial x} \right)_x - x^{n+1} \frac{\partial v}{\partial x} \right] \tag{3.3}
\]

and

\[
\frac{d^n V(q, s)}{d q^n} = \frac{d^n G_1(q)}{s d q^n} + \frac{d^n G(q, s)}{s^2 d q^n} + \left( \frac{-1}{s} \right)^n L_x L_t \left[ \left( x^n \frac{\partial v}{\partial x} \right)_x - x^{n+1} \frac{\partial^2 u}{\partial x \partial t} \right] \tag{3.4}
\]

where \( F(q, s) \), \( G(q, s) \), \( F_1(q) \), \( F_2(q) \) and \( G_1(q) \) are double and single Laplace transforms of \( f(x, t) \), \( g(x, t) \), \( f_1(x) \), \( f_2(x) \) and \( g_1(x) \) respectively, by integrating \( n \) time for both sides of Eq.(2.4) from 0 to \( q \) with respect to \( q \), we obtain

\[
U(q, s) = \int \cdots \int \left( \frac{d^n F_1(q)}{s d q^n} + \frac{d^n F_2(q)}{s^2 d q^n} + \frac{d^n F(q, s)}{s^3 d q^n} \right) d q \ldots d q + \left( \frac{-1}{s} \right)^n \int \cdots \int \left( L_x L_t \left[ \left( x^n \frac{\partial u}{\partial x} \right)_x - x^{n+1} \frac{\partial v}{\partial x} \right] \right) d q \ldots d q \tag{3.5}
\]

and

\[
V(q, s) = \int \cdots \int \left( \frac{d^n G_1(q)}{s d q^n} + \frac{d^n G(q, s)}{s^2 d q^n} \right) d q \ldots d q + \left( \frac{-1}{s} \right)^n \int \cdots \int \left( L_x L_t \left[ \left( x^n \frac{\partial v}{\partial x} \right)_x - x^{n+1} \frac{\partial^2 u}{\partial x \partial t} \right] \right) d q \ldots d q. \tag{3.6}
\]

The double Laplace a domain decomposition methods (DLADM) defines the solution of the system as \( u(x, t) \) and \( v(x, t) \) by the infinite series

\[
u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \quad v(x, t) = \sum_{n=0}^{\infty} v_n(x, t). \tag{3.7}
\]
By applying double inverse Laplace transform for Eq.(3.5) and Eq.(3.6) and use Eq.(3.7) we have

\[ u(x, t) = L_q^{-1} L_s^{-1} \left[ \int \int \int \left( \frac{d^n F_1(q)}{s dq^n} + \frac{d^n F_2(q)}{s^2 dq^n} + \frac{d^n F(q, s)}{s^2 dq^n} \right) dq dq dq \right] \]

\[ + L_q^{-1} L_s^{-1} \left[ \frac{(-1)^n}{s^2} \int \int \int \left( L_x L_t \left[ \left( x^n \frac{\partial u}{\partial x} \right) \right] \right) dq dq dq \right] \]

\[ + L_q^{-1} L_s^{-1} \left[ \frac{(-1)^{n+1}}{s} \int \int \int \left( L_x L_t \left[ x^{n+1} \frac{\partial v}{\partial x} \right] \right) dq dq dq \right] \] (3.8)

\[ v(x, t) = L_q^{-1} L_s^{-1} \left[ \int \int \int \left( \frac{d^n G_1(q)}{s dq^n} + \frac{d^n G(q, s)}{s dq^n} \right) dq dq dq \right] \]

\[ + L_q^{-1} L_s^{-1} \left[ \frac{(-1)^n}{s} \int \int \int \left( L_x L_t \left[ \left( x^n \frac{\partial v}{\partial x} \right) \right] \right) dq dq dq \right] \]

\[ + L_q^{-1} L_s^{-1} \left[ \frac{(-1)^{n+1}}{s} \int \int \int \left( L_x L_t \left[ x^{n+1} \frac{\partial^2 u}{\partial x \partial t} \right] \right) dq dq dq \right] \] (3.9)

in particular, if \( n = 1 \), Eq.(3.8) and Eq.(3.9) becomes

\[ u(x, t) = f_1(x) + t f_2(x) + L_q^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^q dF(q, s) \right] \]

\[ - L_q^{-1} L_s^{-1} \left[ \frac{1}{s^2} \int_0^q L_x L_t \left[ \left( x \frac{\partial u}{\partial x} \right) - x^2 \frac{\partial v}{\partial x} \right] dq \right] \] (3.10)

and

\[ v(x, t) = g_1(x) + L_q^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^q dG(q, s) \right] \]

\[ - L_q^{-1} L_s^{-1} \left[ \frac{1}{s} \int \left( L_x L_t \left[ \left( x \frac{\partial v}{\partial x} \right) - x^2 \frac{\partial^2 u}{\partial x \partial t} \right] \right) dq \right] . \] (3.11)

Using Eq.(3.7) into Eq.(3.10)and Eq.(3.11)we get

\[ \sum_{n=0}^{\infty} u_n(x, t) = f_1(x) + t f_2(x) + L_q^{-1} L_s^{-1} \left[ \frac{1}{s^2} \int_0^q dF(q, s) \right] \]

\[ - L_q^{-1} L_s^{-1} \left[ \frac{1}{s^2} \int_0^q L_x L_t \left( x \left( \sum_{n=0}^{\infty} u_{nx}(x, t) \right) \right) dq \right] \]

\[ + L_q^{-1} L_s^{-1} \left[ \frac{1}{s^2} \int_0^q L_x L_t \left[ x^2 \sum_{n=0}^{\infty} v_{nx}(x, t) \right] dq \right] \] (3.12)

and

\[ \sum_{n=0}^{\infty} v_n(x, t) = g_1(x) + L_q^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^q dG(q, s) \right] \]

\[ - L_q^{-1} L_s^{-1} \left[ \frac{1}{s} \int \left( L_x L_t \left[ \left( x \sum_{n=0}^{\infty} v_{nx}(x, t) \right) \right] \right) dq \right] , \]

\[ + L_q^{-1} L_s^{-1} \left[ \frac{1}{s} \int \left( L_x L_t \left[ x^2 \frac{\partial^2 u}{\partial x \partial t} \left( \sum_{n=0}^{\infty} u_n(x, t) \right) \right] \right) dq \right] \] (3.13)
in particular

\[ u_0(x,t) = f_1(x) + tf_2(x) + L_q^{-1}L_s^{-1}\left[ \frac{1}{s^2} \int_0^q dF(q,s) \right] \]

\[ v_0(x,t) = g_1(x) + L_q^{-1}L_s^{-1}\left[ \frac{1}{s} \int_0^q dG(q,s) \right] \] (3.14)

In general we have

\[ u_{n+1}(x,t) = -L_q^{-1}L_s^{-1}\left[ \frac{1}{s^2} \int_0^q L_q L_t \left[ \left( x \sum_{n=0}^\infty u_{nx}(x,t) \right) \right] dq \right] \]

\[ +L_q^{-1}L_s^{-1}\left[ \frac{1}{s} \int_0^q L_q L_t \left[ x^2 \sum_{n=0}^\infty v_{nx}(x,t) \right] dq \right], \] (3.15)

\[ v_{n+1}(x,t) = -L_q^{-1}L_s^{-1}\left[ \frac{1}{s} \int \left( L_q L_t \left[ \left( \sum_{n=0}^\infty v_{nx}(x,t) \right) \right] \right) dq \right] \]

\[ +L_q^{-1}L_s^{-1}\left[ \frac{1}{s} \int \left( L_q L_t \left[ x^2 \frac{\partial^2}{\partial x \partial t} \left( \sum_{n=0}^\infty u_n(x,t) \right) \right] \right) dq \right]. \] (3.16)

By calculate the terms \( u_0, u_1, \ldots \) and \( v_0, v_1, \ldots \) We obtain the solution of the system as

\[ u(x,t) = u_0 + u_1 + \ldots \text{ and } v(x,t) = v_0 + v_1 + \ldots. \]

In the following example we consider \( n = 1 \) in Eq.(3.1) as:

**Example 2.** Consider the following linear singular one dimensional thermo-elasticity coupled system

\[ \frac{\partial^2 u}{\partial t^2} - \frac{1}{x} \left( x \frac{\partial u}{\partial x} \right) + \frac{\partial v}{\partial t} = -x^2 \sin t - 4 \sin t + 2x^2 e^t, \quad x \in \Omega \]

\[ \frac{\partial v}{\partial t} - x \left( \frac{\partial v}{\partial x} \right) + x^2 \frac{\partial^2 u}{\partial x \partial t} = x^2 e^t - 4e^t + 2x^2 \cos t, \quad t > 0 \] (3.17)

subject to

\[ u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = x^2, \quad v(x,0) = x^2. \] (3.18)

By using modified double Laplace decomposition methods for Eq.(3.17), Eq.(3.18) and apply Eq.(3.10), Eq.(3.11) we have

\[ u(x,t) = x^2 \sin t + 4 \sin t - 4t + 2x^2 e^t - 2x^2 t - 2x^2 \]

\[ -L_q^{-1}L_s^{-1}\left[ \frac{1}{s^2} \int_0^q L_q L_t \left[ \left( x \frac{\partial u}{\partial x} \right) \right] dq \right] \] (3.19)

and

\[ v(x,t) = x^2 e^t - 4e^t + 2x^2 \sin t - 4 \]

\[ -L_q^{-1}L_s^{-1}\left[ \frac{1}{s} \int L_q L_t \left[ \left( x \frac{\partial v}{\partial x} \right) \right] dq \right]. \] (3.20)

On using Eq.(3.14), Eq.(3.15) and apply Eq.(3.16) we get

\[ u_0(x,t) = x^2 \sin t + 4 \sin t - 4t + 2x^2 e^t - 2x^2 t - 2x^2 \]

\[ v_0(x,t) = x^2 e^t - 4e^t + 2x^2 \sin t - 4, \] (3.21)
\[ u_1(x,t) = -4t + 4x^2 \sin t - 4 \sin t + 8e^t - 8 - \frac{4}{3}t^3 - 4t^2 - 2x^2e^t - 2x^2t + 2x^2 \]
\[ v_1(x,t) = 4x^2e^t + 4e^t - 2x^2 \sin t + 4 - 8 \cos t - 4x^2, \]
and
\[ u_2(x,t) = 8t - 4x^2 \sin t - 16 \sin t - 8e^t + 8 + \frac{4}{3}t^3 + 4t^2 - 4x^2e^t + 8x^2t + 4x^2 + 2x^2t^2 \]
\[ v_2(x,t) = 4x^2e^t + 16e^t - 8x^2 \sin t - 24 + 8 \cos t - 4x^2 + 4x^2t - 16t. \]

Therefore, the approximate solution is
\[ u(x,t) = u_0 + u_1 + ... \quad \text{and} \quad v(x,t) = v_0 + v_1 + ... \]

We obtain the closed form solution
\[ u(x,t) = x^2 \sin t \quad \text{and} \quad v(x,t) = x^2 e^t \]

4. Convergence analysis

In this section, we will discuss the convergence analysis of the modified double Laplace decomposition methods for the nonlinear singular one dimensional hyperbolic equation. We propose to extend this idea given in [15, 17]. First of all let us consider the Hilbert space \( H \), defined by
\[ H = L^2_{\mu}((a,b) \times [0,T]), \]
where \( a \ll 0 \) with following scalar product
\[ (u,v) = \int_Q xu(x,t)v(x,t) \, dx \, dt, \]
where \( Q = (a,b) \times [0,T] \) and
\[ H = \left\{ (u,v) : (a,b) \times [0,T], \text{with} \left[ \frac{1}{x^2} \int_a^b \int_0^T L_x L_t [u(x,t)](p,s) \, dp \right](x,t) < \infty \right\}. \]

Problem: We consider the nonlinear singular one dimensional hyperbolic equation:
\[ \frac{\partial^2 u}{\partial t^2} = 1 \frac{\partial}{\partial x} \left( x \frac{\partial u}{\partial x} \right) + u \frac{\partial u}{\partial x} + f(u), \quad t > 0 \quad (4.1) \]
for all \( u, v \in H \). We define \( H \) as \( H = L^2_{\mu}((a,b) \times [0,T]) \) and
\[ u : (a,b) \times [0,T] \to \mathbb{R} \times \mathbb{R}, \text{with} \quad \|u\|^2_H = \int_Q xu(x,t) v(x,t) \, dx \, dt, \]
where \( Q = (a,b) \times [0,T] \) and
\[ H = \left\{ (u,v) : (a,b) \times [0,T], \text{with} \left[ \frac{1}{x^2} \int_a^b \int_0^T L_x L_t [u(x,t)](q,s) \, dq \right](x,t) < \infty \right\}. \]
Multiplying both sides of Eq.(4.1) by $x$, and write the equation in the operator form

$$L(u) = x \frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} x \frac{\partial u^2}{\partial x} + x f(u), \quad u = u(x, t) \quad t > 0,$$

where $|x| \leq b$. For $L$ hemi-continuous operator, by using the following definition:

**Definition 1.**

1. (H1) \((L(u) - L(v), u - v) \geq k \|u - v\|^2 ; k > 0, \forall u, v \in H\)

2. (H2) whatever may be $M > 0$, there exist a constant $C(M) > 0$ such that for $u, v \in H$ with $\|u\| \leq M, \|v\| \leq M$ we have:

\[(L(u) - L(v), w) \leq C(M) \|u - v\| \|w\|,\]

for every $w \in H$. for more details see [15, 17].

In the next Theorem we follows [16, 18, 19].

**Theorem 2.** *(Sufficient condition of convergence)* The Modified double Laplace decomposition methods applied to the nonlinear singular one dimensional hyperbolic equation Eq.(4.2) with homogenous initial condition, converges towards a solution.

**Proof.** First, we will verify the convergence hypothesis H1 for the operator $L(u)$ of Eq.(4.2). we use the definition of our operator $L$, we have the following form

$$L(u) = \frac{\partial}{\partial x} (u - v) + x \frac{\partial^2}{\partial x^2} (u - v) + \frac{1}{2} x \frac{\partial}{\partial x} (u^2 - v^2) + x (f(u) - f(v)),\]

therefore,

$$\begin{align*}
(L(u) - L(v), u - v) &= \left( \frac{\partial}{\partial x} (u - v), u - v \right) \\
&\quad + \left( x \frac{\partial^2}{\partial x^2} (u - v), u - v \right) \\
&\quad + \left( \frac{1}{2} x \frac{\partial}{\partial x} (u^2 - v^2), u - v \right) \\
&\quad + \left( x (f(u) - f(v)), u - v \right).
\end{align*}

\[(4.3)\]

According to the coercive operator the differential operator $\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$ in $H$, then there exist constants $\alpha, \beta, \theta > 0$ such that

$$\left( \frac{\partial}{\partial x} (u - v), u - v \right) \geq \alpha \|u - v\|^2$$

\[(4.4)\]

and by using Cauchy–Schwartz inequality

$$- \left( x \frac{\partial^2}{\partial x^2} (u - v), u - v \right) \leq |x| \left\| \frac{\partial^2}{\partial x^2} (u - v) \right\| \|u - v\|$$

$$\leq \beta b \|u - v\|^2$$

$$\Leftrightarrow$$

$$\left( x \frac{\partial^2}{\partial x^2} (u - v), u - v \right) \geq - \beta b \|u - v\|^2$$

\[(4.5)\]
where \(\|u\| \leq M\) and \(\|v\| \leq M\), and according to the Schwartz inequality, we get
\[
\left( -\frac{1}{2} x \frac{\partial}{\partial x} (u^2 - v^2), u - v \right) \leq \frac{1}{2} |x| \left\| \frac{\partial}{\partial x} (u^2 - v^2) \right\| \left\| u - v \right\|
\]
\[
\leq \frac{1}{2} b \theta \|u^2 - v^2\| \|u - v\|
\]
\[
\leq \frac{1}{2} b \theta \|u + v\| \|u - v\|^2
\]
\[
\leq b \theta \|u - v\|^2,
\]
where \(|x| \leq b\), hence
\[
\left( \frac{1}{2} x \frac{\partial}{\partial x} (u^2 - v^2), u - v \right) \geq -b \theta \|u - v\|^2. \tag{4.7}
\]

By using Schwartz inequality, where \(\sigma > 0\) as \(f\) is Lipschitzian function, we obtain
\[
(-x (f(u) - f(v)), u - v) \leq |x| \|f(u) - f(v)\| \|u - v\|
\]
\[
\leq b \|f(u) - f(v)\| \|u - v\|
\]
\[
\leq b \sigma \|u - v\|^2 \leftrightarrow
\]
\[
(x (f(u) - f(v)), u - v) \geq -b \sigma \|u - v\|^2, \tag{4.8}
\]
substituting Eq.(4.5), Eq.(4.6), Eq.(4.7) and Eq.(4.8) into equation Eq.(4.4) gives
\[
(L(u) - L(v), u - v) \geq (\alpha - \beta b - b \theta M - b \sigma) \|u - v\|^2
\]
\[
(L(u) - L(v), u - v) \geq k \|u - v\|^2.
\]
So the hypothesis (H1) holds. where
\[
k = \alpha - \beta M - b \theta M - b \sigma > 0.
\]

Now we verify the convergence hypotheses (H2) for the operator \(L(u)\), for every \(M > 0\), there exist a constant \(C(M) > 0\) such that for \(u, v \in H\) with \(\|u\| \leq M\), \(\|v\| \leq M\), exist constants \(\alpha_1, \alpha_2, \beta_1, \sigma_1 > 0\) such that
\[
(L(u) - L(v), w) \leq C(M) \|u - v\| \|w\|,
\]
\(\forall w \in H\). we have,
\[
(L(u) - L(v), w) = \left( \frac{\partial}{\partial x} (u - v), w \right)
\]
\[
+ \left( x \frac{\partial^2}{\partial x^2} (u - v), w \right)
\]
\[
+ \left( \frac{1}{2} x \frac{\partial}{\partial x} (u^2 - v^2), w \right)
\]
\[
+ \left( x (f(u) - f(v)), w \right). \tag{4.9}
\]
By using the Schwartz inequality and the fact that $u$ and $v$ are bounded, we obtain the following:

$$
\left( \frac{\partial}{\partial x} (u - v), w \right) \leq \alpha_1 \| u - v \| \| w \| ,
$$

$$
\left( x \frac{\partial^2}{\partial x^2} (u - v), w \right) \leq b \beta_1 \| u - v \| \| w \|
\leq b \beta_1 \| u - v \| \| w \|
$$

$$
\left( \frac{1}{2} x \frac{\partial}{\partial x} (u^2 - v^2), w \right) \leq \frac{1}{2} \alpha_2 \| x \| \| u + v \| \| u - v \| \| w \|
\leq b \alpha_2 M \| u - v \| \| w \|
$$

$$
(x (f(u) - f(v)), w) \leq b \sigma_1 \| u - v \| \| w \| ,
$$

where $|x| \leq b$ we have:

$$
(L(u) - L(v), w) \leq (\alpha_1 + b \beta_1 + b \alpha_2 M + b \sigma_1) \| u - v \| \| w \|
= C(M) \| u - v \| \| w \|
$$

where

$$
C(M) = (\alpha_1 + b \beta_1 + b \alpha_2 M + b \sigma_1).
$$

and therefore (H2) holds. This completes the proof. \hfill \square

**Conclusion 1.** In this work, first: we proposed new modified double Laplace decomposition methods to solve linear singular one dimensional hyperbolic equation and linear singular one dimensional thermo-elasticity coupled system. The efficiency and accuracy of the present scheme are validated through examples. This method can be applied to many complicated linear and non-linear PDEs and also for system of PDEs and does not require linearization. Second, we presented a convergence proof of the (DLADM) applied to the nonlinear singular one dimensional hyperbolic equation.

**References**


1,3Mathematics Department, College of Science, King Saud University, P.O. Box 2455, RIYADH 11451, SAUDI ARABIA
E-mail address: 1hgadain@ksu.edu.sa, 2akilic@upm.edu.my and 3mesloub@ksu.edu.sa
Current address: 2Department of Mathematics, Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia