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Splitting of Gysin extensions

A. J. Berrick
A. A. Davydov

Abstract Let $X \to B$ be an orientable sphere bundle. Its Gysin sequence exhibits $H^*(X)$ as an extension of $H^*(B)$-modules. We prove that the class of this extension is the image of a canonical class that we define in the Hochschild 3-cohomology of $H^*(B)$, corresponding to a component of its A_∞-structure, and generalizing the Massey triple product. We identify two cases where this class vanishes, so that the Gysin extension is split. The first, with rational coefficients, is that where B is a formal space; the second, with integer coefficients, is where B is a torus.

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1 Introduction

The paper is devoted to the description of the module structure of the cohomology $H^*(X)$ of an oriented sphere bundle $X \to B$ over the cohomology of the base $H^*(B)$.

This is a problem with a long history. In the late 'fifties (a decade or so after the discovery of the Gysin sequence), several attempts were made to determine multiplicative properties of the cohomology $H^*(X)$, principally by means of triple Massey products of $H^*(B)$ [7, 14]. An obstacle to a full answer was the partial definiteness of Massey products. W. S. Massey, in his review of a 1959 paper of D. G. Malm [13], summarized the situation as follows:

The author points out that it is misleading to say that 'the cohomology ring of the total space depends on the cohomology of the base space and certain characteristic classes.' Just exactly what aspects of the homotopy type of the base space it does depend on seems like a very complicated question. Certainly various higher order cohomology operations are needed.

Ironically, a mathematical framework for fully defined higher products (A_∞-algebras) appeared just a little bit later, in a different context, in the paper of J. Stasheff [20]. It was however only in 1980 that T. Kadeishvili defined an A_∞-structure of a special kind for the cohomology of a topological space or, more generally, of a differential graded algebra [8]. Informally speaking, an A_∞-structure on cohomology is a collection of multilinear maps, the first of which is multiplication satisfying certain compatibility conditions (see Remark 2.2 for the definition). The Massey products fit into this structure as specializations of the multilinear maps. For example, the triple Massey product is a specialization for Massey triples of the second component of the A_∞-structure (see Remark 2.6). Another problem of higher multiplications, their non-canonical nature, remained unhandled. For Massey products this appears in the peculiar shape of the range of values; while for A_∞-structures it forces them to be defined only up to an isomorphism.

Here we attack the problem by using relations between A_∞-structures and Hochschild cohomology. The compatibility conditions imply that the second component of an A_∞-structure is a Hochschild 3-cocycle, and the corresponding Hochschild cohomology class is invariant under those isomorphisms between A_∞-structures that preserve ordinary multiplication (see Remark 2.2). We give a self-contained construction of this Hochschild 3-cocycle for the cohomology of a differential graded algebra, and prove the canonical nature of the corresponding Hochschild cohomology class (Section 2.2). As in [8], we need some sort of freeness condition on $H^*(B) = H^*(B;k)$ as a module over its coefficient ring k. Throughout we assume that k is commutative and hereditary (for example, a principal ideal domain), so that every submodule of a free module is projective.

The Gysin sequence provides an extension of $H^*(B)$-modules

$$0 \to H^*(B)/(cH^*(B)[|c|]) \to H^*(X) \to \mathrm{Ann}_{H^*(B)}(c)[|c|-1] \to 0,$$

where $c \in H^*(B)$, of degree $|c|$, is the characteristic class of the fibre bundle, and the square brackets indicate grading shifts. We prove that the class of this extension in the extension group

$$\mathrm{Ext}^1_{H^*(B)}(\mathrm{Ann}_{H^*(B)}(c)[|c|-1], H^*(B)/(cH^*(B)[|c|]))$$

is the image of our Hochschild 3-cohomology class of the algebra $H^*(B)$, under a natural homomorphism

$$HH^3_{\mathrm{gr}}(H^*(B), \overline{H^*(B)}[1]) \to \mathrm{Ext}^1_{H^*(B)}(\mathrm{Ann}_{H^*(B)}(c)[|c|-1], H^*(B)/(cH^*(B)[|c|])).$$

To prove this, we use the language of differential graded (dg-) algebras. We use the explicit construction of the Hochschild 3-cohomology class [ ] of the algebra $H^*(C)$ to interpret the class of the $H^*(C)$-module extension

$$0 \to H^*(C)/(cH^*(C)[|c|]) \to H^*(\mathrm{cone}(l_{s(c)})) \to \mathrm{Ann}_{H^*(B)}(c)[|c|-1] \to 0;$$

obtained from the long exact sequence of the mapping cone $\mathrm{cone}(l_{s(c)})$ of left multiplication $l_{s(c)} : C[|c|] \to C$, whenever $s(c)$ denotes a representative cocycle of a cohomology class $c \in H^*(C)$ (section 3). By expressing Thom's Gysin sequence as the long exact sequence of the mapping cone [13], we are able to derive our main result (Theorem 4.3).

Theorem 1.1 *Let $S^m \to X \to B$ be an orientable sphere bundle, where the homology $H_*(B;k)$ of the base is a projective k-module. Let $c \in H^{m+1}(B)$ be its characteristic class and*

$$\in \mathrm{Ext}^1_{H^*(B)}(\mathrm{Ann}_{H^*(B)}(c)[m]; H^*(B)/(cH^*(B)[m+1]))$$

the class of the Gysin extension

$$0 \to H^*(B)/(cH^*(B)[m+1]) \to H^*(X) \to \mathrm{Ann}_{H^*(B)}(c)[m] \to 0.$$

Then the above natural homomorphism sends the Hochschild cohomology class of (B) to the class .

Applications of this theorem depend on calculation of the class $[\ (B)]$ in specific cases. In the case of rational coefficients, there is a large and well-studied family of spaces for which the obstruction is readily seen to vanish, namely formal spaces. This gives the following theorem.

Theorem 1.2 *Let $S^m \to X \to B$ be an orientable sphere bundle having base B a formal space, and characteristic class $c \in H^{m+1}(B)$. Then the Gysin sequence with rational coefficients*

$$0 \to H^*(B)/(cH^*(B)[m+1]) \to H^*(X) \to \mathrm{Ann}_{H^*(B)}(c)[m] \to 0$$

splits as a sequence of right $H^*(B)$-modules.

On the other hand, the case of integer cohomology is far more delicate. There seems to be no existing literature to which one can appeal. We attack the case of a torus base, which is of fundamental importance to understanding the multiplicative cohomology of integer Heisenberg groups, whose additive structure has recently been determined [12]. In Section 5.2 below we outline our proof of triviality of the canonical Hochschild 3-cohomology class for the torus, leading to the second application.

Theorem 1.3 *Let $S^m \to X \to T$ be an orientable sphere bundle with torus base T and characteristic class $c \in H^{m+1}(T)$. Then the Gysin sequence with integer coefficients*

$$0 \to H^*(T)/(cH^*(T)[m+1]) \to H^*(X) \to \mathrm{Ann}_{H^*(T)}(c)[m] \to 0$$

splits as a sequence of $H^*(T)$-modules.

Recently (Gdansk conference, June 2001), D. Benson announced joint work with H. Krause and S. Schwede concerning an obstruction, in 3-dimensional Hochschild cohomology, to decomposability of the cohomology of a dga-module. Presumably their class will be closely related to ours; unfortunately, their arguments seem to be no less lengthy. In applications to the Tate cohomology of certain finite groups, their obstruction turns out to be nontrivial. That finding is consistent with determination of nontrivial higher Massey products with finite coefficients in [10].

The results presented here were announced at the Arolla conference in August 1999, and the second-named author would like to thank D. Arlettaz for providing the opportunity to speak there. Unfortunately, another journal's prolonged inability to find a referee has led to a delay in publication. On the other hand, we are grateful to this journal's referee for several pertinent suggestions that have, we trust, made the paper more readable. In particular, we have replaced the original six page technical proof of triviality of the obstruction with torus base, by a brief summary of the argument. The second author acknowledges the support of National University of Singapore grant RP3970657.

2 Secondary multiplication in cohomology

2.1 Hochschild cohomology of algebras

By the *Hochschild cohomology* of a unital, associative k-algebra R, projective over a commutative ring k, with coefficients in an R-bimodule M, we mean the extension groups

$$HH^*(R;M) = \mathrm{Ext}^*_{R\text{-}R}(R;M)$$

between R-bimodules R and M. It can be identified [1] (IX.6) with the cohomology of the complex

$$CH^*(R;M) = \mathrm{Hom}_k(R^{\otimes *};M),$$

with differential

$$\delta : CH^l(R, M) \to CH^{l+1}(R, M),$$

$$(\delta a)(x_1, \ldots, x_{l+1}) = x_1 a(x_2, \ldots, x_{l+1}) + \sum_{i=1}^{l} (-1)^i a(x_1, \ldots, x_i x_{i+1}, \ldots, x_{l+1}) + (-1)^{l+1} a(x_1, \ldots, x_l) x_{l+1}.$$

While it is well-known in this case that δ is indeed a differential, for future reference we record the simple fact that establishes this property. Its proof is a straightforward induction argument.

Lemma 2.1 (The Differential Lemma) *In the free associative ring*

$$\mathbb{Z}\langle u_1, v_1, u_2, v_2, \ldots \rangle,$$

for all n,

$$\left(\sum_{i=1}^{n-1} u_i\right)\left(\sum_{i=1}^{n} v_i\right) = \sum_{1 \le i \le j \le n} (u_i v_j + u_{j+1} v_i).$$

For the application at hand, it is easy to see that, whenever $i \le j$, we have $\delta_i \delta_j = \delta_{j+1} \delta_i$, so we map u_i and v_i to $(-1)^i \delta_i$ in order to obtain $\delta\delta = 0$.

Remark 2.2 A_∞-structures and Hochschild cohomology [9]

The appearance of Hochschild cohomology in this paper is explained by its connection with cohomological higher multiplications, namely with A_∞-algebras [20]. Recall that a minimal A_∞-algebra is a k-module $R = \bigoplus_{s \ge 0} R_s$ graded by nonnegative integers, together with a collection of graded maps $m_i : R^{\otimes i+2} \to R[i]$ for $i = 0, 1, \ldots,$ satisfying

$$\sum_{i+j=n} \sum_{h=1}^{i+2} (-1)^{h(j-1)+(n+1)j} m_i \circ_h m_j = 0, \tag{1}$$

for any n and suitably defined operations $\circ_h$ (see also[6]).

Note that, for $n = 0$, the condition (1) implies that (R, m_0) is an associative k-algebra. With $\bar{\delta}$ as the differential of the standard Hochschild complex of the algebra $R = (R, m_0)$ with coefficients in the R-bimodule $\overline{R}$, the condition (1) for $n = 1$ has the form $\bar{\delta}(m_1) = 0$. This means that m_1 is a Hochschild 3-cocycle of R with coefficients in $\overline{R}$. The cohomology class $[m_1] \in HH^3(R, \overline{R})$

will be called the *canonical class* (or secondary multiplication) of the A_∞-algebra R.

In the paper [8], T Kadeishvili extended the ordinary multiplication on the cohomology of a differential graded algebra to the minimal A_∞-algebra structure. In this paper we use only the second component of this structure. For the sake of completeness, we give the definition of this secondary multiplication in the next section.

2.2 Secondary multiplication of the cohomology of a differential graded algebra

Let C^* be a *differential graded* algebra (*dg-algebra*) over a commutative hereditary ring k; that is, C^* is a graded k-algebra

$$C^* = \bigoplus_{i\geq 0} C^i; \quad C^iC^j \subseteq C^{i+j};$$

with a graded k-linear derivation (*differential*) $d: C^* \to C^*$ of degree 1

$$B^{i+1} = d(C^i) \subseteq C^{i+1}; \quad d(xy) = d(x)y + (-1)^{|x|}xd(y);$$

such that $d^2 = 0$. Here the formula is given for homogeneous x, y and $|x|$ is the degree of x.

Since Ker(d) is a subalgebra in C^* (subalgebra of *cocycles*) and Im(d) is an ideal in Ker(d) (ideal of *coboundaries*), the cohomology $H^*(C^*)$ = Ker(d)/Im(d) of the dg-algebra C^* has a natural k-algebra structure. Here we consider dg-algebras that arise as the dual of a free chain complex (C_*, ∂). The following is easily checked (cf. [18] p.386). (We write $B_{i-1} = \partial(C_i)$.)

Technical Lemma 2.3 *Suppose that C_* is a free chain complex with $H_*(C_*)$ a projective k-module. Then for each n there is a direct sum decomposition*

$$C_n = H_n(C) \oplus B_n \oplus B_{n-1},$$

which induces by duality a direct sum decomposition

$$\begin{aligned} C^n &= H_n(C)^- \oplus B_n^- \oplus B_{n-1}^- \\ &= H^n(C) \oplus B^{n+1} \oplus B^n. \end{aligned}$$ □

It follows that we can choose graded k-linear sections $s: H^*(C^*) \to$ Ker(d) and q : Im(d) $\to C^*$ of the natural k-linear surjections π : Ker(d) $\to H^*(C^*)$ and

$d : C \to \mathrm{Im}(d)$ respectively. The situation may be summarized as

$$\begin{array}{ccccc} \mathrm{Im}\, d & & & & \\ \downarrow & & & & \\ \mathrm{Ker}\, d & \hookrightarrow & C & \underset{d}{\overset{q}{\twoheadrightarrow}} & \mathrm{Im}\, d[-1]. \\ \downarrow \uparrow s & & & & \\ H^*(C) & & & & \end{array}$$

The map $\pi : \mathrm{Ker}(d) \to H^*(C)$ is a k-algebra homomorphism, hence the difference

$$s(x)s(y) - s(xy)$$

has zero image under π:

$$\pi(s(x)s(y) - s(xy)) = \pi(s(x))\pi(s(y)) - \pi(s(xy)) = xy - xy = 0;$$

that is, lies in $\mathrm{Im}(d)$. Define $q(x;y) = q(s(x)s(y) - s(xy)) \in C$, so that

$$dq(x;y) = s(x)s(y) - s(xy).$$

(Observe that $|q(x;y)| = |x| + |y| - 1$.) Then the expression

$$\theta(x;y;z) = (-1)^{|x|}s(x)q(y;z) - q(xy;z) + q(x;yz) - q(x;y)s(z) \qquad (2)$$

lies in $\mathrm{Ker}(d)$. Indeed,

$$\begin{aligned} d\theta(x;y;z) &= s(x)dq(y;z) - dq(xy;z) + dq(x;yz) - d(q(x;y))s(z) \\ &= s(x)(s(y)s(z) - s(yz)) - (s(xy)s(z) - s(xyz)) \\ &\quad + (s(x)s(yz) - s(xyz)) - (s(x)s(y) - s(xy))s(z) \\ &= 0. \end{aligned}$$

Define $\varphi(x;y;z) = \pi(\theta(x;y;z)) \in H^*(C)$. Then

$$|\varphi(x;y;z)| = |\theta(x;y;z)| = |x| + |y| + |z| - 1;$$

so $\varphi : H^*(C)^{\otimes 3} \to H^*(C)$ has degree -1. This is a k-linear map $H^*(C)^{\otimes 3} \to H^*(C)[1]$, where $H^*(C)[1]$ is a *shifted* graded module $H^*(C)$ such that $(H^*(C)[1])^m = H^{m-1}(C)$. The twisted multiplication $x \circ y = (-1)^{|x|}xy$ allows us to define a new $H^*(C)$-bimodule structure on $H^*(C)$ by setting the left module structure to be twisted and the right module structure to be the ordinary one. We denote this $H^*(C)$-bimodule by $\overline{H^*(C)}$.

Proposition 2.4 *The map* $\varphi : H^*(C)^{\otimes 3} \to H^*(C)$ *defined above is a Hochschild 3-cocycle of degree -1 with respect to the graded algebra* $H^*(C)$.

Proof We check the 3-cocycle property of $\varphi : H_*(C)^{\otimes 3} \to \overline{H_*(C)}$. Define the maps

$$\delta_s : \mathrm{Hom}(H_*(C)^{\otimes l}, C) \to \mathrm{Hom}(H_*(C)^{\otimes l+1}, C)$$

as

$$\begin{aligned}(\delta_s \psi)(x_1, \ldots, x_{l+1}) &= (-1)^{|x_1|} s(x_1)\psi(x_2, \ldots, x_{l+1}) + \sum_{i=1}^{l} (-1)^i \psi(x_1, \ldots, x_i x_{i+1}, \ldots, x_{l+1}) \\ &\quad + (-1)^{l+1} \psi(x_1, \ldots, x_l) s(x_{l+1}).\end{aligned}$$

The restrictions of these maps to $\mathrm{Hom}(H_*(C)^{\otimes l}, \mathrm{Ker}(d))$ are compatible with Hochschild differentials on $\mathrm{Hom}(H_*(C)^{\otimes l}, \overline{H_*(C)})$, so that the diagram

$$\begin{array}{ccccc} \xrightarrow{\delta_s} & \mathrm{Hom}(H_*(C)^{\otimes l}, \mathrm{Ker}(d)) & \xrightarrow{\delta_s} & \mathrm{Hom}(H_*(C)^{\otimes l+1}, \mathrm{Ker}(d)) & \xrightarrow{\delta_s} \\ & \downarrow & & \downarrow & \\ \xrightarrow{\delta} & \mathrm{Hom}(H_*(C)^{\otimes l}, \overline{H_*(C)}) & \xrightarrow{\delta} & \mathrm{Hom}(H_*(C)^{\otimes l+1}, \overline{H_*(C)}) & \xrightarrow{\delta} \end{array} \qquad (3)$$

is commutative. We shall show that

$$\delta_s \varphi(x, y, z, w) = (-1)^{|x|+|y|} d(q(x, y)q(z, w)),$$

which implies that $\bar{\varphi} = \pi\varphi$ is a 3-cocycle.

Since s is not a ring homomorphism in general, the map δ_s does not satisfy the condition $\delta_s \delta_s = 0$. Instead of this, upon writing $\delta_s = \sum_{i=0}^{l+1} (-1)^i \delta_s^i$ as usual, we note that

$$\delta_s^i \delta_s^j = \delta_s^{j+1} \delta_s^i \quad i \le j, \ (i, j) \ne (0, 0), (l+1, l+1).$$

So the Differential Lemma immediately implies that

$$\delta_s \delta_s = (\delta_s^0 \delta_s^0 - \delta_s^1 \delta_s^0) + (\delta_s^{l+1} \delta_s^{l+1} - \delta_s^{l+2} \delta_s^{l+1}),$$

in other words,

$$\begin{aligned}&\delta_s \delta_s \psi(x_1, \ldots, x_{l+2}) \\ &= (-1)^{|x_1|+|x_2|} dq(x_1, x_2)\psi(x_3, \ldots, x_{l+2}) - \psi(x_1, \ldots, x_l) dq(x_{l+1}, x_{l+2}).\end{aligned}$$

Now, by definition, $\varphi = \delta_s q$. Therefore

$$\delta_s \varphi(x, y, z, w) = (-1)^{|x|+|y|} d(q(x, y)q(z, w)). \qquad \square$$

Proposition 2.5 *The cohomology class* $[\varphi] \in HH^3(H_*(C), \overline{H_*(C)}[1])$ *is independent of the choices of* s *and* q.

Proof For another choice q' of the section $q:\mathrm{Im}(d)\to C$, the difference $q'-q$ is a map $\mathrm{Im}(d)\to\mathrm{Ker}(d)$. Hence we are in the situation of diagram (3), and the cocycles θ' and θ differ by the coboundary $\delta(q'-q)$.

For another choice s' of the section $s:H^*(C)\to C$, the difference $s'-s$ is a map $H^*(C)\to\mathrm{Im}(d)$. So we can write $s'-s=da$ for some $a:H^*(C)\to C$ of degree -1. Now because $ds=dd=0$, we have

$$d((-1)^{|x|}s(x)a(y)+a(x)s(y)+a(x)da(y))=s'(x)s'(y)-s(x)s(y).$$

Therefore

$$\begin{aligned}d(q'(x,y)-q(x,y)) &= s'(x)s'(y)-s'(xy)-(s(x)s(y)-s(xy))\\ &= db(x,y),\end{aligned}$$

where

$$b(x,y)=(-1)^{|x|}s(x)a(y)-a(xy)+a(x)s(y)+a(x)da(y).$$

This means that the difference $q'(x,y)-q(x,y)-b(x,y)$ lies in $\mathrm{Ker}(d)$ and, as we saw before, does not affect the cohomology class. Hence we can suppose that $q'(x,y)=q(x,y)+b(x,y)$. Then the difference of cocycles

$$\begin{aligned}&\theta'(x,y,z)-\theta(x,y,z)\\ =\ &(-1)^{|x|}s'(x)q'(y,z)-q'(xy,z)+q'(x,yz)-q'(x,y)s'(z)\\ &-((-1)^{|x|}s(x)q(y,z)-q(xy,z)+q(x,yz)-q(x,y)s(z))\end{aligned}$$

can be rewritten as

$$\begin{aligned}&(-1)^{|x|}da(x)(q(y,z)+b(y,z))-(q(x,y)+b(x,y))da(z)\\ &+(-1)^{|x|}s(x)b(y,z)-b(xy,z)+b(x,yz)-b(x,y)s(z).\end{aligned}$$

Using congruences modulo $\mathrm{Im}(d)$,

$$\begin{aligned}(-1)^{|x|}da(x)q(y,z) &\equiv a(x)dq(y,z)\\ &= a(x)(s(y)s(z)-s(yz)),\end{aligned}$$

$$q(x,y)da(z)\equiv(-1)^{|xy|}(s(x)s(y)-s(xy))a(z),$$

and the definition of b, we can, after cancellation of similar terms, rewrite the expression as

$$(-1)^{|x|}d((-1)^{|y|}a(x)s(y)a(z)-a(x)a(yz)+a(x)a(y)s'(z)),$$

which lies in $\mathrm{Im}(d)$, completing the proof. □

Remark 2.6 Connection with Massey triple product

This connection admits an abstract algebraic setting. For any triple of elements $x, y, z \in R$ of an associative ring R satisfying the conditions $xy = yz = 0$ (*Massey triple*), we can define a homomorphism

$$HH^3(R;M) \to M/(xM + Mz),$$

by sending the class of a 3-cocycle α to the image of its value $\alpha(x, y, z)$ in $M/(xM + Mz)$. Indeed, the value of any coboundary

$$\begin{aligned} \partial a(x,y,z) &= xa(y,z) - a(xy,z) + a(x,yz) - a(x,y)z \\ &= xa(y,z) - a(x,y)z \end{aligned} \tag{4}$$

represents zero in $M/(xM + Mz)$.

Now, for any Massey triple $x, y, z \in H^*(C)$, the expression $\alpha(x, y, z)$ takes the form

$$(-1)^{|x|} s(x)q(y,z) - q(x,y)s(z),$$

which coincides with the usual definition of $\langle x, y, z\rangle$ (*Massey triple product*) [14]. Note that the value of $\langle x, y, z\rangle$ in the quotient $H^*(C)/(xH^*(C) + H^*(C)z)$ is canonically defined, that is, does not depend on the choice of s and q. This property can be deduced from the canonical nature of the definition of the corresponding Hochschild cohomology class (Proposition 2.5).

Let $C^*(X;k)$ be the dg-algebra (under cup-product) of singular cochains of a topological space X. It is the dual of the singular chain complex $C_*(X;k)$, a free graded k-module. If the graded k-module $H^*(X;k)$ is also projective, then (2.3) applies, and we have a linear section

$$q : B^*(X;k) = \mathrm{Im}(d) \to C^*(X;k)$$

of the surjection $d : C^*(X;k) \to B^*(X;k)$, and section $s : H^*(X;k) \to \mathrm{Ker}(d)$ of the surjection $\mathrm{Ker}(d) \to H^*(X;k)$. Hence the secondary multiplication class

$$[\mu(X;k)] \in HH^3_{\mathrm{gr}}(H^*(X;k), \overline{H^*(X;k)}[1])$$

is defined.

3 Splitting of the multiplication map

For a morphism $f : K \to L$ of cochain complexes, we use the mapping cone $\mathrm{cone}(f)$, which fits into a short exact sequence of complexes

$$0 \to L \to \mathrm{cone}(f) \to K[-1] \to 0. \tag{5}$$

In particular, we have a long exact sequence of cohomology groups

$$\cdots \to H^n(L) \to H^n(\mathrm{cone}(f)) \to H^{n+1}(K) \xrightarrow{f^*} H^{n+1}(L) \to \cdots. \tag{6}$$

Recall that $\mathrm{cone}(f)$ is defined by equipping the direct sum $L \oplus K[-1]$ with the differential $D(x, y) = (d(x) + f(y), -d(y))$.

A (right) *dg-module* over a dg-algebra C is a complex M with a module structure over the algebra C

$$M \otimes C \to M, \quad m \otimes x \mapsto mx, \tag{7}$$

such that

$$d(mx) = d(m)x + (-1)^{|m|} m d(x). \tag{8}$$

In other words, a module over a dg-algebra is a module over an algebra such that the module-structure map (7) is a homomorphism of complexes. A *morphism* or *C-linear map* of dg-modules over the dg-algebra C is a morphism of complexes that commutes with the module structure maps. The following is proved by straightforward computation.

Proposition 3.1 *Let $f : M \to N$ be a morphism of right dg-modules over a dg-algebra C. Then the mapping cone $\mathrm{cone}(f)$ has a natural dg-module structure over C such that all maps of the exact sequence (5) are C-linear.* □

The dg-module structure over the dg-algebra C on the complex M induces the $H^*(C)$-module structure on its cohomology $H^*(M)$; any C-linear map between dg-modules induces an $H^*(C)$-linear map between their cohomology modules. In particular, for the C-linear map $f : M \to N$ between dg-modules, the sequence (6) is an exact sequence of $H^*(C)$-modules.

Now we examine some special cases of C-linear map. Note that the dg-algebra C can be considered as a module over itself. For any cocycle $z \in \mathrm{Ker}(d) \subset C$, left multiplication by z is a morphism of complexes

$$l_z : C[|z|] \to C, \tag{9}$$

which is also C-linear with respect to the natural right C-module structure on C.

The map $H^*(C)[|z|] \to H^*(C)$ induced by the left multiplication map l_z is left multiplication by the class $c = \pi(z) \in H^*(C)$ of z. Thus in the case when $f = l_{s(c)}$ for some $c \in H^*(C)$ we can rewrite the long exact sequence (6) as an extension of $H^*(C)$-modules

$$0 \to H^*(C)/(cH^*(C)[|c|]) \to H^*(\mathrm{cone}(l_{s(c)})) \to \mathrm{Ann}_{H^*(C)}(c)[|c|-1] \to 0, \tag{10}$$

where $\mathrm{Ann}_{H_*(C)}(c) = \{x \in H_*(C) \mid cx = 0\}$ is the right annihilator of c.

We shall describe the class of this extension in the group

$$E := \mathrm{Ext}^1_{H_*(C)}(\mathrm{Ann}_{H_*(C)}(c)[|c| - 1]; H_*(C)/(cH_*(C)[|c|])).$$

We first develop an abstract setting for the connection between the groups $HH^3_{\mathrm{gr}}(H_*(C); \overline{H_*(C)}[1])$ and E. The following is easily checked.

Lemma 3.2 $\mathrm{Ker}(\mathrm{Ext}^1_R(N; M) \to \mathrm{Ext}^1_k(N; M))$ *coincides with the group of k-linear maps $\varphi : N \otimes_k R \to M$ satisfying*

$$\varphi(n; xy) = \varphi(nx; y) + \varphi(n; x)y \tag{11}$$

modulo its subgroup of maps of the form

$$\varphi(n; x) = b(nx) - b(n)x; \tag{12}$$

for some k-linear $b : N \to M$. ☐

Let R be a k-algebra and M an R-bimodule. For any $c \in R$ we define a homomorphism

$$HH^3(R; M) \to \mathrm{Ext}^1_R(\mathrm{Ann}_R(c); M/cM). \tag{13}$$

Let $\psi \in ZH^3(R; M)$ be a cocycle of the standard Hochschild complex. Define the map $\varphi : \mathrm{Ann}_R(c) \otimes R \to M/cM$ by setting $\varphi(x; y)$ to be the class of $\psi(c; x; y)$ in M/cM. The cocycle property

$$c\psi(x; y; z) - \psi(cx; y; z) + \psi(c; xy; z) - \psi(c; x; yz) + \psi(c; x; y)z = 0$$

implies the equation (11)

$$\varphi(x; yz) = \varphi(xy; z) + \varphi(x; y)z;$$

for any $x \in \mathrm{Ann}_R(c)$ and $y, z \in R$. Thus, by the lemma above, φ defines the structure of an R-extension on $\mathrm{Ann}_R(c) \oplus M/cM$. For a coboundary $\psi = d(a)$

$$\psi(c; x; y) = ca(x; y) - a(cx; y) + a(c; xy) - a(c; x)y$$

the corresponding φ takes the form $\varphi(x; y) = b(xy) - b(x)y$ where $b(x) = a(c; x)$.

Now let R be a graded k-algebra, M a graded R-bimodule and $c \in R$ homogeneous of degree $|c|$. For ψ of degree zero, the degree of its corresponding φ is $|c|$. Hence we have a map

$$HH^3_{\mathrm{gr}}(R; M) \to \mathrm{Ext}^1_R(\mathrm{Ann}_R(c)[|c|]; M/cM).$$

In particular, when $R = H_*(C)$ and $M = \overline{H_*(C)}[1]$; we have the map

$$HH^3_{\mathrm{gr}}(H_*(C); \overline{H_*(C)}[1]) \to E. \tag{14}$$

Theorem 3.3 *Let C, a dg-algebra over a hereditary ring, be the dual of a projective chain complex with projective homology. Let $c \in H^*(C)$, and let*

$$\in \mathrm{Ext}^1_{H^*(C)}(\mathrm{Ann}_{H^*(C)}(c)[|c|-1], H^*(C)/(cH^*(C)[|c|]))$$

be the class of the extension (10). Then the map (14) sends the Hochschild cohomology class [ ] to the class .

Proof Note that the mapping cone $\mathrm{cone}(l_{s(c)})$ of the multiplication map (9) coincides with $C \oplus C[|c|-1]$ equiped with the differential

$$D(x,y) = (d(x) + s(c)y, (-1)^{|c|-1}d(y)).$$

We construct a section of the surjection

$$H^*(\mathrm{cone}(l_{s(c)})) \to \mathrm{Ann}_{H^*(C)}(c)[|c|-1] \tag{15}$$

from the short exact sequence (10), as follows. For $x \in \mathrm{Ann}_{H^*(C)}(c)$ the product $s(c)s(x)$ lies in $B^*(C)$, so we can write $s(c)s(x) = dq(c,x)$ for a section q of d as in (2.2). Then $(-q(c,x), s(x))$ is a cocycle of $\mathrm{cone}(l_{s(c)})$, because

$$D(-q(c,x), s(x)) = (-dq(c,x) + s(c)s(x), (-1)^{|c|-1}ds(x)) = 0.$$

Obviously, the image of $(-q(c,x), s(x))$ under the natural map $\mathrm{cone}(l_{s(c)}) \to C[|c|-1]$ is $s(x)$. Thus the map $: \mathrm{Ann}_{H^*(C)}(c)[|c|-1] \to H^*(\mathrm{cone}(l_{s(c)}))$ sending x to the class of $(-q(c,x), s(x))$ in $H^*(\mathrm{cone}(l_s(c)))$ is a section of the surjection (15).

Now we calculate the expression $(x)y - (xy)$ which represents the class of the extension (10) in E:

$$\begin{aligned} (x)y - (xy) &= ((-q(c,x), s(x))s(y) - (-q(c,xy), s(xy))) \\ &= (-q(c,x)s(y) + q(c,xy), s(x)s(y) - s(xy)) \\ &= (-q(c,x)s(y) + q(c,xy), dq(x,y)). \end{aligned}$$

Since $(0, dq(x,y)) = ((-1)^{|c|}s(c)q(x,y), 0) + (-1)^{|c|-1}D(0, q(x,y))$, we have

$$(x)y - (xy) = ((-1)^{|c|}s(c)q(x,y) + q(c,xy) - q(c,x)s(y), 0),$$

which from (2) coincides with the image of (c,x,y) under the map $H^*(C) \to H^*(\mathrm{cone}(l_{s(C)}))$. □

4 Cohomology of an orientable sphere bundle

In order to apply the previous algebraic result (3.3) to geometrical situations, we use the following.

Lemma 4.1 *Let $f : C_1 \to C_2$ be a homomorphism of dg-algebras and $z \in C_1$ be a cocycle whose class c lies in the kernel $c \in \operatorname{Ker} H(f)$. Then the homomorphism f of dg-algebras extends to a (right) C_1-linear map of complexes* $\operatorname{cone}(l_z) \to C_2$.

Proof Since the class c of the cocycle $z \in C_1$ lies in the kernel of the homomorphism $H^*(C_1) \to H^*(C_2)$, we can find some cochain $b \in C_2^{|z|-1}$ satisfying $db = f(z)$. Thus we can define a map $\operatorname{cone}(l_z) \to C_2$ sending (x, y) to $f(x) + bf(y)$. This map is compatible with the differentials, for

$$\begin{aligned} d(f(x) + bf(y)) &= df(x) + (db)f(y) + (-1)^{|b|}b(df(y)) \\ &= f(dx) + f(zy) + (-1)^{|z|-1}bf(dy) \end{aligned}$$

coincides with the image of $D(x, y) = (d(x) + zy, (-1)^{|z|-1}d(y))$. Moreover, the map is obviously C_1-linear with respect to the structure of a right dg-module over C_1 on C_2 given by the dg-algebra homomorphism $f : C_1 \to C_2$. □

Now let $S^m \to X \xrightarrow{p_0} B$ be an orientable sphere bundle, with associated disc bundle $D^{m+1} \to E \xrightarrow{p} B$. Denote by $u \in H^{m+1}(E, X)$ its orientation class and by $c \in H^{m+1}(B)$ its characteristic class. Choose some representative $z \in Z^{m+1}(B)$ for c. Write $l_z : C^*(B)[m+1] \to C^*(B)$ for the left multiplication map. The class c lies in the kernel of the homomorphism $p_0^* : H^{m+1}(B) \to H^{m+1}(X)$. Thus we can apply the lemma and extend the homomorphism of dg-algebras $p_0^* : C^*(B) \to C^*(X)$ to a $C^*(B)$-linear map of complexes $p^* : \operatorname{cone}(l_z) \to C^*(X)$.

Proposition 4.2 *The map* $\operatorname{cone}(l_z) \to C^*(X)$ *induces an $H^*(B)$-linear isomorphism in cohomologies, and the long exact sequence (6) is isomorphic under this map to the Gysin sequence of the sphere bundle*

$$\begin{array}{ccccccccc} \longrightarrow & H^*(B) & \longrightarrow & H^*(\operatorname{cone}(l_z)) & \longrightarrow & H^{*-m}(B) & \xrightarrow{l_c} & H^{*+1}(B) & \longrightarrow \\ & \| & & \downarrow{\scriptstyle H(p)} & & \| & & \| & \\ \longrightarrow & H^*(B) & \longrightarrow & H^*(X) & \longrightarrow & H^{*-m}(B) & \xrightarrow{l_c} & H^{*+1}(B) & \longrightarrow \end{array}$$

Proof It is well-known that the Gysin sequence of a sphere bundle can be derived from the long exact sequence of the pair (E, X), using the natural identification $H^*(B) \xrightarrow{p^*} H^*(E)$ and the Thom isomorphism $H^*(B)[m+1] \xrightarrow{l_u} H^*(E, X)$ given by multiplication by the orientation class $u \in H^{m+1}(E, X)$.

We can establish this connection on the level of singular cochain complexes. The short exact sequence $0 \to C^*(E, X) \to C^*(E) \to C^*(X) \to 0$, which gives the long exact sequence of the pair (E, X), can be extended to the diagram

$$\begin{array}{ccccccccc}
 & & C^*(B)[m+1] & \xrightarrow{l_z} & C^*(B) & \xrightarrow{p_0} & C^*(X) & & \\
 & & \downarrow{\scriptstyle l_U p^*} & & \downarrow{\scriptstyle p^*} & & \| & & \\
0 & \longrightarrow & C^*(E, X) & \longrightarrow & C^*(E) & \longrightarrow & C^*(X) & \longrightarrow & 0
\end{array}$$

where the first vertical arrow is a composition of $p^* : C^*(B) \to C^*(E)$ with multiplication by a representative cocycle $U \in C^{m+1}(E, X)$ of the orientation class $u \in H^{m+1}(E, X)$. The right-hand square of the diagram is obviously commutative. We seek to choose cocycles z and U such that the left-hand square of the diagram is commutative up to homotopy. To achieve this, we exploit the following fact. Denote by $s : B \to E$ the zero section of the associated disc bundle. The composition ps is the identity, so $s^* p^* = \mathrm{id}_{C^*(B)}$. Further, sp is homotopic to the identity, so we have $p^* s^* = \mathrm{id}_{C^*(E)} + dh + hd$ for some $h : C^*(E) \to C^*(B)[1]$. Now we can start with a given U and define z to be $s^* U$. Then the homotopy that makes the left square of the diagram commutative coincides, up to sign, with left multiplication by $h(U)$ composed with p^*. Indeed,

$$\begin{aligned}
p^* l_z &= l_{p^* z}\, p^* \\
&= l_{p^* s^* U}\, p^* \\
&= (l_U + l_{dh(U)}) p^* \\
&= l_U p^* + d l_{h(U)} p^* + (-1)^m l_{h(U)} p^* d.
\end{aligned}$$

The upper row of the diagram is not exact. But we can extend it to the morphism (up to homotopy) of two short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(B)[m+1] & \longrightarrow & \mathrm{cyl}(l_z) & \longrightarrow & \mathrm{cone}(l_z) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle p} & & \\
 & & C^*(B)[m+1] & \xrightarrow{l_z} & C^*(B) & \xrightarrow{q} & C^*(X) & & \\
 & & \downarrow{\scriptstyle l_U p^*} & & \downarrow{\scriptstyle p^*} & & \| & & \\
0 & \longrightarrow & C^*(E, X) & \longrightarrow & C^*(E) & \longrightarrow & C^*(X) & \longrightarrow & 0
\end{array}$$

where the upper row is the short exact sequence of the cylinder $\mathrm{cyl}(l_z)$ of the left multiplication l_z. The quasi-isomorphism τ is left inverse to the inclusion of $C^*(B)$ in $\mathrm{cyl}(l_z)$. We conclude with the stock remark that, up to a shift, the long exact sequence corresponding to the short exact sequence of the cylinder $\mathrm{cyl}(l_z)$ coincides with the long exact sequence corresponding to the short exact sequence of the cone. □

Combining this with Theorem 3.3, we obtain the following.

Theorem 4.3 *Let $S^m \to X \to B$ be an orientable sphere bundle, where the homology $H^*(B;k)$ of the base is a projective k-module. Let $c \in H^{m+1}(B)$ be its characteristic class and*

$$\in \mathrm{Ext}^1_{H^*(B)}(\mathrm{Ann}_{H^*(B)}(c)[m], H^*(B)/(cH^*(B)[m+1]))$$

the class of the Gysin extension

$$0 \to H^*(B)/(cH^*(B)[m+1]) \to H^*(X) \to \mathrm{Ann}_{H^*(B)}(c)[m] \to 0.$$

Then the map (14) sends the Hochschild cohomology class of (B) to the class .

5 Applications and extensions

Our information on the vanishing of the secondary multiplication class now leads to two types of results.

5.1 Rational coefficients and formal base

In the rational case, we may apply to formal spaces, as follows.

A space X is called (rationally) *formal* if there is an A_∞-algebra map [9] $H^*(X;\mathbb{Q}) \to C^*(X;\mathbb{Q})$ inducing an isomorphism in cohomology. According to [9], this is equivalent to triviality of the A_∞-structure on $H^*(X;\mathbb{Q})$. In particular (as can be seen directly from the definition), the secondary multiplication class $[\ (X;\mathbb{Q})]$ of a formal space vanishes.

Rational homotopy theory has provided many examples of formal spaces. Among them are simply-connected compact Kähler manifolds [2], complex smooth algebraic varieties [16], $(k-1)$-connected compact manifolds of dimension less than or equal to $4k-2$ (where $k \geq 2$) [15], some flag varieties [11], and some loop spaces [3, 17].

Theorem 5.1 *Let $S^m \to X \to B$ be an orientable sphere bundle having base B a formal space, and characteristic class $c \in H^{m+1}(B)$. Then the Gysin sequence with rational coefficients*

$$0 \to H^*(B)/(cH^*(B)[m+1]) \to H^*(X) \to \mathrm{Ann}_{H^*(B)}(c)[m] \to 0$$

splits as a sequence of right $H^*(B)$-modules.

5.2 Integer coefficients and torus base

Of course, rational formality does not in general imply formality over integer or finite coefficients (see, for example [4]). However, when the base is a torus T and one takes integer coefficients, because the homology is free abelian one can take advantage of the technical lemma (2.3).

Here we briefly indicate our calculation of the secondary multiplication class $[\mu(T;\mathbb{Z})]$ of an n-torus $T = \mathrm{T}^n = (A \otimes \mathbb{R})/(A \otimes 1)$; using the quasi-isomorphism between the dg-algebra of singular cochains $C^*(T)$ and the standard cochain dg-algebra (bar construction) $C^*(A)$ of the free abelian group A of rank n.

Since the graded $\mathbb{Z}$-module $C^*(A)$ is free and also $H^*(A)$ is free abelian, by (2.3) we have a linear section $q : B^*(A) = \mathrm{Im}(d) \to C^*(A)$ of the surjection $d : C^*(A) \to B^*(A)$ and a linear section $s : H^*(A;k) \to \mathrm{Ker}(d)$ of the natural projection. So, by Proposition 2.5, the class

$$[\mu(A)] \in HH^3_{\mathrm{gr}}(H^*(A); \overline{H^*(A)}[1])$$

is defined, and does not depend on the choices of sections q and s made above.

The cohomology $H^*(A)$ of the free abelian group A coincides with the algebra $\mathrm{Hom}(\Lambda^* A;\mathbb{Z})$ of alternating polylinear maps

$$\mathrm{Hom}(\Lambda^l A;\mathbb{Z}) = \left\{ x : A^l \to \mathbb{Z} \mid x(a_1,\dots,a_l) = 0 \text{ whenever } a_i = a_j\ i \neq j \right\}.$$

We can consider $\mathrm{Hom}(\Lambda^* A;\mathbb{Z})$ as the exterior algebra $\Lambda^* V$ of the $\mathbb{Z}$-module $V = A^\vee = \mathrm{Hom}(A;\mathbb{Z})$.

To describe the graded Hochschild cohomology $HH^l_{\mathrm{gr}}(R;R[1])$ of the exterior algebra $R = \Lambda^* V$, we start with construction of a map

$$HH^l_{\mathrm{gr}}(\Lambda^* V; \overline{\Lambda^* V}[1]) \to \mathrm{Hom}(S^l V; \Lambda^{l-1} V) \tag{16}$$

for each l, where $S^l V = (V^{\otimes l})_{S_l}$ is the module of coinvariants of the natural action of the symmetric group S_l on $V^{\otimes l}$. So, for an abelian group M, $\mathrm{Hom}(S^l V; M)$ can be identified with the abelian group of symmetric polylinear maps from V to M. Now the above map is obtained from a homomorphism of complexes induced from symmetrization, and can thereby be reduced to a homomorphism involving a Koszul-type complex as in [1](IX.6). Then, using induction on the rank of A, it can be shown to be a monomorphism. Explicit computation of the image of $[\ (A)]$ in $\mathrm{Hom}(S^3V,\ {}^2V)$, exploiting symmetrization, gives this image zero. Thus we obtain the desired conclusion.

Proposition 5.2 *The class* $[\ (A)] \in HH^3_{\mathrm{gr}}(H\ (A), \overline{H\ (A)}[1])$ *is trivial.* □

As a direct consequence of the quasi-isomorphism of dg-algebras between $C\ (T)$ and $C\ (A)$, we obtain the vanishing of the topological obstruction.

Corollary 5.3 *For a torus* T, *the class of* $\ (T)$ *in* $HH^3_{\mathrm{gr}}(H\ (T), \overline{H\ (T)}[1])$ *is trivial.* □

The vanishing of the class of $\ (T)$ gives the desired splitting.

Theorem 5.4 *Let* $S^m \to X \to T$ *be an orientable sphere bundle with torus base* T *and characteristic class* $c \in H^{m+1}(T)$. *Then the Gysin sequence with integer coefficients*

$$0 \to H\ (T)/(cH\ (T)[m+1]) \to H\ (X) \to \mathrm{Ann}_{H\ (T)}(c)[m] \to 0$$

splits as a sequence of right $H\ (T)$*-modules.* □

5.3 Extensions

Fibrations Although we have, for simplicity, presented our results in terms of sphere bundles, they are also applicable to the generalized Gysin sequence of an orientable fibration with fibre F a cohomology m-sphere [19](9.5.2). In the more general setting, E may be obtained by the fibrewise cone construction on the fibration $F \to X \xrightarrow{p_0} B$ [5](1.F). In order to repeat our argument on the singular cochain complexes for the pair (E, X), we need to show that the Gysin sequence is again the cohomology exact sequence of the pair (E, X). Chasing these two sequences, one observes that, because $E \to B$ is a homotopy equivalence and by assumption $H^{m+1}(B)$ is k-torsion-free, both of the groups $H^0(B)$ and $H^{m+1}(E, X)$ are isomorphic to the kernel of $p_0 : H^{m+1}(B) \to H^{m+1}(X)$ when the characteristic class in $H^{m+1}(B)$ is nonzero, and to the cokernel of $p_0 : H^m(B) \to H^m(X)$ otherwise. Corresponding to a generator of $H^0(B)$ we therefore obtain an orientation class u in $H^{m+1}(E, X)$. Since the two exact sequences coincide when B is restricted to any point, there results a cohomology extension of the fibre as in [19](5.7), and so a generalized Thom isomorphism $H^i(B) \to H^{i+m+1}(E, X)$ as required.

Bimodules Skew-commutativity of singular cohomology implies that our main results (Theorems 4.3, 5.4) are valid in the context of bimodules, and not only for right modules. Corresponding results of Section 3 can also be proven for this context; however, the proofs are much more involved and need some additional conditions for considering dg-algebras, such as stronger homotopy commutativity. That is the reason we restricted ourselves there to the case of right modules.

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Department of Mathematics
National University of Singapore
2 Science Drive 2, Singapore 117543, SINGAPORE
and
Department of Mathematics
Macquarie University, Sydney, NSW 2109
AUSTRALIA

Email: berrick@math.nus.edu.sg, davydov@ics.mq.edu.au

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