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Bihomogeneity of solenoids

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Abstract Solenoids are inverse limit spaces over regular covering maps of closed manifolds. M.C. McCord has shown that solenoids are topologically homogeneous and that they are principal bundles with a profinite structure group. We show that if a solenoid is bihomogeneous, then its structure group contains an open abelian subgroup. This leads to new examples of homogeneous continua that are not bihomogeneous.

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A topological space X is *homogeneous* if for every pair of points $x, y \in X$ there is a homeomorphism $h : X \to X$ satisfying $h(x) = y$. The space is *bihomogeneous* if for each such pair there is a homeomorphism satisfying $h(x) = y$ and $h(y) = x.$ A compact and connected space is called a *continuum*. Knaster and Van Dantzig asked whether a homogeneous continuum is necessarily bihomogeneous. This was settled in the negative by Krystyna Kuperberg [5]. Subsequent counterexamples were given by Minc, Kawamura and Greg Kuperberg [8, 2, 4]. The counterexamples in [5, 4] are locally connected. Ungar [15] has studied stronger types of homogeneity conditions and showed that these conditions imply local connectivity.

A solenoid M_∞ is an inverse limit space over closed connected manifolds with bonding maps that are covering maps. We shall silently assume that the bonding maps are not $1-1$, so that M_∞ is not locally connected. McCord [7] has shown that solenoids are homogeneous provided that compositions of the bonding covering maps are regular. Minc [8] presented an example of a homogeneous but not bihomogeneous infinite-dimensional continuum similar to a solenoid, and Krystyna Kuperberg [6] observed that a similar construction could be used to construct a finite-dimensional solenoid which is homogeneous but not bihomogeneous. We shall show that M_∞ is bihomogeneous only if a certain condition related to commutativity (or lack thereof) of $\pi_1(M_i)$ is met. In case the solenoid is 2-dimensional, the condition is both necessary and sufficient.

1 Path-components of solenoids as left-cosets of the structure group

A *(strong) solenoid* M_∞ is an inverse-limit space of closed manifolds M_i with bonding maps $p_i\colon M_{i+1} \to M_i$ for $i \in \mathbb{N}$ which are covering maps, such that any composition $p_{i+k} \circ \cdots \circ p_i$ is regular. Solenoids are homogeneous spaces and they have dense path-components.

A G-bundle $(E; B; p; F)$ is *principal* if the structure group G acts effectively on the fibers. As a consequence, the fiber F is homeomorphic to G, and G is naturally equivalent to the group of deck-transformations.

Theorem 1 (McCord, [7]) *Suppose that $M_\infty = \lim_\leftarrow (M_i; p_i)$ is a solenoid. Let $\pi_0\colon M_\infty \to M_0$ be the projection onto the first coordinate and let $\Gamma_0 = \pi_0^{-1}(m_0)$ be a fiber. Then $(M_\infty; M_0; \pi_0; \Gamma_0)$ is a principal-bundle.*

The projection π_0 is not to be confused with a homotopy group. Note that a solenoid $\lim_\leftarrow (M_i; f_i)$ is a principal bundle over any M_i and we have singled out M_0. The spaces M_i are called the *factor spaces* of the solenoid. We think of the fundamental groups $\pi_1(M_i)$ as (normal) subgroups of $\pi_1(M_0)$. The structure group Γ_0 is isomorphic to the profinite group $\lim_\leftarrow \pi_1(M_0)/\pi_1(M_i)$.

Choose base-points $m_i \in M_i$ such that $p_i(m_i) = m_{i-1}$, so $m_\infty = (m_i)$ is an element of M_∞. We identify the structure group Γ_0 with the fiber of m_0 and we identify m_∞ with the unit element of Γ_0. The fundamental group $\pi_1(M_0)$ acts on the base-point fiber Γ_0 by path lifting: for $g \in \Gamma_0$ and $\gamma \in \pi_1(M_0; m_0)$, define $g \cdot \gamma$ as the end-point of the lifted path $\tilde{\gamma}$ starting from the initial-point g. One verifies that this right action of $\pi_1(M_0)$ commutes with left multiplication of Γ_0. More precisely, suppose that h is a deck-transformation and that $\tilde{\gamma}$ is a lifted path with initial-point g. Then $h(\tilde{\gamma})$ has initial point $h(g)$ and end-point $h(g \cdot \gamma)$. Identify the structure group with the group of deck-transformations, so we get that $(hg) \cdot \gamma = h(g \cdot \gamma)$.

Definition 2 Suppose that $(M_\infty; M_0; \pi_0; \Gamma_0)$ is a solenoid. We shall call the $\pi_1(M_0)$-orbit of $e \in \Gamma_0$ the characteristic group and we shall denote it by γ_0. Let $K_\infty \subset \pi_1(M_0)$ be the intersection of all $\pi_1(M_i)$. Then γ_0 is isomorphic to $\pi_1(M_0)/K_\infty$ and we shall refer to K_∞ as the kernel of $\pi_1(M_0)$.

Our definition deviates from the common terminology, as in [14], where the equivalence class of γ_0 under inner automorphisms of Γ_0 is called the characteristic class. Note that γ_0 inherits a topology from Γ_0.

Lemma 3 *The path components of a solenoid are naturally equivalent to the left cosets* Γ_0/γ_0.

Proof Suppose that $x, y \in \Gamma_0$ are elements of the base-point fiber. Then $x \cdot \gamma = y$ for some $\gamma \in \pi_1(M_0)$ if and only if there exists a path $\gamma \subset M_\infty$ that connects x to y. ☐

If we replace the base space M_0 by M_i for some index i, then we get a principal bundle $(M_\infty, M_i, \pi_i, \Gamma_i)$, where $\Gamma_i \subset \Gamma_0$ is the subgroup of transformations that leave M_i invariant. The topology of Γ_0 is induced by taking the Γ_i as an open neighborhood base of the identity. One verifies that the charateristic group of the bundle, denoted γ_i, is equal to $\gamma_0 \cap \Gamma_i$. Hence the γ_i are open subgroups of γ_0.

Lemma 4 *For* $j > i$ *the inclusion* $\Gamma_j \subset \Gamma_i$ *induces a natural isomorphism between* Γ_j/γ_j *and* Γ_i/γ_i.

Since path components are dense in M_∞, the characteristic subgroups γ_i are dense in Γ_i.

2 The permutation of path-components by self-homeomorphisms

A solenoid M_∞ can be represented as a subspace of $\prod M_i$, the Cartesian product of its factor spaces. We identify M_i with the subspace of $\prod M_i$ defined by:

$$M_i = \{(x_j) : x_j \in M_j,\ x_j = p^i_j(x_i) \text{ if } j \le i,\ x_j = m_j \text{ if } j > i\}$$

where $p^i_j : M_i \to M_j$ is a composition of bonding maps. In this representation, the factor spaces M_i and M_∞ all have the same base-point.

A *morphism* between fiber bundles can be represented by a commutative diagram:

$$\begin{array}{ccc} E_1 & \xrightarrow{h} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

We shall say that h is the *lifted map* and that f is the *base-map*. We say that morphisms are homotopic if their base-maps are. By the unique path-lifting property, a morphism between bundles with a totally disconnected fiber is determined by the base-map $f\colon B_1 \to B_2$ and the image under h of a single element of E_1. For pointed spaces, therefore, a bundle-morphism is determined by the base-map only. This implies that, for principal bundles with a totally disconnected fiber, bundle-morphisms commute with deck-transformations; i.e., for a lifted map h and a deck-transformation $\varphi\colon E_1 \to E_1$, we have that $h \circ \varphi = \psi \circ h$ for some deck-transformation $\psi\colon E_2 \to E_2$.

Lemma 5 *Suppose that $(E_i, B_i, p_i, \Gamma_i)$ are principal Γ_i-bundles with a totally disconnected fiber (for $i = 1, 2$). Then a base-point preserving bundle-morphism induces a homomorphism of the structure group. Furthermore, homotopic morphisms induce the same homomorphism.*

Proof First note that the lifted map h maps Γ_1 to Γ_2. Deck-transformations are (left) translations $x \to ax$ of the base-point fiber Γ_i ($i = 1, 2$). Since a bundle-morphism commutes with deck-transformations, $h\colon \Gamma_1 \to \Gamma_2$ satisfies $h(ax) = f(a)h(x)$ for some $f\colon \Gamma_1 \to \Gamma_2$. Substitute $x = e$ to find that $h(ax) = h(a)h(x)$. Now homotopic bundle-morphisms give homotopic homomorphisms $h\colon \Gamma_1 \to \Gamma_2$. Since the groups are totally disconnected, the homomorphisms are necessarily the same. □

We shall say that a bundle morphism of a solenoid is an *automorphism* if the commutative diagram can be extended on the right-hand side

$$\begin{array}{ccccc} M_\infty & \xrightarrow{h_1} & M_\infty & \xrightarrow{h_2} & M_\infty \\ \pi_j \downarrow & & \downarrow \pi_i & & \downarrow \pi_k \\ M_j & \xrightarrow{f_1} & M_i & \xrightarrow{f_2} & M_k \end{array}$$

such that $f_2 \circ f_1$ is homotopic to p^j_k. We shall say that h_1 is the inverse of h_2. For instance, the covering projection $p^j_i\colon M_j \to M_i$ with lifted map id_{M_∞} yields an automorphism. We show that for every self-homeomorphism of a solenoid, there is an automorphism that acts in the same way on the space of path-components.

Theorem 6 *Suppose that h is a base-point preserving self-homeomorphism of a solenoid M_∞. Then h is homotopic to the lifted map of an automorphism of M_∞.*

Proof Since M_0 is an ANR, the composition $\pi_0 \circ h\colon M_\infty \to M_0$ extends to $H\colon U \to M_0$ for a neighborhood of $M_\infty \subset U$ in $\prod M_i$. The restriction $H\colon M_i \to M_0$ is well-defined for sufficiently large i. Note that H preserves the base-point of M_i. For sufficiently large i, the maps $H \circ \pi_i$ and $\pi_0 \circ h$ are homotopic. By the homotopy lifting property, $H \circ \pi_i$ can then be lifted to $\tilde{H}\colon M_\infty \to M_\infty$, which is homotopic to h.

Now apply the same argument to $\pi_i \circ h^{-1}$ to find a map $G\colon M_j \to M_i$ for sufficiently large j which can be lifted to $\tilde{G}\colon M_\infty \to M_\infty$. By choosing j and i sufficiently large, the composition $H \circ G\colon M_j \to M_0$ gets arbitrarily close to and hence homotopic to the covering map p^j_0. □

Theorem 6 and Lemma 5 describe how a self-homeomorphism acts on path-components of a solenoid (provided that it preserves the base-point).

Lemma 7 *Suppose that h is the lifted map of an automorphism of a solenoid M_∞. For some index i, h induces a monomorphism $\hat{h}\colon \Gamma_i \to \Gamma_0$ such that $\hat{h}^{-1}(\gamma_0) = \gamma_i$ and $\hat{h}(\Gamma_i)$ is an open subgroup of Γ_0.*

Proof By Lemma 5 we know that h induces a homomorphism $\hat{h}\colon \Gamma_i \to \Gamma_0$. Since homeomorphisms preserve path-components, Lemma 3 implies that h induces a homomorphism $\Gamma_i/\gamma_i \to \Gamma_0/\gamma_0$. Since h is the lifted map of an automorphism, it has an inverse g which induces a homomorphism $\hat{g}\colon \Gamma_j \to \Gamma_0$. The composition $\hat{g} \circ \hat{h}$, which is defined on an open subgroup, is equal to the identity. By Lemma 5, $\hat{g} \circ \hat{h}$ is equal to the homomorphism induced by p^j_i, which is the identity. □

3 An algebraic condition for bihomogeneity

Definition 8 Suppose that Γ_0 is the structure group of a solenoid with characteristic group γ_0. We define $\mathrm{Mon}(\Gamma_0; \gamma_0)$ as the set of monomorphisms $f\colon \Gamma_i \to \Gamma_0$, such that $f(\gamma_i) = \gamma_0 \cap f(\Gamma_i)$.

We say that an element of $\mathrm{Mon}(\Gamma_0; \gamma_0)$ is a *characteristic automorphism*. A self-homeomorphism H of M_∞ need not preserve the base-point. It can however be represented as a composition of a homeomorphism h that preserves the path-component of the base-point and a deck-transformation. Since h is homotopic to a base-point preserving homeomorphism, H permutes the path-components in the same way as a composition of a base-point preserving homeomorphism and a deck-transformation. In terms of Γ_0 / y_0, this is a composition of a characteristic automorphism φ and a left translation $z \to wz$ of Γ_0.

Definition 9 We say that a solenoid is *algebraically bihomogeneous* if it satisfies the following condition. For every $x, y \in \Gamma_0$ there are elements $w \in \Gamma_0$ and $\varphi \in \mathrm{Mon}(\Gamma_0; y_0)$ such that $z \to w\varphi(z)$ switches the residue classes $x \bmod y_0$ and $y \bmod y_0$.

Obviously, bihomogeneity implies algebraic bihomogeneity. The condition of algebraic bihomogeneity may seem awkward, but fortunately there is a simpler characterization as we shall see below. We denote $x \sim y$ if x, y are in the same residue class of y_0.

Lemma 10 *A solenoid M_∞ is algebraically bihomogeneous if and if only for every $z \in \Gamma_0$ there is a characteristic automorphism φ such that $\varphi(z) \sim z^{-1}$.*

Proof Suppose that M_∞ is algebraically bihomogeneous. For every $z \in \Gamma_0$, we can switch the cosets of z and e. More precisely, there exists a $w \in \Gamma_0$ and a $\varphi \in \mathrm{Mon}(\Gamma_0; y_0)$ such that $zg = w\varphi(e)$ and $eg' = w\varphi(z)$ for $g, g' \in y_0$. Since $\varphi(e) = e$; it follows that $w = zg$ and $\varphi(z) = g^{-1}z^{-1}g'$. Compose φ with the inner automorphism $x \to gxg^{-1}$ to obtain $\psi \in \mathrm{Mon}(\Gamma_0; y_0)$ satisfying $\psi(z) \sim z^{-1}$:

If $\varphi(z) = z^{-1}g$ for some $g \in y_0$, then compose φ with the inner automorphism $x \to gxg^{-1}$ to get $\psi \in \mathrm{Mon}(\Gamma_0; y_0)$ satisfying $\psi(z) = gz^{-1}$. Then $x \to zg^{-1}\psi(x)$ switches the cosets of e and z. This implies algebraic bihomogeneity. □

Since $z \to z^{-1}$ is a homomorphism if and only if the group is abelian, we have the following corollary.

Corollary 11 *A solenoid with an abelian structure group Γ_i is algebraically bihomogeneous.*

This condition is automatically met if $\pi_1(M_i)$ is abelian.

Lemma 12 *Suppose that y_0 is a characteristic group. Then* $\mathrm{Mon}(\Gamma_0; y_0)$ *is countable.*

Proof There are only countably many subgroups γ_i and each of these is finitely generated. Hence, there are only finitely many homomorphisms $f\colon \gamma_i \to \gamma_0$. Since characteristic automorphisms are determined by their action on some γ_i, the result follows. □

Theorem 13 *Let M_∞ be a bihomogeneous solenoid with structure group Γ_0. Then Γ_0 contains an open abelian subgroup.*

Proof Suppose that $\varphi\colon \Gamma_j \to \Gamma_0$ is a characteristic automorphism. For $g \in \gamma_0$ define the subset $V(\varphi, g) = \{z \in \Gamma_j : z\varphi(z) = g\} \subset \Gamma_0.$ As φ ranges over $\mathrm{Mon}(\Gamma_0, \gamma_0)$ and g ranges over γ_0, the countable family of all $V(\varphi, g)$ covers Γ_0 by Lemma 10. Hence one of these sets, say $V(\varphi_0, g_0)$, is of second category in Γ_0. It follows that $K = \{z \in \Gamma_0 : z\varphi_0(z) = g_0\}$ is closed with non-empty interior in $\Gamma_0.$ Since K has non-empty interior, there exist a $z_0 \in K$ and a neighborhood V of e such that $\varphi(z_0\eta) = \eta^{-1}z_0^{-1}g_0$ for all $\eta \in V$. It follows that $\varphi(\eta) = g_0^{-1}z_0\eta^{-1}z_0^{-1}g_0$. By composition with the inner automorphism $x \to z_0^{-1}g_0xg_0^{-1}z_0,$ we get a homomorphism ψ such that $\psi(\eta) = \eta^{-1}$ for $\eta \in V$. The group generated by V is an open abelian subgroup of Γ_0. □

For any neighborhood V of the identity, $\gamma_j \subset V$ for large enough j. Hence there exists an open abelian subgroup of Γ_0 if and only if γ_j is abelian for some $j.$

Corollary 14 *Let M_∞ be a solenoid and let K_∞ be the kernel of $\pi_1(M_0)$. Then M_∞ is algebraically bihomogeneous if and only if $\pi_1(M_j)/K_\infty$ is abelian for sufficiently large index j, or, equivalently, Γ_j is abelian for sufficiently large index j.*

4 An application

Our algebraic condition for (topological) bihomogeneity in Corollary 14 is necessary but not sufficient. For this, there should exist a homeomorphism $h\colon M_i \to M_i$ which induces an isomorphism $h_*\colon \pi_1(M_i) \to \pi_1(M_i)$ such that $h_*(x) = x^{-1}$ (modulo K_∞). The problem whether homomorphisms between fundamental groups are realized by continuous maps is known as the geometric realization problem. It is a classical result of Nielsen [9] that closed surfaces admit geometric realizations. This can be extended to certain three-dimensional manifolds [16]. The following result now follows from Nielsen's theorem.

Theorem 15 *A two-dimensional solenoid S_∞ with kernel $K_\infty \subset \pi_1(S_0)$ is bihomogeneous if and only if $\pi_1(S_i)/K_\infty$ is abelian for sufficiently large index i.*

One easily constructs two-dimensional solenoids that are not bihomogeneous, using results from geometric group theory. The fundamental group $\pi_1(S)$ of a closed surface is subgroup separable, see [13]; i.e., for every subgroup $H \subset \pi_1(S)$ there is a descending chain of subgroups of finite index with kernel H. Hence, there exists a solenoid with base-space S and kernel H. For a closed surface S of genus greater than 1, the fundamental group contains no abelian subgroup of finite index. Therefore, a solenoid with base-space S and kernel $\{e\}$ is a (simply-connected) continuum which is not bihomogeneous.

5 Final remarks

One-dimensional solenoids are indecomposable continua. It is not difficult to show that higher-dimensional solenoids are not. Rogers [10] has shown that a homogeneous, hereditarily indecomposable continuum is at most one-dimensional. His question whether there exists a homogeneous, indecomposable continuum of dimension greater than one remains open.

Our example of a non-bihomogeneous space is based on obstructions of the fundamental group, which seems to be characteristic for all examples so far. So it is natural to ask whether there exists a simply-connected Peano continuum that is homogeneous but not bihomogeneous. More generally, it is natural to ask whether there exists a continuum with trivial first Čech cohomology that is homogeneous but not bihomogeneous.

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