ISSN 1472-2739 (on-line) 1472-2747 (printed)

Algebraic & Geometric Topology
Volume 2 (2002) 1147–1154
Published: 19 December 2002

Equivalences to the triangulation conjecture

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Abstract We utilize the obstruction theory of Galewski-Matumoto-Stern to derive equivalent formulations of the Triangulation Conjecture. For example, every closed topological manifold M^n with $n \geq 5$ can be simplicially triangulated if and only if the two distinct combinatorial triangulations of RP^5 are simplicially concordant.

AMS Classification 57N16, 55S35; 57Q15

Keywords Triangulation, Kirby-Siebenmann class, Bockstein operator, topological manifold

1 Introduction

The Triangulation Conjecture (TC) affirms that every closed topological manifold M^n of dimension $n \geq 5$ admits a simplicial triangulation. The vanishing of the Kirby-Siebenmann class $KS(M)$ in $H^4(M;Z/2)$ is both necessary and sufficient for the existence of a combinatorial triangulation of M^n for $n \geq 5$ by [7]. A combinatorial triangulation of a closed manifold M^n is a simplicial triangulation for which the link of every i-simplex is a combinatorial sphere of dimension $n-i-1$. Galewski and Stern [3, Theorem 5] and Matumoto [8] independently proved that a closed connected topological manifold M^n with $n \geq 5$ is simplicially triangulable if and only if

$$\delta KS(M) = 0 \quad \text{in} \quad H^5(M; \ker \mu) \qquad (1.1)$$

where δ denotes the Bockstein operator associated to the exact sequence $0 \to \ker \mu \to \Theta^H_3 \xrightarrow{\mu} Z/2 \to 0$ of abelian groups. Moreover, the Triangulation Conjecture is true if and only if this exact sequence splits by [3] or [11, page 26]. The Rochlin invariant morphism μ is defined on the homology bordism group Θ^H_3 of oriented homology 3-spheres modulo those which bound acyclic compact PL 4-manifolds. Fintushel and Stern [1] and Furuta [2] proved that Θ^H_3 is infinitely generated.

We freely employ the notation and information given in Ranicki's excellent exposition [11]. The relative boundary version of the Galewski-Matumoto-Stern obstruction theory in [11] produces the following result. Given any homeomorphism $f : |K| \to |L|$ of the polyhedra of closed m-dimensional PL manifolds K and L with $m \geq 5$, f is homotopic to a PL homeomorphism if and only if $KS(f)$ vanishes in $H^3(L;\mathbb{Z}/2)$. More generally, a homeomorphism $f : |K| \to |L|$ is homotopic to a PL map $F : K \to L$ with acyclic point inverses if and only if

$$(1.2) \qquad (KS(f)) = 0 \quad \text{in} \quad H^4(L; \ker\ ) .$$

Concordance classes of simplicial triangulations on M^n for $n \geq 5$ correspond bijectively to vertical homotopy classes of liftings of the stable topological tangent bundle $: M \to$ BTOP to BH by [3, Theorem 1] and so are enumerated by $H^4(M; \ker\ )$. The classifying space BH for the stable bundle theory associated to combinatorial homology manifolds in [11] is denoted by BTRI in [3] and by BHML in [8]. We employ obstruction theory to derive some known and new results and generalizations of [4] and [13] on the existence of simplicial triangulations in section 2 and to record some equivalent formulations of TC in section 3. Although some of these formulations may be known, they do not seem to be documented in the literature.

2 Simplicial Triangulations

Let denote the integral Bockstein operator associated to the exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$. We proceed to derive some consequences of the vanishing of on Kirby-Siebenmann classes. The coefficient group for cohomology is understood to be $\mathbb{Z}/2$ whenever omitted. Matumoto knew in [8] that the vanishing of $KS(M)$ implied the vanishing of $KS(M)$. Let $_m$ denote the fundamental class of the Eilenberg-MacLane space $K(\mathbb{Z}; m)$. Since $H^{m+1}(K(\mathbb{Z}; m); G) = 0$ for all coefficient groups G, trivially $(_m) = 0$ in $H^{m+1}(K(\mathbb{Z}; m); \ker\ )$. Thus vanishes on $KS(M)$ in (1.1) or $KS(f)$ in (1.2) whenever does. This observation together with (1.1) and (1.2) justifies the following well-known statements. Every closed connected topological manifold M^n with $n \geq 5$ and $KS(M) = 0$ admits a simplicial triangulation. Let $f : |K| \to |L|$ be any homeomorphism of the polyhedra of closed m-dimensional PL manifolds K and L with $m \geq 5$. If $KS(f) = 0$, then f is homotopic to a PL map $F : K \to L$ with acyclic point inverses.

Proposition 2.1 *All k-fold Cartesian products of closed 4-manifolds are simplicially triangulable for $k \geq 2$. All products $M^4 \times S^1$ with non-orientable closed*

4-manifolds M^4 are simplicially triangulable. Let N^4 be any simply connected closed 4-manifold with $KS(N)$ trivial and also $b = \text{rank of } H_2(N;Z) \geq 1$. Let $f : |K| \to |L|$ be any homeomorphism with $KS(f)$ nontrivial and $|K| = |L| = N \times S^1$. Then f is homotopic to a PL map $F : K \to L$ with acyclic point inverses.

Proof of 2.1 Since $KS(\gamma)$ is a primitive cohomology class for the universal bundle γ on BTOP, we have $KS(M_1 \times M_2) = KS(M_1) \otimes 1 + 1 \otimes KS(M_2)$ in $H^4(M_1 \times M_2)$. Triviality of β on $H^4(M^4)$ by dimensionality yields triangulability of all k-fold products of closed 4-manifolds for $k \geq 2$, and of $M^4 \times S^1$ by (1.1).

The product $N^4 \times S^1$ admits 2^b distinct combinatorial structures by [7]; moreover, for every non-zero class u in $H^3(N \times S^1)$, there is a homeomorphism of polyhedra with distinct combinatorial structures whose Casson-Sullivan invariant is u by [11, page 15]. The vanishing of $\beta KS(f)$ follows from the triviality of β on $H^3(N \times S^1) = \beta(H^2(N;Z) \otimes H^1(S^1;Z))$. □

No closed 4-manifold M^4 with $KS(M)$ non-zero can be simplicially triangulated. Yet k-fold products of such manifolds M^4 by (2.1) and their products with spheres or tori produce infinitely many distinct non-combinatorial, yet simplicially triangulable closed manifolds in every dimension ≥ 5. In contrast, there are no known examples of non-smoothable closed 4-manifolds which can be simplicially triangulated, according to Problem 4.72 of [6, page 287].

Theorem 2.2 *Let M^n be any closed connected topological manifold with $n \geq 5$ such that the stable spherical fibration determined by the tangent bundle $\tau(M)$ has odd order in $[M; BSG]$. Suppose that either $H_2(M;Z)$ has no 2-torsion or else all 2-torsion in $H_4(M;Z)$ has order 2. Then M is simplicially triangulable.*

Proof The Stiefel-Whitney classes of M are trivial by the hypothesis of odd order. We first consider the special case that $\tau(M)$ is stably fiber homotopically trivial. Let $g : M \to SG/STOP$ be any lifting of a classifying map $\tau(M) : M \to$ BSTOP in the fibration

$$(2.3) \qquad SG/\mathrm{STOP} \xrightarrow{j} \mathrm{BSTOP} \longrightarrow \mathrm{BSG}$$

The Postnikov 4-stage of $SG/$STOP is $K(Z/2;2) \times K(Z;4)$. Now $j^* KS(\gamma) = \iota_2^2 + \beta(\iota_4)$ by Theorem 15.1 of [7, page 328] where γ denotes the universal bundle over BSTOP. Clearly $\beta(j^* KS(\gamma)) = \beta(\iota_2^2) = 2u$ where u generates $H^5(K(Z/2;2);Z) \cong Z/4$. If all nonzero 2-torsion in $H_4(M;Z)$ has order 2, then $KS(M) = 2g\ u = 0$. If $H_2(M;Z)$ has no 2-torsion, then $(g\ _2) = 0$ so again $KS(M) = 0$. Thus $KS(M) = 0$.

We suppose now that the stable spherical bration of (M) has order $2a + 1$ in $[M;BSG]$ with $a\ \ 1$. Let $s : M\ !\ S(2a\ (M))$ be a section to the sphere bundle projection $p : S(2a\ (M))\ !\ M$ associated to $2a\ (M)$. Now $S(2a\ (M))$ is a stably ber homotopically trivial manifold, since its stable tangent bundle is $(2a + 1)p\ \ (M)$. Since $KS(M) = (2a + 1)KS(M) = s\ (KS(S(2a\ (M))))$ we conclude that

$$(2{:}4) \qquad KS(M) = s\ (\ KS(S(2a\ (M)))) = s\ 0 = 0\ :$$

We consider the following homotopy commutative diagram of principal brations.

$$
(2{:}5) \qquad
\begin{array}{ccccccccc}
 & & K(\ker\ ;4) & \xrightarrow{i} & (K(\ker\ ;4);\ ) & = & (K(\ker\ ;4);\ ) \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow i \\
 & & BH & \xrightarrow{i} & (BH;BPL) & \xrightarrow{t} & (K(\ _3;4);\ ) \\
 & & \big\downarrow & & \big\downarrow \wedge & & \big\downarrow \\
S^4 & \xrightarrow{ks} & BTOP & \xrightarrow{i} & (BTOP;BPL) & \xrightarrow{\widehat{KS}} & (K(Z{=}2;4);\ ) \\
 & & & & \big\downarrow \widehat{KS} & & \big\downarrow \\
 & & & & (K(\ker\ ;5);\ ) & = & (K(\ker\ ;5);\ )
\end{array}
$$

The ber map is induced from the path-loop bration on $K(\ker\ ;5)$ via the Bockstein operator on the fundamental class of $K(Z{=}2;4)$. The induced morphism on $_4$ is the Rochlin morphism $:\ _3\ !\ Z{=}2$ by construction. The relative principal bration $\wedge$ is induced from via the map $\widehat{KS}$ classifying the relative universal Kirby-Siebenmann class. Thus $(\widehat{KS}\ \ i) = KS(y)$. Inclusion maps are denoted by i in (2.5). The induced morphisms t and $(\widehat{KS})$ are isomorphisms on $_4$. We employ (2.5) in the proof of Theorem 3.1.

3 Equivalent formulations to TC

Galewski and Stern constructed a non-orientable closed connected 5-manifold M^5 in [4] such that $Sq^1KS(M)$ generates $H^5(M)\ \ Z{=}2$. They also proved that any such M^5 is "universal" for TC. Moreover, Theorem 2.1 of [4] essentially a rms that either TC is true or else no closed connected topological n-manifold M^n with $Sq^1KS(M) \neq 0$ and $n\ \ 5$ can be simplicially triangulated.

Theorem 3.1

The following statements are equivalent to the Triangulation Conjecture.

1. *Any (equivalently all) of the classes* $KS(\gamma)$, $\widetilde{ks}$*, and* *in (2.5) is trivial if and only if any (equivalently all) of the fiber maps* , $\hat{}$*, and* *in (2.5) admits a section.*
2. *The essential map* $f : S^4 \cup_2 e^5 \to$ BTOP *lifts to* BH *in (2.5).*
3. $Sq^1 KS(\tilde{\gamma}) \neq 0$ *in* $H^5(BH)$ *for the universal bundle* $\tilde{\gamma} =$ γ *on* BH.
4. *Any closed connected topological manifold* M^n *with* $Sq^1 KS(M) \neq 0$ *and* n 5 *admits a simplicial triangulation.*
5. *Every homeomorphism* $f : |K| \to |L|$ *with* $KS(f)$ *non-trivial is homotopic to a* PL *map with acyclic point inverses where* K *and* L *are any combinatorially distinct polyhedra with* $|K| = |L| = N^4$ RP^2. *Here* N^4 *denotes any simply connected, closed 4-manifold with* $KS(N)$ *trivial and positive rank for* $H_2(N; Z)$.
6. *All combinatorial triangulations of each closed connected* PL *manifold* M^n *with* n 5 *are concordant as simplicial triangulations.*
7. *The two distinct combinatorial triangulations of* RP^5 *are simplicially concordant.*
8. *Every closed connected topological manifold* M^n *with* n 5 *that is stably fiber homotopically trivial admits a simplicial triangulation.*

Proof TC , (1) Statement (1) is equivalent to the splitting of the exact sequence $0 \to \ker \to {}_3 \longrightarrow Z/2 \to 0$ through the induced morphisms on homotopy in dimension 4.

TC , (2) Let $ks : S^4 \to$ BTOP represent the Kirby-Siebenmann class in homotopy. That is, $[ks]$ has order 2 and is dual to $KS(\gamma)$ under the mod 2 Hurewicz morphism. Now ks admits an extension $f : S^4 \cup_2 e^5 \to$ BTOP, since the cofibration exact sequence

$$(3.2) \qquad {}_5(\mathrm{BTOP}) \longrightarrow [S^4 \cup_2 e^5; \mathrm{BTOP}] \to {}_4(\mathrm{BTOP}) \xrightarrow{2} {}_4(\mathrm{BTOP})$$

corresponds to $0 \longrightarrow Z/2 \longrightarrow Z \quad Z/2 \xrightarrow{2} Z \quad Z/2$. If $g : S^4 \cup_2 e^5 \to BH$ is any lifting of f, the composite map using (2.5)

$$(3.3) \qquad h : S^4 \quad S^4 \cup_2 e^5 \xrightarrow{g} BH \xrightarrow{i} (BH; BPL) \xrightarrow{t} (K({}_3; 4); \ )$$

produces $u = [h]$ in ${}_3$ with $2u = 0$ and $(u) = 1$, since $(u) = [\quad h] = [\widetilde{ks} \quad ks]$ generates ${}_4(K(Z/2; 4))$. Thus TC is true. Conversely, if TC is true, a section $s :$ BTOP $\to$ BH to in (2.5) gives a lifting $s \quad f$ of f.

TC , (3) Properties of $KS(y)$ are enumerated in [9] and [10]. Since $Sq^1KS(y) \neq 0$, a section s to in (2.5) gives $Sq^1(KS(\gamma)) \neq 0$ so TC implies 3. We now assume that TC is false and claim that the generator Sq^1 for $H^5(K(Z/2;4))$ $Z/2$ lies in the image of

$$H^5(K(\ker\ ;5)) \quad Hom(\ _5(K(\ker\ ;5));Z/2) \quad Hom(\ker\ ;Z/2):$$

The Serre exact sequence then gives $(Sq^1\ ) = 0$ in $H^5(K(\ _3;4))$ so

$$Sq^1KS(\gamma) = (t \quad i)\ (\quad Sq^1\ ) = 0:$$

Thus we must construct a morphism $\ker \to Z/2$ which does not extend to $_3$. We consider the sequence $\ker \xrightarrow{2} \ker \longrightarrow \ker \quad Z/2$ and define $h : \ker \quad Z/2 \to Z/2$ as follows. $h(v) = 1$ if and only if $v = (2z)$ for some $z \in \ _3$ with $(z) = 1$. Now h is a well-defined and non-trivial morphism, since $_3$ does not have an element u with $2u = 0$ and $(u) = 1$ by hypothesis. The composite morphism h : $\ker \to Z/2$ does not extend to $_3$.

TC , (4) Suppose M^n with $Sq^1KS(M) \neq 0$ admits a simplicial triangulation. Now $Sq^1KS(M) = g\ Sq^1KS(\gamma)$ for any lifting $g : M \to BH$ of $: M \to$ BTOP. Since $Sq^1KS(\gamma) \neq 0$, TC holds by (3).

TC , (5) Clearly triviality of $K\!S$ in (2.5) gives $KS(f) = 0$ via naturality for every f. Suppose that $KS(f) = 0$ for any such f in 5. Now $KS(f) = (v) \quad i\ a$ in $(H^2(M;Z)) \quad H^1(RP^2) \quad H^3(L)$. Here a generates $H\ (RP^{\infty})$ and $i : RP^2 \quad RP^{\infty}$. Naturality via the universal example $CP^{\infty} \quad RP^{\infty}$ for $(v) \quad i\ a$ gives $KS(f) = v \quad (i\ a)$. Since $i : H^2(RP^{\infty};\ker\ ) \to H^2(RP^2;\ker\ )$ is a monomorphism, $(i\ a) = 0$ if and only if $(a) = 0$. Now $(a) = 0$ if and only if TC is true via the fibration

$$K(\ker\ ;1) \longrightarrow K(\ _3;1) \longrightarrow RP^{\infty}:$$

TC , (6) , (7) TC holds if and only if $= 0$ for the fundamental class of $K(Z/2;3)$. Concordance classes of simplicial triangulations of M^n arising from combinatorial triangulations differ by classes in $H^3(M)$. This subgroup of $H^4(M;\ker\ )$ is trivial by naturality if $= 0$. Conversely, $H^3(RP^5) = 0$ if the two distinct combinatorial triangulations of RP^5 given by Theorem 16.5 in [7, pages 332 and 337] are simplicially concordant. But $(a^3) = 0$ if and only if $= 0$ via the skeletal inclusion $RP^5_3 \quad K(Z/2;3)$ and naturality for $RP^5 \to RP^5_3$.

TC , (8) Similar to Theorem 5.1 of [12], we consider a regular neighborhood of the 9-skeleton of $SG/$STOP embedded in R^m for some $m \quad 19$ in order to obtain a smoothly parallelizable manifold W with boundary and a map $g : W \to SG/\mathrm{STOP}$ which is a homotopy equivalence through dimension 7. The double DW is smoothly parallelizable and admits an extension $\bar{g} : DW \to SG/\mathrm{STOP}$. Note that $(\bar{g})$ is a monomorphism through dimension 7. Let $h : M \to DW$ be a degree one normal map. Now M is stably fiber homotopically trivial and h is a monomorphism in cohomology. In particular, $(\bar{g} \; h)$ is a monomorphism on $H^5(SG/\mathrm{STOP}; \ker \;)$. We conclude that $KS(M) = (\bar{g} \; h)(\; {}^2_2) = 0$ if and only if ${}^2_2 = 0$ for the fundamental class ${}_2$ of $K(Z/2;2)$. So statement (8) yields ${}^2_2 = 0$.

Let $f : K(Z/2;2) \to K(Z/2;4)$ classify ${}^2_2$. Since ${}^2_2 = 0$ assuming statement (8), f admits a lifting $h : K(Z/2;2) \to K({}_3;4)$ in (2.5) such that $f = \; h$. The diagram

$$(3.4) \qquad \begin{array}{ccccccc} & & & & [CP^3; K({}_3;4)] & & {}^3 \\ & & & {}^{h}\nearrow & \Big\downarrow & & \Big\downarrow \\ Z/2 & & [CP^3; K(Z/2;2)] & \xrightarrow{f} & [CP^3; K(Z/2;4)] & & Z/2 \end{array}$$

yields a splitting to the exact sequence $0 \to \ker \; \to {}_3 \to Z/2 \to 0$ so TC holds.

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Received: 19 July 2002