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Ideal triangulations of 3–manifolds II; taut and angle structures

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Abstract This is the second in a series of papers in which we investigate ideal triangulations of the interiors of compact 3–manifolds with tori or Klein bottle boundaries. Such triangulations have been used with great effect, following the pioneering work of Thurston [22]. Ideal triangulations are the basis of the computer program SNAPPEA of Weeks [3] and the program SNAP of Coulson, Goodman, Hodgson and Neumann [4]. Casson has also written a program to find hyperbolic structures on such 3–manifolds, by solving Thurston's hyperbolic gluing equations for ideal triangulations. In this second paper, we study the question of when a taut ideal triangulation of an irreducible atoroidal 3–manifold admits a family of angle structures. We find a combinatorial obstruction, which gives a necessary and sufficient condition for the existence of angle structures for taut triangulations. The hope is that this result can be further developed to give a proof of the existence of ideal triangulations admitting (complete) hyperbolic metrics. Our main result answers a question of Lackenby. We give simple examples of taut ideal triangulations which do not admit an angle structure. Also we show that for 'layered' ideal triangulations of once-punctured torus bundles over the circle, that if the manodromy is pseudo Anosov, then the triangulation admits angle structures if and only if there are no edges of degree 2. Layered triangulations are generalizations of Thurston's famous triangulation of the Figure–8 knot space. Note that existence of an angle structure easily implies that the 3–manifold has a CAT(0) or relatively word hyperbolic fundamental group.

AMS Classification 57M25; 57N10

Keywords Normal surfaces, 3–manifolds, ideal triangulations, taut and angle structures

1 Introduction

We will work in the smooth category. For simplicity, all 3–manifolds M will be the interior of compact manifolds N with tori or Klein bottle boundary components.

A map $f\colon T \to M$ from a surface T into M is called π_1–injective if the induced map $\pi_1(T) \to \pi_1(M)$ is one-to one. By an abuse of notation, we will call T (or $f(T)$) incompressible, if f is a π_1–injective embedding. We will suppose throughout that all the boundary components of N are incompressible.

All 3–manifolds will be assumed to be irreducible and $\mathbf{P}^2$–irreducible, ie, every embedded 2–sphere bounds a 3–ball and there are no embedded 2–sided projective planes. Such a 3–manifold M will be called atoroidal, if given any π_1–injective map $f\colon T \to M$ from a torus or Klein bottle into M, f is homotopic to a map into one of the boundary components of N. Any surface or map which is homotopic into a boundary surface of N will be called peripheral.

For basic 3–manifold theory, see either [9] or [10].

An ideal triangulation Γ of M will be a cell complex which is a decomposition of M into tetrahedra $\Delta_1, \Delta_2, ..., \Delta_k$ glued along their faces and edges, so that the vertices of the tetrahedra are all removed. Moreover the link of each such missing vertex will be a Klein bottle or torus. We call these links the peripheral surfaces of M. Note that tetrahedra may have faces and edges self-identified. Using Moise's construction of triangulations of 3–manifolds [18], one can convert a triangulation of N into such an ideal triangulation, by collapsing the boundary surfaces to ideal vertices and also collapsing edges which join the ideal vertices to the interior vertices. See [11] for a discussion of such collapsing procedures. One has to ensure that at each stage of such collapsings, that the topological type of M does not change.

We now summarize Haken's theory of normal surfaces [7], as extended by Thurston to deal with spun normal surfaces in ideal triangulations (see also [13] and [14]). Given an abstract tetrahedron Δ with vertices $ABCD$, there are four normal triangular disk types, cutting off small neighborhoods of each of the four vertices. There are also three normal quadrilateral disk types, which separate pairs of opposite edges, such as AB, CD. Each tetrahedron Δ_i of Γ contributes 7 coordinates which are the numbers n_j of each of the normal disk types. We can form a vector of integers of length $7k$ from a list of these coordinates n_j, $1 \le j \le 7k$.

A normal surface S is formed by gluing finitely many normal disk types together and its coordinate vector is denoted by $[S]$. $[S]$ is called the normal class of S. There are $6k$ compatibility equations for the coordinates of a normal surface, each of the form $n_i + n_j = n_m + n_p$, where the left side of the equation gives the number of normal triangles and quadrilaterals with a particular normal arc type in the boundary, eg, the arc running between edges AB and AC in Δ. If the face ABC is glued to $A'B'C'$ of the tetrahedron Δ', then n_m, n_p are the number of normal triangles and quadrilaterals with the boundary normal arc type running between $A'B'$ and $A'C'$ in Δ'. Note that we allow self identifications of tetrahedra and hence also of normal disk types. Note that normal surfaces may be embedded, immersed or branched.

It turns out that the solution space $\mathcal{V}$ of these compatibility equations in $\mathbf{R}^{7k}$ has dimension $2k$, ie, there are k redundant compatibility equations. The non-negative integer solutions in $\mathcal{V}$ are then normal surfaces and we can regard $2k$ as the dimension of the space of these surfaces. For a proof, see [15]. Also in [15], the dimension of the space $\mathcal{W}$ of spun and ordinary normal surfaces is computed. In fact, if c is the number of tori and Klein bottle boundary components of N, then it is shown there that the dimension of $\mathcal{W}$ is $2k+c$.

In an ideal triangulation Γ, a spun normal surface S is formed by gluing infinitely many normal disk types together. By definition, there are finitely many quadrilaterals and infinitely many triangular disks in such a spun normal surface. A connected neighborhood (in S) of these quadrilaterals can be formed by adding finite regions of triangles, yielding a compact core $\mathbf{C}$ of S. Then the closure of $S\setminus\mathbf{C}$ is a collection of non-compact triangular regions of S. It is easy to see that these regions must then all be half open annuli. The reason is that any such region projects onto a boundary surface of N, which becomes a triangulated Klein bottle or torus, when pushed into M as a normal surface. The projection is locally one-to-one and so the region must be an annulus winding around the boundary surface.

Now to form a vector space $\mathcal{W}$ of spun and ordinary normal surfaces S, we will consider only the quadrilateral coordinates of each S. So $\mathcal{W}$ will be a subspace of $\mathbf{R}^{3k}$. This idea has been studied previously in [24], in the case of ordinary normal surface theory in standard (closed) triangulations and is called $\mathbf{Q}$ normal surface theory. For spun normal surfaces, $\mathbf{Q}$ theory has been investigated in [13] and [14]. There are k compatibility equations for the quadrilaterals and in [15], it is shown there are c redundancies. In an ideal triangulation, the solutions to these equations are naturally either normal or spun normal surfaces. The only surfaces which are not 'seen' by this theory, are the boundary Klein bottles and tori, formed entirely of triangular disk types. If we added these in also, the theory would have dimension $2k+2c$. However these boundary surfaces play no significant role, so it is reasonable to leave them out of consideration. Spun normal surfaces have been used in an interesting way by Stefan Tillmann [23], to study essential splitting surfaces arising from representation varieties in Culler–Shalen theory.

Finally we briefly discuss the theory of generalized almost normal surfaces, which turns out to give an elegant way of describing the combinatorial obstruction to deform a taut structure into an angle structure. The disks of generalized almost normal surface theory are properly embedded in the tetrahedra and have boundary loops consisting of normal arcs. It is an elementary exercise to check that such loops can be described as the boundary of a regular neighborhood of an embedded arc in the boundary of a tetrahedron, where the latter arc runs between two vertices and consists of normal arcs plus two arcs from the vertices to interior points of edges not containing the vertices (see Figure 1). As a consequence, every such disk, which is not a triangle or quadrilateral, has length $4k$ and we will refer to it as a $4k$–gon. We will find an interesting connection between these $4k$–gons, for $k \geq 2$, and branch points of normal classes.

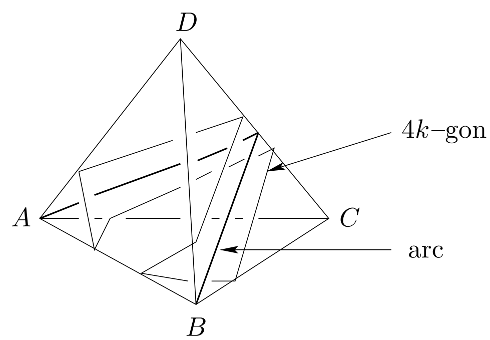

Figure 1: An elementary disk in an almost normal surface

In our work, it turns out to be sufficient to use normal and generalised almost normal surfaces. However, there is an interesting interaction with spun normal surface theory, which we will mention for completeness.

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2 Efficiency, tautness and angle structures of ideal triangulations

Taut triangulations were introduced by Lackenby [16], based on Gabai's theory of taut foliations, as developed by Scharlemann using sutured hierarchies. Lackenby showed that any irreducible atoroidal orientable compact 3–manifold with tori boundary has ideal triangulations which admit taut structures. Angle structures have been discussed by Casson and Rivin and are sometimes called semi-hyperbolic structures. One way of viewing an angle structure, is to associate all the tetrahedra with ideal hyperbolic simplices, in such a way that the sums of the dihedral angles about each edge of the triangulation are 2π. The latter is one of Thurston's three hyperbolic gluing conditions. One can then view a taut structure as a limit of such angle structures. For our purposes, we introduce a slightly weaker version of tautness than used by Lackenby [16]. This fits very conveniently with angle structures and also enables us to consider non-orientable manifolds.

We also introduce the notion of a semi-angle structure, as a convenient interpolation between angle structures and taut structures.

Definition 2.1 Given an ideal triangulation Γ of M, a taut structure is an assignment of angles 0 or π to the dihedral angles at edges between pairs of faces in each tetrahedron $\Delta_1, \Delta_2, ..., \Delta_k$ of Γ. These angles satisfy two conditions:

- For each Δ_i there are four 0 dihedral angles and two π angles. The π angles are at an opposite pair of edges of Δ_i.
- For every edge E of Γ, the sum of all the dihedral angles around E is exactly 2π.

Next, we discuss the concept of angle structures as introduced by Casson and Rivin.

Definition 2.2 An angle structure is an assignment of non-zero dihedral angles α, β, γ to each tetrahedron Δ_i of an ideal triangulation of M with the following conditions:

- Each opposite pair of edges of Δ_i has the same dihedral angle, so the 3 pairs of opposite edges have dihedral values α, β, γ.
- $\alpha + \beta + \gamma = \pi$.
- The sum of all the dihedral angles around an edge of M is 2π.

Notice that an ideal hyperbolic tetrahedron Δ has all 4 vertices on the 2–sphere at infinity of hyperbolic 3–space $\mathbf{H}^3$. Each face is an ideal hyperbolic triangle and it is well-known ([22]) that the dihedral angles for such a tetrahedron are equal for opposite pairs of edges of Δ and sum to π. So the conditions of an angle structure are part of the compatibility conditions for gluing together choices of hyperbolic metrics on the tetrahedra $\Delta_1, \Delta_2, ..., \Delta_k$ of Γ to form a (complete) hyperbolic metric.

Finally we relax the definition of angle structure to give the new definition of a semi-angle structure. Note that both angle structures and taut structures are examples of semi-angle structures.

Definition 2.3 A semi-angle structure satisfies the same conditions as an angle structure except that all dihedral angles are non-negative rather than strictly positive.

Combinatorial restrictions on an ideal triangulation Γ, are related to the possibility of finding an angle structure using Γ. The following discussion is based on an inspiring talk given by Casson in Montreal in 1995. In [11] and [12], these conditions on ideal triangulations are developed for the more difficult case of triangulations of closed 3–manifolds.

Definition 2.4 We say that an ideal triangulation Γ of M is 0–efficient, if there are no embedded normal spheres or projective planes. We say that Γ is 1–efficient, if Γ is 0–efficient and there are no embedded normal tori or Klein bottles, except for the boundary tori and Klein bottles of N. Finally we say that Γ is strongly 1–efficient if there are no singular or embedded normal spheres, projective planes, tori or Klein bottles, except for coverings of the boundary surfaces, realized as normal surfaces in M.

An initial connection between these concepts is given by the following result, due to Casson and Rivin.

Theorem 2.5 *Suppose that M is the interior of a compact 3–manifold N with tori and Klein bottle boundary components and has an ideal triangulation Γ with an angle structure. Then M is irreducible, $\mathbf{P}^2$–irreducible and atoroidal. Moreover Γ is strongly 1–efficient.*

Proof Suppose that M has an embedded essential 2–sphere S (which does not bound a 3–cell), an embedded 2–sided projective plane P or a π_1–injective map $f\colon T \to M$ of a torus or Klein bottle, which is not homotopic into a boundary component of N. We claim that S or P or $f(T)$ can be isotoped or homotoped to be an embedded or immersed normal surface. This follows by standard arguments, initially due to Haken (see [8]). Note that for the case of the immersed torus, one can use the method of Freedman, Hass, Scott [5], to lift a map homotopic to f to an embedding in a covering space of M and so Haken's method applies also in this covering space.

Now the dihedral angles associated with edges of ideal tetrahedra can also be given to corresponding vertices of the triangular and quadrilateral normal disk types. The condition for an angle structure that the sum of the 3 dihedral angles of a tetrahedron is π, means that the angle sum for the vertices of any normal triangle is also π. On the other hand, the angle sum for the vertices of any normal quadrilateral is of the form $2\alpha + 2\beta$ where α, β, γ are the 3 dihedral angles for pairs of opposite edges of the tetrahedron containing the quadrilateral. Therefore, we conclude that this angle sum is $2\pi - 2\gamma$ and is therefore $< 2\pi$ (see Figure 2).

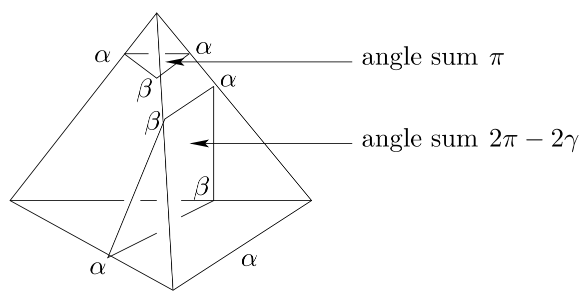

Figure 2: Vertex angle sums for normal disks

Next, notice that the other angle structure condition that the dihedral angles around an edge add up to 2π, implies the same is true for any vertex of any immersed normal surface $f\colon S' \to M$. Therefore we find that the Euler characteristic of any immersed normal surface $f(S')$ is non-positive and is strictly negative if there are any quadrilaterals.

In fact, by Gauss–Bonnet, the Euler characteristic $\chi(S')$ can be calculated by summing $\alpha+\beta+\gamma-\pi$ and $2\alpha+2\beta-2\pi$ over all normal triangles with angles α, β, γ and normal quadrilaterals with angles $\alpha, \beta, \alpha, \beta$ and dividing the total by 2π.

So this shows that M cannot have any embedded (or immersed) normal spheres or projective planes and is therefore irreducible and $\mathbf{P}^2$–irreducible. Moreover, any immersed normal torus or Klein bottle must consist only of triangular normal disks and is therefore a covering of one of the peripheral normal surfaces of M, as required in an atoroidal manifold. This also establishes that M is strongly 1–efficient. □

Remarks Note that a similar argument also shows that for the case of an angle structure, there are no non boundary parallel embedded or immersed spun normal annuli or Mobius bands. The reason is that any spun normal surface with some quadrilaterals would have negative Euler characteristic. Hence for a spun normal annulus or Mobius band, there can only be triangular normal disks and the surface is then boundary parallel.

To complete this section, we also look at the situation of a taut or semi-angle structure. This extends a result due to Lackenby [16].

Theorem 2.6 *Suppose that M is the interior of a compact 3–manifold N with tori and Klein bottle boundary components and has an ideal triangulation Γ with a taut or semi-angle structure. Then M is irreducible, $\mathbf{P}^2$–irreducible and Γ is 0–efficient. Moreover any embedded normal torus or Klein bottle must be incompressible. If M is also atoroidal, then Γ is strongly 1–efficient.*

Proof For simplicity, we assume a taut structure and leave it to the reader to make the necessary simple modifications for the case of a semi-angle structure. For the first part of this theorem, we can follow exactly the same method as in the previous theorem. Namely, angles can be associated to the vertices of any embedded or immersed normal surface in M using 0 and π as dihedral angles at edges of tetrahedra of Γ, coming from the taut structure. As in Theorem 2.4, it then follows that the angle sum of any normal triangle is π and for quadrilaterals is either π or 2π. Therefore there cannot be any embedded normal spheres or projective planes and M is irreducible, $\mathbf{P}^2$–irreducible and Γ is 0–efficient exactly as before.

Next, consider an embedded normal torus or Klein bottle T in M which is not π_1–injective. By the loop theorem and Dehn's lemma, there must be an embedded compressing disk for T. (If T is one-sided, we can work instead with the boundary of a small regular neighborhood, which is a two-sided torus or Klein bottle). It easily follows that either T is a torus bounding a solid torus or cube-with-knotted hole or T is a Klein bottle bounding a non-orientable solid torus. (We know by the previous paragraph that there are no projective planes in M so this rules out a disk cutting the Klein bottle into two projective planes). The case of cube-with-knotted hole can be easily ruled out, since an embedded 2–sphere bounding a 3–cell containing T can be shrunk relative to Γ using T as a barrier as in [11] to give a normal 2–sphere. But this contradicts our previous observation that Γ is 0–efficient.

To rule out T bounding a solid torus (orientable or not, depending on whether T is a torus or Klein bottle), we use the method of sweepouts or thin position (see [19] or [21]). Using T as a barrier, we can sweep across the normal solid torus getting a minimax value of complexity for a moving torus or Klein bottle. This minimax surface T' is almost normal, so since we have a 0–efficient triangulation, must be normal except for a single octagonal disk properly embedded in one of the tetrahedra. The other possibility for the minimax torus or Klein bottle is a normal sphere with a tube attached parallel to an edge and 0–efficiency rules this out. Now we can do the same Euler characteristic calculation for T' using the angles induced by the taut structure. It is easy to see for the octagon, the vertex angle sum Σ is either 2π or 4π (see Figure 3). The contribution towards $2\pi\chi(T')$ from the octagon is then $\Sigma - 6\pi$ so is always negative. Since all the normal triangles and quadrilaterals also make non-positive contributions, it follows that $\chi(T') < 0$ and so T' cannot be a Klein bottle or torus. This completes the proof that any embedded normal torus or Klein bottle must be π_1–injective.

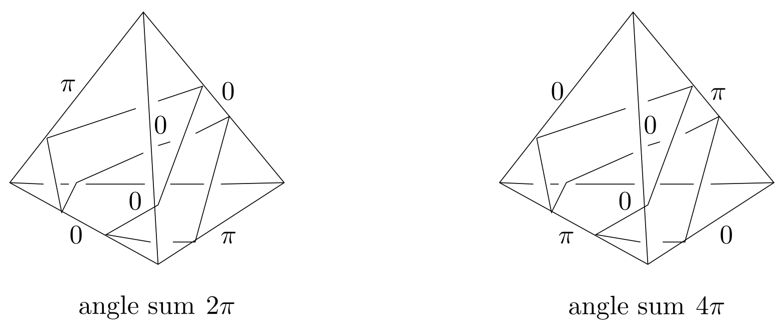

Figure 3: Vertex angle sums for an octagonal disk

Finally if M is atoroidal, we need to prove that Γ is strongly 1–efficient. The idea is to use a covering space approach similar to the one in [5], together with the argument in the preceding paragraph about non-π_1–injective (compressible) normal tori and Klein bottles. Assume first that there is an embedded or immersed π_1–injective normal torus or Klein bottle $f\colon T \to M$ which is not a covering of a peripheral torus or Klein bottle. By the atoroidal assumption, we know that the map f is homotopic to a map into such a peripheral normal surface, which we denote by T_1. Let N_0 be the covering of N (the compactification of M) corresponding to the peripheral subgroup $\pi_1(T_1)$ of $\pi_1(N)$. Denote by T_0 an embedded lift of T_1 to N_0. Now it is well-known (see eg, [20]) that this covering space N_0 is almost compact, ie, is the result of removing part of one boundary component of a product of a torus or Klein bottle and an interval. It is then immediate that we can find a new embedded torus or Klein bottle T_2

which is parallel to the lifted surface T_0 in N_0, so that there is a lift f_0 of the map f so that $f_0(T)$ is contained in the product region between T_0 and T_2. But then, by the usual barrier argument as in [11], we can isotopically shrink T_2 to an embedded normal surface T_3 using $f_0(T)$ as the barrier (see Figure 4). But now a sweepout across the product region between T_3 and T_0 in N_0 gives an almost normal torus or Klein bottle in the interior M_0 of N_0. The taut structure on Γ lifts to a taut structure on the lifted triangulation Γ_0 on M_0. Since we have shown previously there cannot be an embedded almost normal torus or Klein bottle in a taut triangulation, so this case is done.

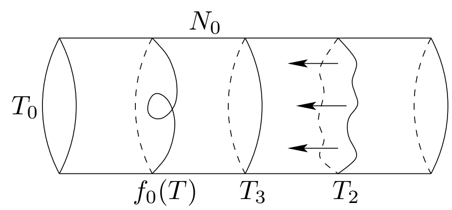

Figure 4: Isotoping T_2 to an embedded normal surface T_3

The last case to consider is a compressible immersed normal torus or Klein bottle
$f\colon T \to M$. Suppose first that the image $f_*\pi_1(T)$ in $\pi_1(M)$ is trivial. In this case, the map f lifts to $\widetilde{f}$ in the universal covering space $\widetilde{M}$ of M. But the taut structure on Γ obviously lifts to a taut structure on the lifted triangulation $\widetilde{\Gamma}$ on $\widetilde{M}$. Also $\widetilde{M}$ is almost compact (see eg, [20]), so is an open 3–cell. We can find an embedded 2–sphere S in $\widetilde{M}$ which bounds a 3–cell containing $\widetilde{f}(T)$ and as previously, can use $\widetilde{f}(T)$ as a barrier and shrink S to a normal 2–sphere. But we have shown before there are no such normal 2–spheres in a taut triangulation $\widetilde{\Gamma}$ so this gives a contradiction.

Assume secondly that the image $f_*\pi_1(T)$ in $\pi_1(M)$ is non trivial, but not an isomorphic copy of $\pi_1(T)$. Then it is easy to see that the only possibility is a cyclic image isomorphic to $\mathbf{Z}$, since $\pi_1(M)$ has no torsion. Consequently, there is an essential simple curve C in T with homotopy class in $\pi_1(T)$ generating the kernel of f_*. We can define a continuous map of a disk $\bar{f}\colon D \to M$, so that $\bar{f}(\partial D) = f(C)$.

To complete this case, the idea is to pull back the triangulation Γ of M to D, using $\bar{f}$. Moreover, there is a natural way to also pull back the taut structure on Γ to a 'taut' structure on D. Then a Gauss–Bonnet argument similar to that in Theorem 2.5 gives a contradiction.

After a small perturbation, it can be assumed that $\bar{f}$ is transverse to the triangulation Γ. So the pull back of the edges of Γ are interior vertices in D and the pull back of the faces of Γ are arcs and loops in D. We can simplify the map $\bar{f}$ by an obvious homotopy to eliminate any loops in the preimage of the faces. For if C_0 is an innermost such loop bounding a subdisk D_0 of D, we can homotop $\bar{f}$ into the face of Γ containing $\bar{f}(C_0)$. It is then easy to slightly perturb the map to push $\bar{f}$ off this face on a small neighborhood of D_0, thus eliminating C_0 (and any other arcs and vertices inside D_0).

Let $\mathcal{G}$ denote the graph on D with these vertices and arcs. If $\mathcal{G}$ is disconnected, we can again homotop the map $\bar{f}$ to eliminate some of the components of the graph. Just choose a loop C_1 disjoint from $\mathcal{G}$ and bounding a subdisk D_1 containing some components. Then $\bar{f}(C_1)$ lies inside some tetrahedron of Γ. So we can homotop $\bar{f}|D_1$ into this tetrahedron and eliminate any pieces of the graph inside D_1. Consequently we may assume that the graph is connected and it defines a cell decomposition of D.

Next, any polygonal face P of the cell decomposition of D which is a bigon can be removed by a further homotopy of $\bar{f}$. For it is easy to see that first $\bar{f}|_P$ can be homotoped into an edge or face of Γ and then $\bar{f}$ can be pushed off this edge or face on a small neighborhood of P in D. The boundary arcs of an interior bigon P of D map to arcs with ends on an edge E of Γ. The two boundary arcs then bound a bigon in the boundary of the tetrahedron, lying in the two faces containing E. For a bigon P adjacent to the boundary of D, note that one of its boundary arcs lies on a normal triangular disk or quadrilateral and the other on a face of Γ, so it follows that the ends must be on the same edge E of Γ. Hence a similar picture is obtained to the interior bigon case (see Figure 5). This will eliminate P as claimed and so decrease the number of faces of the cell decomposition. In a similar manner, any boundary edge λ of the cell decomposition with image having both ends on the same edge of Γ can be eliminated by a homotopy of $\bar{f}$, by homotoping the map of λ into the edge and then perturbing the image of λ off the edge, simplifying the cell decomposition of D. Therefore we can assume that every boundary edge λ of the cell decomposition of D, maps to a spanning arc in a disk of the normal surface $f(T)$ running between two different edges of Γ.

By transversality of $\bar{f}$ relative to the edges and faces of Γ, we see that all vertices of $\mathcal{G}$ have the same degree as the corresponding edges of Γ. We can therefore pull back the angles of the taut structure on Γ to the cell decomposition of D. Therefore every polygonal face has vertices with angles either 0 or π. Moreover the condition that boundary edges λ of the cell decomposition of D map to spanning arcs in faces of Γ, means that if the vertex at one end of λ has angle

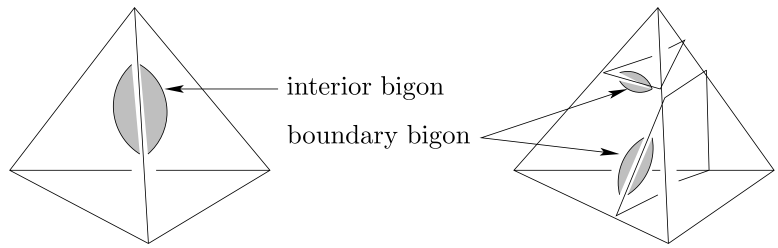

Figure 5: Bigon faces of the cell decomposition of D

π, then the vertex at the other end of λ must have angle 0. Two edges of a tetrahedron which share a vertex, cannot both have dihedral angle π in a taut structure.

To complete this discussion, we need to consider boundary edges and polygonal faces adjacent to ∂D. Note that if a triangular polygonal face F with one arc σ on ∂D, has the property that $\bar{f}|F$ is homotopic into a vertex of the normal structure on the torus or Klein bottle $f(T)$, then we can use such a homotopy to remove this face and simplify the cell decomposition on D. To be more specific, we mean that the arc σ cuts off a triangular corner of a normal triangular disk or quadrilateral of $f(T)$ and $\bar{f}|F$ can be homotoped to have image equal to this triangular corner (see Figure 6). After removing all such triangular faces, we claim that any remaining triangular face F with one arc σ on ∂D must have the angle of 0 at the vertex v opposite to σ, since σ must run between opposite edges of a quadrilateral of T and the edges of Γ at the four corners of this quadrilateral *must* have angles $0, \pi, 0, \pi$. Then v is mapped by $\bar{f}$ into an edge of the tetrahedron disjoint from the quadrilateral and so the dihedral angle at this edge must be 0, since the three dihedral angles in the tetrahedron are $0, 0, \pi$ for a taut structure (see Figure 6). The property of angles at the corners of a normal quadrilateral follows by the usual Gauss–Bonnet argument – if such a quadrilateral had all angles 0, then $\chi(T)$ would be negative and so the normal surface would not be a torus or Klein bottle.

We can now do a similar Gauss–Bonnet calculation for D. For vertices on ∂D of faces of D, we assign an angle of $\frac{\pi}{2}$. For any polygonal face F of D with n sides, the contribution to $\chi(D)$ is $\Sigma - \frac{(n-2)}{2}$, where Σ denotes the sum of the angles at the vertices of F divided by 2π. Now any triangular face F has $\Sigma = \frac{1}{2}$ and so contributes 0 to $\chi(D)$, by our discussion of boundary triangular faces above. (Any boundary triangular face has angles $0, \frac{\pi}{2}, \frac{\pi}{2}$.) Moreover since angles at the vertices of an interior polygonal face F, with $n \geq 4$ edges, are

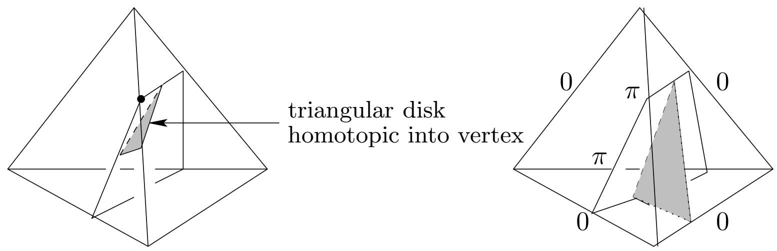

Figure 6: Triangular faces of the cell decomposition of D

either π or 0 and no two adjacent vertices around ∂F are π, we see that the contribution to $\chi(D)$ is non-positive. Similarly for polygonal faces with at least one boundary edge on ∂D and $n \geq 4$ edges, there are angles of $\frac{\pi}{2}$ at the vertices on ∂D and the other angles are π or 0. Again as there are no adjacent angles of π, the contribution to $\chi(D)$ is also non-positive. But then the Euler characteristic of D will not be one, and this contradiction establishes strong 1–efficiency of Γ. □

Corollary 2.7 *Any immersed or branched normal surface with non-negative Euler characteristic is normally boundary parallel, if M has an angle structure or is atoroidal and has a taut or semi-angle structure on Γ.*

Proof This follows easily by the same method as for Theorems 2.4 and 2.5. By normally boundary parallel, we mean that the surface is a collection of triangular disks, with no quadrilaterals. The key point in the case of a branched normal surface $\bar{f}\colon \bar{T} \to M$, is that in the Gauss–Bonnet formula, a branch point of degree $d > 1$ contributes an amount of $2\pi(1-d)$ to the calculation of $2\pi\chi(\bar{T})$. Consequently it follows immediately that for either an angle structure, or a taut or semi-angle structure, that any branched normal surface gets a negative contribution to Euler characteristic from its branch points. □

Remarks Notice that these two results give an important bridge between taut structures and angle structures for ideal triangulations of atoroidal 3–manifolds M which are also irreducible and $\mathbf{P}^2$–irreducible. In the next section, we will show that strong 1–efficiency is one of the two key conditions needed to deform a taut structure to an angle structure on an ideal triangulation. However it turns out that certain special branched normal surfaces with negative Euler characteristic can occur in taut triangulations but not in angle structures and so non existence of such surfaces is the obstruction to solving Lackenby's question positively.

For completeness, we note a connection between strong 1–efficiency and the existence of spun normal surfaces with zero Euler characteristic. It is elementary to check by the same method as in Theorem 2.4 that if Γ has an angle structure, then there cannot be any such spun normal surfaces. However by the next result, strong 1–efficiency guarantees this.

Theorem 2.8 *Suppose that Γ is a strongly 1–efficient ideal triangulation of M. Then there are no spun normal surfaces U with zero Euler characteristic in M.*

Proof By passing to 2–fold covering spaces, we can assume without loss of generality that M is orientable and U is an annulus, rather than possibly a Mobius band. Let $f\colon U \to M$ denote the immersion or embedding of U and let T_0 denote one of the peripheral tori with one of the half open annuli ends of U projecting onto T_0. Also let $\widetilde{f}\colon U \to \widetilde{M}$ denote the lift to the covering space $\widetilde{M}$, where $\pi_1(\widetilde{M})$ corresponds to the subgroup $f_*\pi_1(T_0)$. Finally let $\widetilde{N}$ be the covering space of the compact manifold N also corresponding to $f_*\pi_1(T_0)$ (see Figure 7).

As in Theorem 2.5, we know that $\widetilde{N}$ (and hence $\widetilde{M}$) is almost compact. So if $\widetilde{f}(U)$ has both half open annuli (its two ends) covering the same lift $\widetilde{T}_0$ of the peripheral surface T_0 for M_0, then we can use $\widetilde{f}(U) \cup \widetilde{T}_0$ as a barrier as in [11] and find an embedded normal torus in $\widetilde{M}$ which is not peripheral. This contradicts our assumption that M is strongly 1–efficient. On the other hand, if $\widetilde{f}(U)$ has its second half open annulus (other end) covering a second peripheral surface, then clearly $f(U)$ is not homotopic, keeping its structure at infinity fixed, into a neighborhood of T_0. Consequently we can replace $f(U)$ by a compact properly immersed annulus denoted V in N, by replacing the half open annular ends by compact annular ends finishing at boundary components of N. Then either this annulus V has both ends on T_0 and is not homotopic, keeping its boundary fixed, into T_0, or the annulus has ends on two different peripheral tori. But then by the classical characteristic variety theorem (see [10], [9]), either N is a Seifert fibred space with two exceptional fibers or N has an embedded π_1–injective torus which is not homotopic to a peripheral torus. In the first case, there are many immersed π_1–injective tori in M which are not homotopic into a peripheral torus. So in either case, we can homotop such tori to be normal in M and this contradicts our hypothesis that M is strongly 1–efficient. □

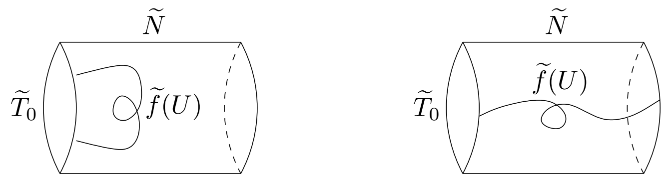

Figure 7: The lift of $f(U)$ to the covering space $\widetilde{M}$

3 Tautness and angle structures of ideal triangulations

In this section, our objective is to show that taut structures can be deformed to angle structures for atoroidal manifolds M if certain branched normal surfaces do not occur. The two major ideas needed are strong 1–efficiency as in the previous section, together with the observation that angle structures can be written in terms of a collection of compatibility equations, which are 'dual' to the canonical basis for normal surface theory (see [15]), when written in $\mathbf{Q}$–coordinates. So our first task is to review some more facts for normal surface theory and also to write down the equations for angle structures.

From [15], it follows that for an ideal triangulation Γ of M with tetrahedra $\Delta_1, \Delta_2, ..., \Delta_k$, the solution space $\mathcal{V}$ for the $6k$ compatibility equations using standard normal coordinates, has a canonical basis $\mathcal{B}$ consisting of k tetrahedral and k edge solutions. A tetrahedral solution d_i for the tetrahedron Δ_i is $d_i = q_1 + q_2 + q_3 - t_1 - t_2 - t_3 - t_4$, where the q_j, t_k denote the quadrilateral and triangular disk types in Δ_i (see Figure 8). Given an edge E_i of Γ, the corresponding edge solution e_i is $e_i = q_{j_1} + ... + q_{j_r} - t_{m_1} - ... - t_{m_s}$, where the q_{j_u} are quadrilaterals in the tetrahedra adjacent to the edge E_i with E_i at least one of the edges disjoint from the quadrilateral disk type and the t_{m_v} are triangular disk types in the same collection of tetrahedra but which have at least one vertex on E_i (see Figure 8). Note that if in some tetrahedron adjacent to E_i, both the edges disjoint from some quadrilateral disk type are E_i, then we must take this quadrilateral with multiplicity 2, or equivalently take q_{j_u} and q_{j_v} as being equal, for some pair $u \neq v$.

It is easy to see that the number of edges is the same as the number of tetrahedra in Γ, since $\chi(M) = 0$. So there are $2k$ 'formal' normal surfaces $d_1, e_1, ..., d_k, e_k$ which are tetrahedral or edge solutions. These form the canonical basis $\mathcal{B}$. It is often convenient to talk about formal normal surfaces as vectors of integers satisfying the compatibility equations, for which the coordinates need not be non-negative. If all the coordinates are non-negative, then the vector

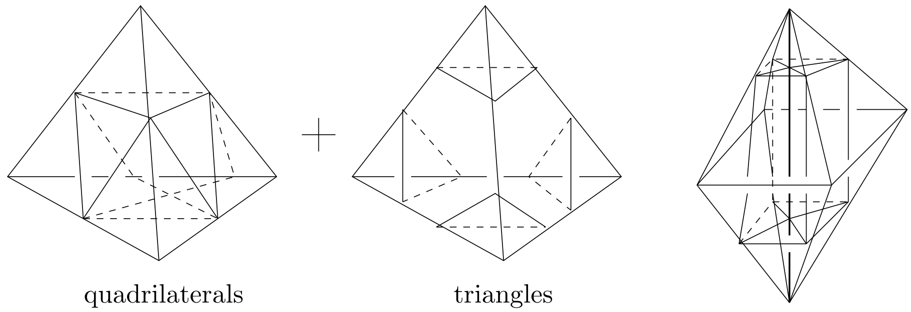

Figure 8: Canonical basis for the solution space

corresponds to a standard normal surface. If we work in **Q**–coordinates, then normal surfaces are vectors of length $3k$ and a tetrahedral surface d_i has three 'ones' and all remaining coordinates 0, whilst an edge surface e_i has 'ones' at the entries corresponding to the quadrilateral types in adjacent tetrahedra to E which have E as a disjoint edge. Notice that if there are self identifications of edges in the tetrahedra of Γ, then some 'ones' might be twos as noted in the previous paragraph. Note however that the sum of the coordinates of an edge surface is precisely the degree of that edge in the triangulation.

Next, we introduce the compatibility equations for angle structures on Γ. Let $\alpha_i, \beta_i, \gamma_i$ denote the 3 angles for the tetrahedron Δ_i of Γ, so that opposite edges are assigned the same angle. The angle equations are the following system:

$$\begin{aligned} \alpha_i + \beta_i + \gamma_i &= \pi \\ \alpha_{j_1} + \ldots + \alpha_{j_r} &= 2\pi \end{aligned} \qquad (*)$$

There are k equations of the first kind, one for each tetrahedron Δ_i of Γ and k equations of the second kind, one for each edge E_i of Γ. Every α_{j_u} is an angle at the edge E_i for one of the adjacent tetrahedra to E_i. So α_{j_u} is one of α, β, γ. An angle structure is then a solution for the system $(*)$, where every angle has a positive value. The 'duality' between the angle equations $(*)$ and the canonical basis $\mathcal{B}$ is the observation that the coefficients of the angle variables in $(*)$ are precisely the vectors making up $\mathcal{B}$ in **Q**–coordinates. Note that as previously discussed for edge solutions, we can have $\alpha_{j_u} = \alpha_{j_v}$, for $u \neq v$, in case two opposite edges in the same tetrahedron are identified with E_i.

The main result of this paper is the following;

Theorem 3.1 *Suppose that M is atoroidal and has a taut or semi-angle ideal triangulation Γ. Then there is a non-empty k–dimensional space $\mathcal{A}$ of angle* *structures for* Γ *if and only if there are no branched normal classes containing some quadrilaterals with angle sum* 2π *in the taut or semi-angle structure and no quadrilaterals with angle sum* $< 2\pi$*.*

Proof The argument proceeds by a series of steps. For simplicity, we again discuss the case of a taut structure and leave it to the reader to check the simple modifications needed for semi-angle structures.

Claim 1 *Given an immersed normal surface* S*, if it has normal class* $[S]$ *given by* $[S] = \Sigma_i(n_i d_i + m_i e_i)$*, a linear combination of solutions in the canonical basis* $\mathcal{B}$*, then the Euler characteristic of* S *is given by* $\chi(S) = -\Sigma_i(n_i + 2m_i)$*.*

The explanation is simple; we just note that $\chi(d_i) = -1$ and $\chi(e_i) = -2$. In fact, calculating $2\pi\chi(d_i)$ for example, we sum $2\pi(\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} - \frac{1}{2})$, for each triangle of d_i or e_i, where the u_j are the degrees of the three edges met by the triangle, and also sum $2\pi(\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} + \frac{1}{u_4} - 1)$ for every quadrilateral of d_i or e_i, where the u_j are the degrees of the four edges met by the quadrilateral. This gives the total -2π for $2\pi\chi(d_i)$ and -4π for $2\pi\chi(e_i)$. Note that we have used the very convenient device of extending χ to a linear functional on the total space of formal solutions to the normal surface equations $\mathcal{W}$, by defining values of χ for triangles and quadrilaterals. This gives the correct result for χ on a vector with non-negative coordinates, so long as the corresponding normal surface is an immersion.

For the next steps, we need to define $\chi^*(S)$ for surfaces S with branch points as well as for immersions. This is *not* the usual Euler characteristic when branch points occur, but $\chi^*(S) = \chi(S)$ for embeddings or immersions. So $\chi^*(S)$ is defined for a vector exactly as in Claim 1; we just add all the contributions of triangles and quadrilaterals with signs as for the vector satisfying the normal surface equations. Specifically, if the normal class $[S] = \Sigma_i n_i t_i + \Sigma_j m_j q_j$, where the t_i are the normal triangles and the q_j are the normal quadrilaterals, then $\chi^*(S) = \Sigma_i n_i \chi(t_i) + \Sigma_j m_j \chi(q_j)$, where the Euler characteristic contributions of the triangles and quadrilaterals are computed as in Claim 1 above. Note that Claim 1 extends then to surfaces with branch points, ie, the formula $\chi^*(S) = -\Sigma_i(n_i + 2m_i)$ is valid if $[S] = \Sigma_i(n_i d_i + m_i e_i)$, since χ^* is a linear functional.

It is often convenient to compute $\chi^*(S)$ by a Gauss–Bonnet angle sum approach, but ignoring branch points. So every triangle contributes zero angle sum and every quadrilateral contributes its angle sum minus 2π, for angles induced from a taut or semi-angle structure. The total angle sum divided by 2π is then $\chi^*(S)$. An easy way to prove this gives the previous formula for χ^* is to note that for the canonical basis $\mathcal{B} = \{d_1, \ldots, d_k, e_1, \ldots, e_k\}$, this angle sum approach also yields -1 for tetrahedral solutions d_i and -2 for edge solutions e_j. So since a linear functional is determined by its values on a basis, this establishes that the angle sum formula is the same as that given previously.

Note that for a normal surface S with branch points in a taut structure, having $\chi^*(S) = 0$ is the same as requiring that S can contain quadrilaterals with two dihedral angles π, but cannot have a quadrilateral with all angles zero (see Figure 9).

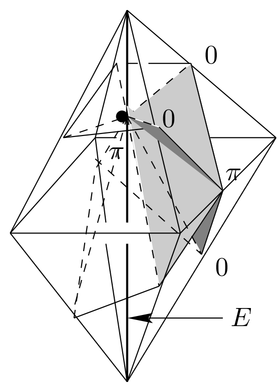

Figure 9: Quadrilaterals in a normal surface with a branch point

Such surfaces will then have their 'real' Euler characteristic $\chi(S) < 0$, if they are not peripheral tori or Klein bottles. For branch points contribute a strictly negative amount to the Euler characteristic, as noted in the previous section.

Claim 2 *There is a normal surface S, which may be embedded, immersed or branched and is non peripheral, with $\chi^*(S) = 0$, if and only if there is a linear combination of the equations of the system $(*)$ with right side having value 0 and all coefficients of the variables $\alpha_i, \beta_i, \gamma_i$ of the left side being non negative, with at least one coefficient being positive.*

We can write S as a linear combination $\Sigma_i(n_i d_i + m_i e_i)$ of solutions in the canonical basis $\mathcal{B}$, where the n_i, m_i are integers. By Claim 1, $\chi^*(S) = 0$ if and only if $\Sigma_i(n_i + 2m_i) = 0$. Notice then that if we compute the linear combination of the equations $(*)$ by adding multiples n_i of the first type of equation (angle sum in the tetrahedron Δ_i is π) and m_i of the second type of equation (angle sum around the edge E_i is 2π), then the equation $\Sigma_i(n_i + 2m_i) = 0$ means precisely that the right hand side of the resulting equation is zero.

Next, for this particular linear combination of the system $(*)$, the condition that all the coefficients of the variables on the left side are non negative corresponds to the requirement that all the quadrilaterals in S are taken with non negative multiplicity. Adding up the contributions from the individual equations is the same as adding up the corresponding multiples of the coordinates in the basis vectors d_i, e_i. This follows since as noted above that these coordinates are exactly the same as the coefficients of the angle variables in the equations of $(*)$. But then as all the quadrilateral coordinates of S must be non negative and at least one must be strictly positive, since we are dealing with a surface which is not peripheral, the claim follows.

Claim 3 *Let $\mathcal{A}^*$ denote the affine space of solutions of the system $(*)$ of angle equations and let $\mathcal{O}$ be the positive octant consisting of vectors with all coordinates strictly positive. Then $\mathcal{A}^*$ intersects $\mathcal{O}$ if and only if there are no linear combinations of the equations of the system $(*)$, with right side zero and all coefficients of the angle variables of the left side being non negative, with at least one coefficient being positive.*

Note that this claim completes the proof of the theorem, since the angle space $\mathcal{A} = \mathcal{A}^* \cap \mathcal{O}$. In one direction this claim is easy – if there is such a linear combination, we know by Claim 2 that there is a non peripheral normal surface S with $\chi^*(S) = 0$. But this contradicts Theorem 2.5 if S is embedded or immersed. Otherwise we have precisely the condition that there are no non peripheral branched classes with only quadrilaterals having two angles π and no quadrilaterals with all angles zero. So it remains to show that if there are no linear combinations as in claim 3, that $\mathcal{A}$ is non empty.

Firstly, the assumption that Γ has a taut structure implies that the affine space $\mathcal{A}^*$ intersects $\partial\mathcal{O}$ at the vector of angles corresponding to the taut structure. If $\mathcal{A}^*$ misses $\mathcal{O}$, then we would like to construct a hyperplane U in $\mathbf{R}^{3k}$ which contains $\mathcal{A}^*$ and misses $\mathcal{O}$. So let us suppose that $\mathcal{A}^* \cap \mathcal{O} = \emptyset$. By taking all multiples of vectors in the affine space $\mathcal{A}^*$, we get a subspace U which includes $\mathcal{A}^*$ (see Figure 10). We claim that U does not intersect $\mathcal{O}$. To verify this, notice that there is at least one vector of $\partial\mathcal{O}$ in $\mathcal{A}^*$ (corresponding to the taut structure) and certainly all multiples L of this vector will also be in $\partial\mathcal{O}$. Let $\bar{\mathcal{O}}$ denote the closure of $\mathcal{O}$. If U intersected $\mathcal{O}$, then $U \cap \bar{\mathcal{O}}$ would be a cone with cross section a polytope of dimension one less than the dimension of U. But then the line L would be in this cone and would therefore be in the closure of $U \cap \mathcal{O}$. But this would imply that $\mathcal{O}$ intersected $\mathcal{A}^*$, contrary to assumption. For U is a cone on $\mathcal{A}^*$ locally near L. Since L crosses $\mathcal{A}^*$, lines of U through the origin, nearby to L, clearly meet $\mathcal{A}^*$ in nearby points to L and so such lines would be in $U \cap \mathcal{O}$. So the description of U is verified.

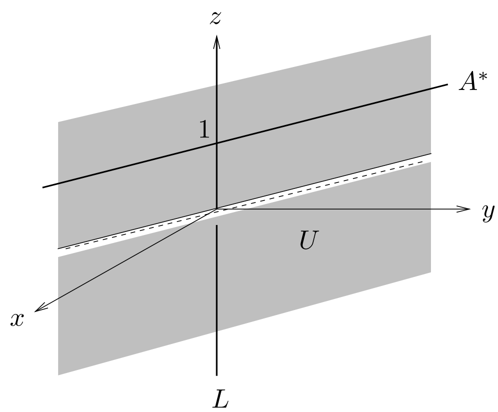

Figure 10: Hyperplane containing $\mathcal{A}^*$ and missing $\mathcal{O}$

Consider all subspaces U' which contain U as a hyperplane. If such a subspace intersects $\mathcal{O}$, then this must be inside one of the two half spaces bounded by U in U', since the intersection is convex and misses U. As U' rotates around U, the two half spaces bounded by U interchange positions. Consequently, by continuity, for some intermediate position, $U' \cap \mathcal{O}$ must be empty. In this way we can find subspaces of increasing dimension including $\mathcal{A}^*$ and disjoint from $\mathcal{O}$. So the procedure terminates with the required hyperplane V.

It is an elementary fact from linear algebra that any hyperplane containing the affine space of solutions of the system $(*)$ comes from a linear equation obtained by taking a linear combination of the equations of $(*)$, so that the right side is zero. Hence there is such a linear equation which yields V as solution space. Now a normal vector to V is given by the coefficients of the variables in this linear equation. By our assumption above that there are no branched normal surfaces with some quadrilaterals having two dihedral angles π and no quadrilaterals with all dihedral angles, Claim 2 implies that any non zero linear equation which is a linear combination of the system $(*)$ with right side zero, must have some coefficients of the left side being negative and some must be positive. Hence we conclude that the normal vector to V has some negative and some positive coordinates. Consequently there is a vector in V with all positive entries perpendicular to the normal vector, contradicting our construction of $V \cap \mathcal{O}$ being empty. Therefore the proof of Claim 3 and the theorem is complete. □

Remarks (1) In [2], we investigate existence of taut and angle structures 'experimentally' for examples where the number of tetrahedra in Γ is at most 8. Examples are found where Γ admits a complete hyperbolic metric of finite volume but no taut structure. Other examples are described where Γ has an angle structure but no taut structures and similarly where Γ has angle structures but no complete hyperbolic structure of finite volume. In particular, these examples show that strong 1–efficiency on its own is not enough to guarantee existence of a taut structure or hyperbolic structure.

(2) In [1], [17] it is shown that if Γ has an angle structure then $\pi_1(M)$ is CAT(0) and also relatively word hyperbolic.

(3) We next discuss the connection between normal classes with branch points which have only quadrilaterals with two angles π and embedded generalized almost normal surfaces. In fact, the normal arcs belonging to a collection of p, q of the two different quadrilaterals with two angles π and two angles zero in a tetrahedron Δ, where p, q are relatively prime, precisely correspond to the boundary of a $4(p+q-1)$–gon in Δ (see Figure 11). If p', q' are not relatively prime, we get n copies of such a disk where n is the g.c.d. of p', q' and $p' = np, q' = nq$, given p', q' quadrilaterals with two angles π and two angles zero. Consequently we can do regular branch cuts between compatible normal disk types, ie, pairs of quadrilaterals of the same type or triangles and quadrilaterals or between triangles, and replace sets of incompatible quadrilaterals with $4(p+q-1)$–gons. So we can find an embedded generalized almost normal surface corresponding to a normal class S with branch points and only quadrilaterals with angle sum 2π in a taut structure.

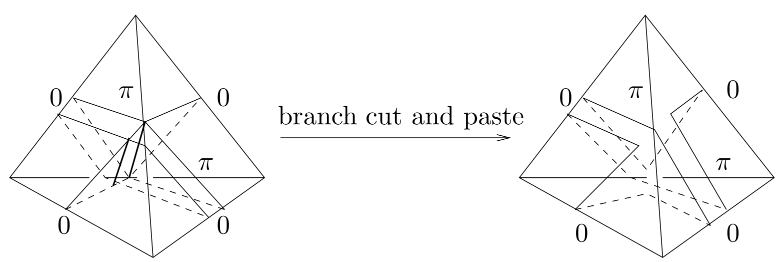

Figure 11: Regular branch cuts between compatible disk types

Notice that each $4(p+q)$–gon produced in this manner has $2(p+q)$ angles π and $2(p+q)$ angles zero. It is easy to see that one cannot deform the taut structure to an angle structure, given any embedded almost normal surface containing such $4(p+q)$–gons. For the formula for Euler characteristic of such a surface will decrease, under the deformation of a taut structure to an angle structure, by the same argument as in Theorem 2.4. Since Euler characteristic cannot change, this is an interesting illustration of why such surfaces are an obstruction to the existence of angle structures near the taut structure.

4 Examples

Example 1

We begin with a simple example of a taut triangulation which is strongly 1–efficient but has a normal classes with branch points, which is the obstruction to finding an angle structure. Start with the standard ideal triangulation of the Figure–8 knot space M as in [22] and 'blow up' two faces into two new tetrahedra. By this we mean split open along two faces which belong to one of the two tetrahedra and which glue together into a copy of the once punctured torus which is a Seifert surface for M. Now glue one tetrahedron to these two faces and then another onto the free two faces of the first one. Do this so that the boundary pattern of the two free faces of the second tetrahedron is the same as the original faces (see Figure 12). So we can glue back together to get a new taut triangulation with 4 tetrahedra, which comes from the taut triangulation corresponding to the fibred structure on M. Note there are two other taut structures on M which we are not considering here. Next, it is easy to find a normal class with branch points by taking one of each quadrilateral type with two angles π in both of the new tetrahedra and complete the normal class with triangles. Using branch cuts, one can also convert the two quadrilaterals with two angles π into a single quadrilateral with all angles 0 in each of the new tetrahedra. The remainder of the surface is then triangles. It can be seen that this is an embedded surface of genus 2 given by adding a tube to the peripheral torus along the common edge between the two faces shared by the two new tetrahedra. So this is a simple taut triangulation which does not admit an angle structure.

Example 2

Next, take any ideal taut triangulation of an orientable punctured surface bundle over a circle with pseudo Anosov monodromy, where there can be several punctures and the surface can have higher genus. Assume that the triangulation is formed by taking an ideal triangulation of the surface and adding tetrahedra along two faces at a time, similar to the structure on the Figure–8 knot space. Suppose that in the sequence of tetrahedra, we find two at different positions which are added along two faces with edge loops which are isotopic. Now we

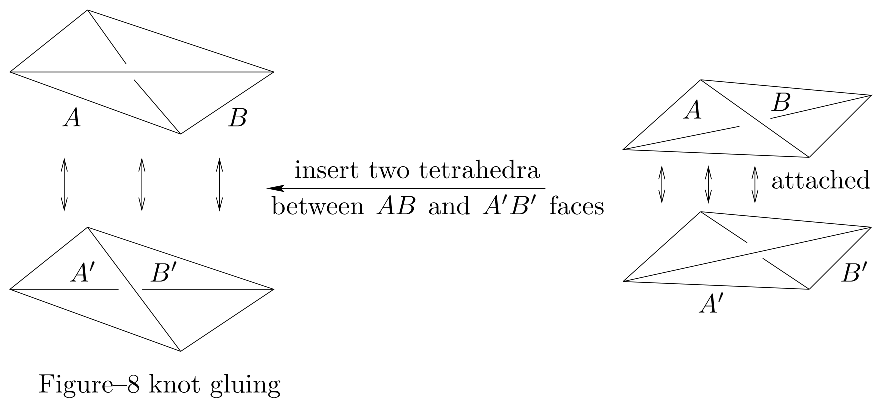

Figure 12: New taut triangulation of the Figure–8 knot space

can perform a similar construction to the previous paragraph, assuming that the isotopy between edge loops is as in the Figure–8 knot example. Namely take the corresponding quadrilaterals with all angles 0 in each of these two tetrahedra (see Figure 13). We can connect these by disks and annuli, since the arcs at the top and bottom of the octagons are isotopic. These disks and annuli can be made normal and so we again get a genus two generalized almost normal surface which has a (big) tube attached to the peripheral torus. So this shows that many taut triangulations of an orientable punctured surface bundle over a circle with pseudo Anosov monodromy do not admit angle structures.

Example 3

Take M as any once-punctured torus bundle over the circle with monodromy as a matrix A in $SL(2,\mathbf{Z})$ having $|traceA| > 2$. As is well known, these are precisely the bundles which are irreducible and atoroidal and so admit complete hyperbolic structures of finite volume. We want to consider the canonical ideal triangulation as in the first example, again given by triangulating the once punctured torus by two ideal triangles glued together and then adding a sequence of ideal tetrahedra along two faces at a time. Moreover we are only interested in such sequences which do not admit a 'canceling pair' as in the first example, ie, there are no edges of degree two in the triangulation.

In a recent paper on the arXiv [6], Gueritaud shows directly that these triangulations can be given hyperbolic structures which match to produce the complete hyperbolic metric of finite volume. Here we wish to show that the obstruction to deforming the taut structure coming from the fiber bundle to an angle structure always vanishes. So this gives another proof that these triangulations all admit

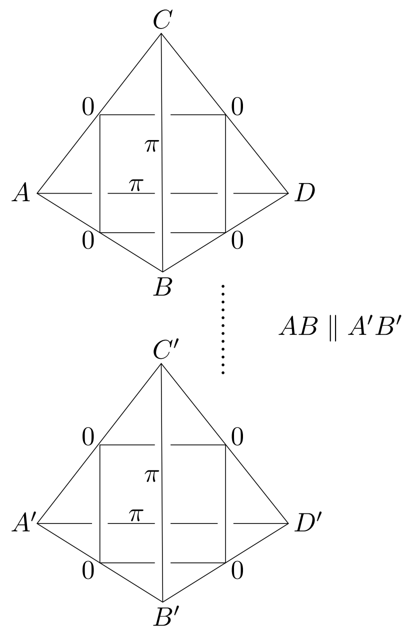

Figure 13: Connecting two quadrilaterals with all angles 0

angle structures and also gives some insight into the behavior of the obstruction normal classes.

Start with a square $v_1v_2v_3v_4$ which is glued up in the usual way to form a once-punctured torus T, so after opposite sides are identified together, the single vertex is removed. Suppose also that the diagonal edge v_1v_3 is included so as to produce an ideal triangulation of T. We also suppose that an ideal tetrahedron Δ is glued onto T so that the two triangles of T become two faces of Δ. A second tetrahedron Δ' with vertices $w_1w_2w_3w_4$ is then glued onto the two bottom faces of Δ which are $v_1v_2v_4$ and $v_2v_3v_4$ by either; $w_1w_3w_2 \to v_1v_4v_2$ and $w_1w_3w_4 \to v_2v_3v_4$ or $w_1w_3w_2 \to v_4v_3v_2$ and $w_1w_3w_4 \to v_1v_2v_4$ (see Figure 14).

Note that these are the only two possibilities since the third way of gluing on Δ' would make the two tetrahedra Δ, Δ' a canceling pair with an edge of degree 2, contrary to hypothesis. There are precisely three possibilities since we are gluing together two once-punctured tori divided into three triangles with three edges. So once a single edge matching is chosen, the gluing is determined.

We now study how a normal class with quadrilaterals with two dihedral angles π but none with all angles 0 can look inside these two tetrahedra. In the triangle

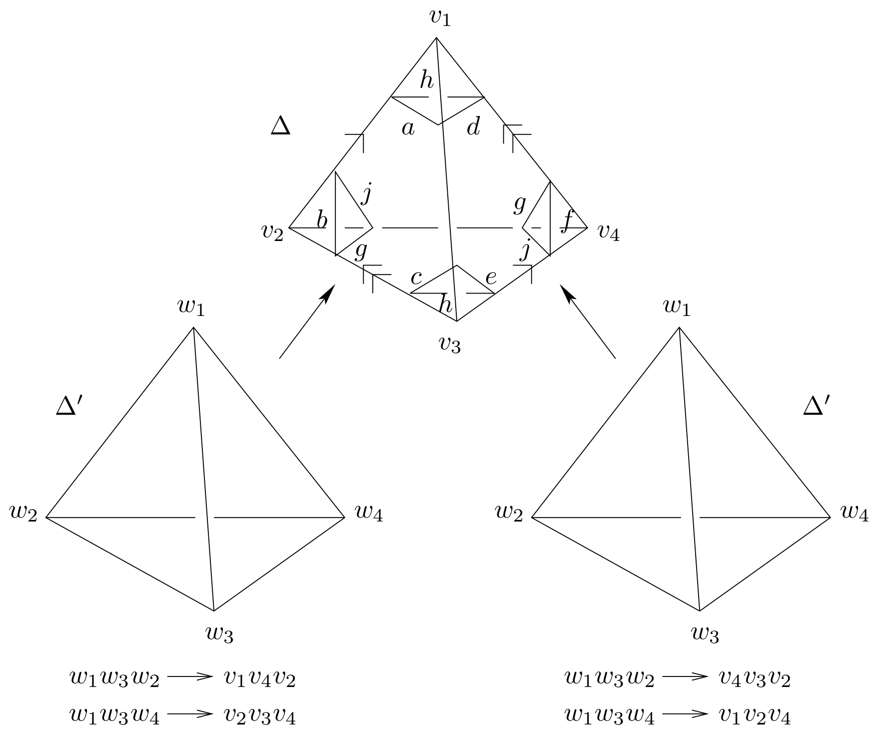

Figure 14: Gluing of an ideal tetrahedron onto T

$v_1v_2v_3$ (respectively $v_1v_3v_4$), let a, b, c (respectively d, e, f) denote the number of normal arcs which cut off the vertices v_1, v_2, v_3 (respectively v_1, v_3, v_4). Then it is straightforward to check that by the usual compatibility equations for normal surface theory, the normal arcs on the once-punctured torus T must glue up to form normal curves, so satisfy the compatibility equations that the number of ends of normal arcs on the two sides of an edge must agree. We see immediately that this forces $d = c, e = a, f = b$.

Next suppose that the numbers of triangular disks at the vertices v_1, v_2, v_3, v_4 in Δ are n_1, n_2, n_3, n_4 respectively and the number of quadrilaterals disjoint from the edges v_1v_2 and v_1v_4 are m_1, m_2 respectively also. Finally let g, h, j denote the number of normal arcs cutting off vertices v_4, v_1, v_2 in the triangle $v_4v_1v_2$, so that j, g, h are the number of normal arcs cutting off the vertices v_4, v_2, v_3 in the triangle $v_4v_2v_3$.

We get the system of equations

$$a = n_1 + m_2 = n_3 + m_2, b = n_2 = n_4, c = n_3 + m_1 = n_1 + m_1.$$

So $n_1 = n_3$ and $n_2 = n_4$. We can write our system then as

$$a = n_1 + m_2, b = n_2, c = n_1 + m_1.$$

Let $x = n_2 - n_1$. Then

$$g = n_2 + m_1 = c + x, h = n_1 = b - x, j = n_2 + m_2 = a + x.$$

Therefore, if $x > 0$ we conclude that the total number of normal arcs $g+h+j > a+b+c$.

Notice that if b is larger than either a or c, then consequentially $n_2 > n_1$, so $x > 0$. After gluing on Δ', there are two possibilities for which of the three integers g, h, j plays the role of b in the new tetrahedron. By this, we mean the normal arc cutting off w_2 (or w_4) on the top two faces of Δ', which does not have ends on the top diagonal. The first gluing pattern above gives j and the second g for this normal arc. However notice now that $h = n_1 < n_2 \leq min\{g, j\}$ and so neither g nor j is the smallest of the three numbers g, h, j. So the computation for the second tetrahedron will result in the same conclusion as the first, namely the total number of normal arcs in the bottom two faces is larger than the number in the top two faces. But we are now trapped in a cycle and so cannot go around the bundle and return to the top of the tetrahedron Δ. At every stage the total number of normal arcs is increasing, so we cannot glue the top to the bottom to close up the normal class. So this shows that there cannot be a normal class with only quadrilaterals which are disjoint from the edges v_1v_2 and v_1v_4, in this case.

The case when $n_2 < n_1$ is entirely similar and corresponds to 'turning the bundle upside down', ie, interchanging the roles of the top and bottom two faces of each tetrahedron. We see by symmetry that the total number of normal arcs must monotonically decrease as we traverse the bundle and again we cannot close up the normal class around the bundle, as in the previous case.

If $n_2 = n_1$, ie $x = 0$, then the total number of normal arcs does not change across the first tetrahedron. If this total increases across the second tetrahedron, we are back in the first case above, and if it decreases, we are in the second situation. So it remains to consider the case where in traversing the second tetrahedron Δ', we also have the same total number of normal arcs. This leads to the conclusion that either $m_1 = 0$ or $m_2 = 0$ or $m_1 = m_2 = 0$. Consequently, either there are no quadrilaterals at all, which is not allowed for the normal classes we are interested in constructing, or there is a single quadrilateral class in each tetrahedron. The latter is easily checked to correspond to the case of monodromy with trace ± 2 which has been excluded, so the discussion is complete.

5 Epilogue

We set out to try to construct angle structures from Lackenby's taut structures [16]. In our next paper [2], we find that taut structures are very common amongst minimal small ideal triangulations. In some sense, there are too many taut triangulations and the obstruction found in section 2 above, shows that many of these do not admit angle structures. There are also most likely too many ideal triangulations with angle structures and further work needs to be done to identify other obstructions to finding hyperbolic structures via this approach. To conclude, we make some observations about our obstruction. Here we restrict to irreducible, $\mathbf{P}^2$–irreducible and atoroidal manifolds with incompressible tori and Klein bottle boundary components.

Observation 1 *If M has a taut ideal triangulation which has a non empty obstruction to find angle structures, then any covering space of M has the same properties.*

Observation 2 *If M has a number of different taut and semi angle structures on the same ideal structure, then the obstruction for one such structure vanishes if and only if the obstruction vanishes for any other one.*

Note that it may be interesting to have a more direct understanding of why this is true, rather than just through the general results above that all such structures lie on the boundary of the same angle space, if and only if any one such structure has vanishing obstruction.

Observation 3 *If M has a taut ideal triangulation Γ, then so does Γ' for at least $\frac{2}{3}$ of the possible choices of a single $2 \to 3$ Pachner move, to change Γ to Γ'. Note that if two tetrahedra Δ, Δ' in Γ are chosen with a common face, then so long as the π angles at the edges of the two tetrahedra do not occur at the same edge of the common face, then there is an easy way of putting the 'same' taut structure on Γ' as on Γ.*

Observation 4 *With the same setup as for Observation 3, there is an obstruction to deform the taut structure on Γ to an angle structure if and only if there is a similar obstruction for Γ'. The proof is by enumerating cases and we will omit it.*

One would like to see how the obstruction to an angle structure changes, in passing between the standard two tetrahedra triangulation of the Figure–8 knot space and the simple four tetrahedra Example 1 above, by $2 \to 3$ and $3 \to 2$ Pachner moves. The conclusion is this can only be done in a way that destroys the taut structure. So unfortunately this means that such moves will not give an easy way of keeping the taut structure but eliminating the obstruction.

References

[1] **I Agol**, *Bounds on exceptional Dehn filling*, Geom. Topol. 4 (2000) 431–449

[2] **B Burton**, **E Kang**, **J H Rubinstein**, *Ideal triangulations of 3–manifolds III; Taut structures in low census manifolds*, in preparation

[3] **P J Callahan**, **M V Hildebrand**, **J R Weeks**, *A census of cusped hyperbolic 3–manifolds*, Math. Comp. 68 (1999) 321–332

[4] **D Coulson**, **O A Goodman**, **C D Hodgson**, **W D Neumann**, *Computing arithmetic invariants of 3–manifolds*, Experiment. Math. 9 (2000) 127–152

[5] **M Freedman**, **J Hass**, **P Scott**, *Least area incompressible surfaces in 3–manifolds*, Invent. Math. 71 (1983) 609–642

[6] **F Gueritaud**, *On canonical triangulations of the mapping tori over the punctured torus*, `arXiv:math/0406242`

[7] **W Haken**, *Theorie der Normalflächen*, Acta Math. 105 (1961) 245–375

[8] **W Haken**, *Some results on surfaces in 3–manifolds*, from: "Studies in Modern Topology", Math. Assoc. Amer. (distributed by Prentice-Hall, Englewood Cliffs, N.J.) (1968) 39–98

[9] **J Hempel**, *3–Manifolds*, Princeton University Press, Princeton, N. J. (1976)

[10] **W Jaco**, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Mathematics 43, American Mathematical Society, Providence, RI (1980)

[11] **W Jaco**, **J H Rubinstein**, *0–efficient triangulations of 3–manifolds*, J. Differential Geom. 65 (2003) 61–168

[12] **W Jaco**, **J H Rubinstein**, *1–efficient triangulations of 3–manifolds*, in preparation

[13] **E Kang**, *Normal surfaces in knot complements*, PhD thesis, University of Connecticut (1999)

[14] **E Kang**, *Normal surfaces in non-compact 3–manifolds*, J. Aust. Math. Soc. 78 (2005) 305–321

[15] **E Kang**, **J H Rubinstein**, *Ideal triangulations of 3–manifolds I; spun normal surface theory*, from: "Proceedings of the Casson Fest (Arkansas and Texas 2003)", Geom. Topol. Monogr. 7 (2004) 235–265

[16] **M Lackenby**, *Taut ideal triangulations of 3–manifolds*, Geom. Topol. 4 (2000) 369–395

[17] **M Lackenby**, *Word hyperbolic Dehn surgery*, Invent. Math. 140 (2000) 243–282

[18] **E E Moise**, *Affine structures in 3–manifolds. V. The triangulation theorem and Hauptvermutung*, Ann. of Math. (2) 56 (1952) 96–114

[19] **J H Rubinstein**, *Polyhedral minimal surfaces, Heegaard splittings and decision problems for 3–dimensional manifolds*, from: "Geometric topology (Athens, GA, 1993)", AMS/IP Stud. Adv. Math. 2, Amer. Math. Soc., Providence, RI (1997) 1–20

[20] **J Simon**, *Compactification of covering spaces of compact 3–manifolds*, Michigan Math. J. 23 (1976) 245–256 (1977)

[21] **M Stocking**, *Almost normal surfaces in 3–manifolds*, Trans. Amer. Math. Soc. 352 (2000) 171–207

[22] **W Thurston**, *Three-dimensional geometry and topology, Vol 1*, (Silvio Levy, editor), Princeton Mathematical Series 35, Princeton University Press, Princeton, NJ (1997)

[23] **S Tillmann**, *Degenerations and normal surface theory*, PhD thesis, University of Melbourne (2002)

[24] **J L Tollefson**, *Normal surface Q–theory*, Pacific J. Math. 183 (1998) 359–374

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