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Degree one maps between small 3-manifolds and Heegaard genus

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Abstract We prove a rigidity theorem for degree one maps between small 3-manifolds using Heegaard genus, and provide some applications and connections to Heegaard genus and Dehn surgery problems.

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Keywords Degree one map, small 3-manifold, Heegaard genus

1 Introduction

All terminology not defined in this paper is standard, see [He] and [Ja].

Let M and N be two closed, connected, orientable 3-manifolds. Let H be a (not necessarily connected) compact 3-submanifold of N. We say that a degree one map $f : M \to N$ is a *homeomorphism outside* H if $f : (M, M - \mathrm{int}f^{-1}(H), f^{-1}(H)) \to (N, N - \mathrm{int}H, H)$ is a map between the triples such that the restriction $f| : M - \mathrm{int}f^{-1}(H) \to N - \mathrm{int}H$ is a homeomorphism. We say also that f is a *pinch* and N is obtained from M by *pinching* $W = f^{-1}(H)$ *onto* H.

Let H be a compact 3-manifold (not necessarily connected). We use $g(H)$ to denote the *Heegaard genus* of H, that is the minimal number of 1-handles used to build H.

We define $mg(H) = max\{g(H_i),\ H_i \textit{ runs over components of } H\}$. It is clear that $mg(H) \leq g(H)$ and $mg(H) = g(H)$ if H is connected.

A path-connected subset X of a connected 3-manifold is said to carry $\pi_1 M$ if the inclusion homomorphism $\pi_1 X \to \pi_1 M$ is surjective.

In this paper, any incompressible surface in a 3-manifold is 2-sided and is not the 2-sphere. A closed 3-manifold M is *small* if it is orientable, irreducible and if it contains no incompressible surface.

It has been observed by Kneser, Haken and Waldhausen ([Ha], [Wa], see also [RW] for a quick transversality argument) that a degree one map $M \to N$ between two closed, orientable 3-manifolds is homotopic to a map which is a homeomorphism outside a handlebody corresponding to one side of a Heegaard splitting of N. This fact is known as "any degree one map between 3-manifolds is homotopic to a pinch".

A main result of this paper is the following rigidity theorem.

Theorem 1 *Let M and N be two closed, small 3-manifolds. If there is a degree one map $f: M \to N$ which is a homeomorphism outside an irreducible submanifold $H \subset N$, then either:*

(1) *There is a component U of H which carries $\pi_1 N$ and such that $g(U) \geq g(N)$, or*

(2) *M and N are homeomorphic.*

Remark 1 Given M and N two non-homeomorphic small 3-manifolds , Theorem 1 implies that N cannot be obtained from M by a sequence of pinchings onto submanifolds of genus smaller than $g(N)$. However Theorem 1 does not hold when M is not small. Below are easy examples:

- Let $f: P\#N \to N$ be a degree one map defined by pinching P to a 3-ball in N. Then f is a homeomorphism outside the 3-ball, which is genus zero and does not carry $\pi_1 N$.
- Let k be a knot in a closed, orientable 3-manifold N and let F be a once punctured closed surface. Let M be the 3-manifold obtained by gluing the boundaries of $F \times S^1$ and of $E(k)$ in such a way that $\partial F \times \{x\}$ is matched with the meridian of k, $x \in S^1$. Then a degree one map $f: M \to N$ pinching $F \times S^1$ to a tubular neigborhood $\mathcal{N}(k)$ of k, is a homeomorphism outside a handlebody of genus 1. If $\pi_1 N$ is not cyclic or tivial, then $g(\mathcal{N}(k)) < g(N)$ and $\mathcal{N}(k)$ does not carry $\pi_1 N$.

The pinched part of a degree one map between closed, orientable non-homeomorphic surfaces has incompressible boundary [Ed]. The following straigtforward corollary of Theorem 1 gives an analogous result for small 3-manifolds:

Corollary 1 *Let M and N be two closed, small, non-homeomorphic 3-manifolds. Let $f: M \to N$ be a degree one map and let $V \cup H = N$ be a minimal genus Heegaard splitting for N. Then the map f can be homotoped to be a homeomorphism outside H such that $f^{-1}(H)$ is ∂-irreducible.*

Remark 2 Corollary 1 remains true for any strongly irreducible heegaard splitting of N. Then the argument, using Casson-Gordon's result [CG], is essentially the same as [Le, Theorem 3.1], even if in [Le] it is only proved for the case $M = S^3$ and N a homotopy 3-sphere. The proof in [Le] is based on his main result [Le, Theorem 1.3], but one can also use a direct argument from degree one maps.

Theorem 1 follows directly from two rather technical Propositions (Proposition 1 and Proposition 2). Theorem 1 and its proof lead to some results about Heegaard genus of small 3-manifolds and Dehn surgery on null-homotopic knots.

Theorem 2 *Let M be a closed, small 3-manifold. Let $F \subset M$ be a closed, orientable surface (not necessary connected) which cuts M into finitely many compact, connected 3-manifolds $U_1, \dots, U_n$. Then there is a component U_i which carries $\pi_1 M$ and such that $g(U_i) \geq g(M)$.*

Remark 3 In general (see [La]) one has only the upper bound:

$$g(M) \leq \sum_{i=1}^{n} g(U_i)) - g(F).$$

Suppose that k is a null-homotopic knot in a closed orientable 3-manifold M. Its unknotting number $u(k)$ is defined as the minimal number of self-crossing changes needed to transform it into a trivial knot contained in a 3-ball in M.

Theorem 3 *Let k be a null-homotopic knot in a closed, small 3-manifold M. If $u(k) < g(M)$, then every closed 3-manifolds obtained by a non-trivial Dehn surgery along k is not small. In particular k is determined by its complements.*

This article is organized as follows.

In Section 2 we state and prove Proposition 1 which is the first step in the proof of Theorem 1. The second step, given by Proposition 2 is proved in Section 3; then Theorem 1 follows from these two propositions. Section 4 is devoted to the proof of Theorem 2, and Section 5 to the proof of Theorem 3.

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2 Making the preimage of H ∂-irreducible

The first step of the proof of Theorem 1 is given by the following proposition:

Proposition 1 *Let M and N be two closed, connected, orientable, irreducible 3-manifolds which have the same first Betti number, but are not homeomorphic.*

Suppose there is a degree one map $f_0 : M \to N$ which is a homeomorphism outside a compact irreducible 3-submanifold $H_0 \subset N$ with $\partial H_0 \neq \emptyset$. Then there is a degree one map $f : M \to N$ which is a homeomorphism outside an irreducible submanifold $H \subset H_0$ such that:

- $\partial H \neq \emptyset$;
- $mg(H) \leq mg(H_0)$,
- *Any connected component of $f^{-1}(H)$ is either ∂-irreducible or a 3-ball, and there is at least one component of $f^{-1}(H)$ which is ∂-irreducible.*

Remark 4 Since M is not homeomorphic to N it is clear that at least one component of $f^{-1}(H)$ is not a 3-ball.

Proof In the whole proof, 3-manifolds M and N are supposed to meet all hypotheses given in the first paragraph of Proposition 1.

By the assumption there is a degree one map $f_0 : M \to N$ which is a homeomorphism outside an irreducible submanifold $H_0 \subset N$ with $\partial H_0 \neq \emptyset$.

Let $\mathcal{H}_0$ be the set of all 3-submanifolds $H \subset H_0$ such that:

(1) There is a degree one map $f : M \to N$ which is a homeomorphism outside H;

(2) $\partial H \neq \emptyset$;

(3) $mg(H) \leq mg(H_0)$;

(4) H is irreducible.

For an element $H \in \mathcal{H}_0$, its complexity is defined as a pair

$$c(H) = (\sigma(\partial H), \pi_0(H))$$

with the lexicographic order, and where $\sigma(\partial H)$ is the sum of the squares of the genera of the components of ∂H, and $\pi_0(H)$ is the number of components of H.

Remark on $c(H)$ The second term of $c(H)$ is not used in this section, but will be used in the next two sections.

Clearly $\mathcal{H}_0$ is not the empty set, since by assumption $H_0 \in \mathcal{H}_0$.

A compressing disk for ∂H in H is a properly embedded 2-disk $(D, \partial D) \subset (H, \partial H)$ such that $\partial D = D \cap \partial H$ is an essential simple closed curve on ∂H (i.e. does not bound a disk on ∂H). In the following we shall denote by $H \backslash \mathcal{N}(D)$ the compact 3-manifold obtained from H by removing an open product neighborhood of D. The operation of removing such neighborhood is called *splitting* H *along* D.

Lemma 1 *Let* H *be a compact orientable 3-manifold and let* $(D, \partial D) \subset (H, \partial H)$ *be a compressing disk . Then* $mg(H_*) \leq mg(H)$*, where* $H_* = H \backslash \mathcal{N}(D)$ *is obtained by splitting* H *along* D*. Moreover* $c(H_*) < c(H)$*.*

Proof By Haken's lemma for boundary-compressing disk ([BO], [CG]), a minimal genus Heegaard surface for H can be isotoped to meet D along a single simple closed curve. It follows that $mg(H_*) \leq mg(H)$.

Since ∂D is an essential simple closed curve on ∂H, it is easy to see that $\sigma(\partial H_*) < \sigma(\partial H)$, therefore $c(H_*) < c(H)$. □

The proof of Proposition 1 follows from the following:

Lemma 2 *Let* $H \in \mathcal{H}_0$ *be an element which realizes the minimal complexity, then any component of* $f^{-1}(H)$ *which is not a 3-ball is* ∂*-irreducible.*

Proof Let $W_0 \subset W = f^{-1}(H)$ be a component which is not homeomorphic to a 3-ball. Such a component exits since M is not homeomorphic to N. To prove that W_0 is ∂-irreducible, we argue by contradiction.

If ∂W_0 is compressible in W, there is a compressing disc $(D, \partial D) \to (W, \partial W)$ whose boundary is an essential simple closed curve on ∂W.

Since $f : M \to N$ is a homeomorphism outside the submanifold $H \subset N$ the restriction $f| : (W, \partial W) \to (H, \partial H)$ maps ∂W homeomorphically onto ∂H. Therefore $f(\partial D)$ is an essential simple closed curve on ∂H which bounds the immersed disk $f(D)$ in H. By Dehn's Lemma, $f(\partial D)$ bounds an embedded disc D^* in H.

Lemma 3 *By a homotopy of* f*, supported on* $W = f^{-1}(H)$ *and constant on* ∂W*, we can achieve that:*

- *$f| : W \to H$ is a homeomorphism in a collar neighborhood of $\partial W \cup D$,*
- *$f|^{-1}(D^*) = D \cup S$, where S is a closed orientable surface.*

Proof We define a homotopy $F : W \times [0,1] \to H$ by the following steps:

(1) $F(x,0) = f(x)$ for every $x \in W$;

(2) $F(x,t) = F(x,0)$ for every $x \in \partial f^{-1}(H) = \partial W$ and for every $t \in [0,1]$;

(3) Then we extend $F(x,1) : D \times \{1\} \to D^*$ by a homeomorphism.

We have defined F on $D \times \{0\} \cup \partial D \times [0,1] \cup D \times \{1\}$ which is homeomorphic to a 2-sphere S^2. Since H is irreducible, by the Sphere theorem $\pi_2(H) = \{0\}$. Hence:

(4) We can extend F to $D \times [0,1]$;

Now F has been defined on $W \times \{0\} \cup \partial W \times [0,1] \cup D \times [0,1]$, which is a deformation retract of $W \times [0,1]$, therefore:

(5) We can finally extend F on $W \times [0,1]$.

After this homotopy we may assume that $f(x) = F(x,1)$, for every $x \in W$. Then by construction this new f sends $\partial W \cup D$ homeomorphically to $\partial H \cup D^*$. By transversality, we may further assume that $f| : W \to H$ is a homeomorphism in a collar neighborhood of $\partial W \cup D$ and that $f|^{-1}(D^*) = D \cup S$, where S is a closed surface. □

The following lemma will be useful:

Lemma 4 *Suppose $f : M \to N$ is a degree one map between two closed orientable 3-manifolds with the same first Betti number $\beta_1(M) = \beta_1(N)$. Then $f_\star : H_2(M;\mathbb{Z}) \to H_2(N;\mathbb{Z})$ is an isomorphism.*

Proof Since $f : M \to N$ is a degree one map, by [Br, Theorem I.2.5], there is a homomorphism $\mu : H_2(N;\mathbb{Z}) \to H_2(M;\mathbb{Z})$ such that $f_\star \circ \mu : H_2(N;\mathbb{Z}) \to H_2(N;\mathbb{Z})$ is the identity, where $f_\star : H_2(M;\mathbb{Z}) \to H_2(N;\mathbb{Z})$ is the homomorphism induced by f.

In particular $f_\star : H_2(M;\mathbb{Z}) \to H_2(N;\mathbb{Z})$ is surjective. Then the injectivity follows from the fact that $H_2(M;\mathbb{Z})$ and $H_2(N;\mathbb{Z})$ are torsion free abelian groups with the same finite rank $\beta_2(M) = \beta_1(M) = \beta_1(N) = \beta_2(M)$. □

Since the degree one map $f : M \to N$ is a homeomorphism outside H, the Mayer-Vietoris sequence and Lemma 4 imply that $f_\star : H_2(W;\mathbb{Z}) \to H_2(H;\mathbb{Z})$ is an isomorphism.

Let S' be a connected component of S. Since $f(S') \subset D^*$, the homology class $[f(S')] = f_\star([S'])$ is zero in $H_2(H,\mathbb{Z})$. Hence the homology class $[S']$ is zero in $H_2(W,\mathbb{Z})$, because $f_\star : H_2(W,\mathbb{Z}) \to H_2(H,\mathbb{Z})$ is an isomorphism. It follows that S' is the boundary of a compact submanifold of W. Therefore S' divides W into two parts W_1 and W_2 such that $\partial W_2 = S'$ and W_1 contains $\partial W \cup D$.

We can define a map $g : W \to H$ such that:

(a) $g|_{W_1} = f|_{W_1}$ and $g(W_2) \subset D^*$.

Then by slightly pushing the image $g(W_2)$ to the correct side of D^*, we can improve the map $g : W \to H$ such that:

(b) $g|\partial W = f|\partial W$,

(c) $g^{-1}(D^*) = D \cup (S \setminus S')$ and $g : \mathcal{N}(D) \to \mathcal{N}(D^*)$ is a homeomorphism.

After finitely many such steps we get a map $h : W \to H$ such that:

(b) $h|\partial W = f|\partial W$,

(d) $h^{-1}(D^*) = D$ and $h : \mathcal{N}(D) \to \mathcal{N}(D^*)$ is a homeomorphism.

Let $H_* = H\backslash\mathcal{N}(D)$ obtained by splitting H along D. Then H_* is still an irreducible 3-submanifold of N with $\partial H_* \neq \emptyset$.

Now $f|_{M-\mathrm{int}W}$ and $h|_W$ together provide a degree one map $h : M \to N$. The transformation from f to h is supported in W, hence h is a homeomorphism outside the irreducible submanifold H_* of N.

Since H_* is obtained by splitting H along a compressing disk, we have $H_* \subset H_0$ and H_* belongs to $\mathcal{H}_0$. Moreover $mg(H_*) \leq mg(H)$ and $c(H_*) < c(H)$ by Lemma 1.

This contradiction finishes the proof of Lemma 2 and thus the proof of Proposition 1. □

3 Finding a closed incompressible surface in the domain

Since closed, orientable, small 3-manifolds are irreducible and have first Betti number equal to zero, Theorem 1 is a direct corollary of the following proposition:

Proposition 2 *Let M and N be two closed, connected, orientable, irreducible 3-manifolds whith the same first Betti number. Suppose that there is a degree one map $f : M \to N$ which is a homeomorphism outside an irreducible submanifold $H_0 \subset N$ such that for each connected component U of H_0, either $g(U) < g(N)$ or U does not carry $\pi_1 N$. Then either M contains an incompressible orientable surface or M is homeomorphic to N.*

Let (M, N) be a pair of closed orientable 3-manifolds such that there is a degree one map from M to N. We say that condition $(*)$ holds for the pair (M, N) if:

$$(*) \qquad \pi_1 N = \{1\} \quad \text{implies} \quad M = S^3.$$

For the proof we first assume that condition $(*)$ holds for the pair (M, N).

Proof of Proposition 2 under condition $(*)$

By the assumptions, there is a degree one map $f : M \to N$ which is a homeomorphism outside an irreducible submanifold $H_0 \subset N$ with $\partial H_0 \neq \emptyset$ and such that for each connected component U of H_0 either $g(U) < g(N)$ or U does not carry $\pi_1 N$. We assume moreover that M is not homeomorphic to N. Our goal is to show that M contains an incompressible surface.

Similar to Section 2, let $\mathcal{H}$ be the set of all 3-submanifolds $H \subset N$ such that:

(1) There is a degree one map $f : M \to N$ which is a homeomorphism outside H.

(2) ∂H is not empty.

(3) For each component U of H, either $g(U) < g(N)$ or U does not carry $\pi_1 N$.

(4) H is irreducible.

The set $\mathcal{H}$ is not empty by our assumptions.

The complexity $c(H) = (\sigma(\partial H), \pi_0(H))$ for the elements of $\mathcal{H}$ is defined like in Section 2.

Lemma 5 *Assume that there is a degree one map $f : M \to N$ which is a homeomorphism outside a submanifold $H \subset N$. If H contains 3-ball component B^3, then f can be homotoped to be a homeomorphism outside H_*, where $H_* = H - B^3$. Moreover if H is irreducible, then H_* is also irreducible.*

Proof By our assumption, there is a degree one map $f : M \to N$ which is a homeomorphism outside a submanifold $H \subset N$ and H contains a B^3 component. Since $f| : f^{-1}(\partial H) \to \partial H$ is a homeomorphism, then $f^{-1}(\partial B^3)$ is a 2-sphere $S^2_* \subset M$. Since M is irreducible, S^2_* bounds a 3-ball B^3_* in M. Then either

(a) $M - \mathrm{int} f^{-1}(B^3) = B^3_*$, or

(b) $f^{-1}(B^3) = B^3_*$.

In case (a), $N = f(B^3_*) \cup B^3$ is a union of two homotopy 3-balls with their boundaries identified homeomorphically, and clearly $\pi_1 N = \{1\}$. So $M = S^3$ by assumption $(*)$. Hence (b) holds in either case.

In case (b), by a homotopy of f supported in $f^{-1}(B^3)$, we can achieve that $f| : f^{-1}(B^3) \to B^3$ is a homeomorphism. Then f becomes a homeomorphism outside the irreducible 3-submanifold $H_* \subset N$, obtained from H by deleting the 3-ball B^3.

The last sentence in Lemma 5 is obviously true. □

Let $H \in \mathcal{H}$ be an element which realizes the minimal complexity. By Lemma 5 no component of H is a 3-ball, hence no component of ∂H is a 2-sphere since H is irreducible. Therefore no component of $f^{-1}(H)$ is a 3-ball and $\partial f^{-1}(H)$ is incompressible in $f^{-1}(H)$ by the proof of Lemma 2.

Since $f : M - \mathrm{int} f^{-1}(H) \to N - \mathrm{int} H$ is a homeomorphism, $\partial f^{-1}(H)$ is incompressible in $M - \mathrm{int} f^{-1}(H)$ if and only if ∂H is incompressible in $N - \mathrm{int} H$. For simplicity we will set $V = N - \mathrm{int} H$, then $N = V \cup H$.

Then the proof of Proposition 2 under condition $(*)$ follows from:

Lemma 6 *If ∂H is compressible in V, then there is $H_* \in \mathcal{H}$ such that $c(H_*) < c(H)$.*

Proof Suppose ∂H is compressible in V. Let $(D, \partial D) \subset (V, \partial V)$ be a compressing disc. By surgery along D, we get two submanifolds H_1 and V_1 as follows:

$$H_1 = H \cup \mathcal{N}(D), \qquad V_1 = V \backslash \mathcal{N}(D).$$

Since H_1 is obtained from H by adding a 2-handle, for each component U' of H_1 there is a component U of H_0 such that $g(U') \le g(U)$ and $\pi_1 U'$ is a quotient of $\pi_1 U$, hence H_1 verifies the defining condition (3) of $\mathcal{H}$. Moreover f is still a homeomorphism outside H_1 because H_1 contains H as a subset.

Clearly $\partial H_1 \neq \emptyset$. Hence H_1 satisfies also the defining conditions (1) and (2) of $\mathcal{H}$. We notice that $c(H_1) < c(H)$ because $\sigma(\partial H_1) < \sigma(\partial H)$.

We will modify H_1 to become $H_* \in \mathcal{H}$ with $c(H_*) \leq c(H_1)$. The modification will be divided into two steps carried by Lemma 8 and Lemma 9 below. First the following standard lemma will be useful:

Lemma 7 *Suppose U is a connected 3-submanifold in N and let $B^3 \subset N$ be a 3-ball with $\partial B^3 = S^2$.*

(i) *Suppose $S^2 \subset \partial U$. If $\mathrm{int}\, U \cap B^3 \neq \emptyset$, then $U \subset B^3$. Otherwise $U \cap B^3 = S^2$.*

(ii) *if $\partial U \subset B^3$, then either $U \subset B^3$, or $N - \mathrm{int}\, U \subset B^3$.*

Proof For (i): Suppose first $\mathrm{int}\, U \cap B^3 \neq \emptyset$. Let $x \in \mathrm{int}\, U \cap B^3$. Since U is connected, then for any $y \in U$, there is a path $\alpha \subset U$ connecting x and y. Since S^2 is a component of ∂U, α does not cross S^2. Hence $\alpha \subset B^3$ and $y \in B^3$, therefore $U \subset B^3$.

Now suppose $\mathrm{int}\, U \cap B^3 = \emptyset$. Let $x \in \partial U \cap B^3$. If $x \in \mathrm{int}\, B^3$, then there is $y \in \mathrm{int}\, U \cap B^3$. It contradicts the assumption. So $x \in \partial B^3 = S^2$.

For (ii): Suppose that U is not a subset of B^3, then there is a point $x \in U \cap (N - \mathrm{int} B^3)$. Let $y \in N - \mathrm{int}\, U$. If $y \in N - \mathrm{int}\, B^3$, there is a path α in $N - \mathrm{int}\, B^3$ connecting x and y, since $N - \mathrm{int}\, B^3$ is connected. This path α does not meet ∂U, because $\partial U \subset B^3$. This would contradict that $x \in U$ and $y \in N - \mathrm{int}\, U$. Hence we must have $y \in B^3$, and therefore $N - \mathrm{int}\, U \subset B^3$. □

Lemma 8 *Suppose H_1 meets the defining conditions (1), (2) and (3) of the set $\mathcal{H}$. Then H_1 can be modified to be a 3-submanifold $H_* \subset N$ such that:*

(i) *∂H_* contains no 2-sphere;*

(ii) *$c(H_*) \leq c(H_1)$;*

(iii) *H_* still meets the the defining conditions (1) (2) (3) of $\mathcal{H}$.*

Proof We suppose that ∂H_1 contains a 2-sphere component S^2, otherwise Lemma 8 is proved. Then S^2 bounds a 3-ball B^3 in N since N is irreducible. We consider two cases:

Case (a) *S^2 does bound a 3-ball B^3 in H_1.*

In this case B^3 is a component of H_1. By Lemma 5, f can be homotoped to be a homeomorphism outside $H_2 = H_1 - B^3$.

Case (b) *S^2 does not bound a 3-ball B^3 in H_1.*

Let H_1' be the component of H_1 such that $S^2 \subset \partial H_1'$. By Lemma 7 (i), either:

(b$'$) $H_1' \subset B^3$, or

(b$''$) $H_1' \cap B^3 = S^2$.

In case (b$'$), let $H_2 = H_1 - B^3$. By Lemma 5 f can be homotoped to be a homeomorphism outside H_2. Note $H_2 \neq \emptyset$, otherwise M and N are homeomorphic, which contradicts our assumption.

In case (b$''$), let $H_2 = H_1 \cup B^3$, then ∂H_2 has at least one component less than ∂H_1. Since we are enlarging H_1, f is a homeomorphism outside H_2.

It is easy to check that in each case (a), (b$'$), (b$''$) the components of H_2 verify the defining condition (3) of $\mathcal{H}$ and $c(H_2) \leq c(H_1) < c(H)$. Moreover H_2 is not empty because M and N are not homeomorphic, and $\partial H_2 \neq \emptyset$ since $g(H_2) \leq g(H_1) < g(N)$. Hence each of the transformations (a), (b$'$) and (b$''$) preserves properties (ii) and (iii) in the conclusion of Lemma 8. Since each one strictly reduces the number of components of H_1 or of ∂H_1, after a finite number of such transformations we reach a 3-submanifold H_* of N such that H_* meets the properties (ii) and (iii) of Lemma 8, and ∂H_* contains no 2-sphere components. This proves Lemma 8. □

Lemma 9 *Suppose that H_1 meets conditions (i), (ii) and (iii) in the conclusion of Lemma 8. Then H_1 can be modified to be a 3-submanifold H_* of N such that:*

(a) *H_* is irreducible;*

(b) *$c(H_*) \leq c(H_1)$ is not increasing;*

(c) *H_* still meets the the defining conditions (1), (2), (3) of $\mathcal{H}$.*

In particular H_* belongs to $\mathcal{H}$.

Proof If there is an essential 2-sphere S^2 in H_1, it must separate N since N is irreducible. Let H_1' be the component of H_1 containing S^2. The 2-sphere S^2 induces a connected sum decomposition of H_1': it separates H_1' into two connected parts $K_\circ$ and $K_\circ'$, such that:

$$H_1' = K \#_{S^2} K' = K_\circ \cup_{S^2} K_\circ',$$

$K_\circ \subset H_1$ (resp. $K_\circ' \subset H_1$) is homeomorphic to a once punctured K (resp. a once punctured K').

By Haken's Lemma, we have:

$$g(H_1') = g(K) + g(K').$$

Neither $K_\circ$ nor $K_\circ'$ is a n-punctured 3-sphere, $n \geq 0$, because ∂H_1 contains no 2-sphere component, hence:

$$g(K) < g(H_1') \qquad \text{and} \qquad g(K') < g(H_1')$$

Since N is irreducible, S^2 bounds a 3-ball B^3 in N. We may assume that $\mathrm{int}K_\circ \cap B^3 = \emptyset$ and $\mathrm{int}K_\circ' \cap B^3 \neq \emptyset$. By Lemma 7 (i), we have $K_\circ \cap B^3 = S^2$ and $K_\circ' \subset B^3$.

Moreover $\partial H_1' \cap B^3 \neq \emptyset$, otherwise $K_\circ'$ is homeomorphic to B^3, in contradiction with the assumption that S^2 is a 2-sphere of connected sum.

Lemma 10 *$\partial H_1'$ is not a subset of B^3.*

Proof We argue by contradiction. If $\partial H_1'$ is a subset of B^3, we have $N - \mathrm{int}H_1' \subset B^3$ by Lemma 7 (ii), since H_1' is not a subset of B^3. Then:

$$N = H_1' \cup (N - \mathrm{int}H_1') = H_1' \cup B^3 = (K_\circ \#_{S^2} K_\circ') \cup B^3 = K_\circ \cup_{S^2} B^3 = K.$$

Hence K is homeomorphic to the whole N. If $g(H_1') < g(N)$, this contradicts the fact that $g(K) < g(H_1') < g(N)$. If H_1' does not carry $\pi_1 N$ this contadicts the fact that $K \subset H_1'$. □

By Lemma 10, $\partial H_1'$ (and therefore ∂H_1) has components disjoint from B^3. Therefore if we replace H_1 by $H_2 = H_1 \cup B^3$, then ∂H_2 is not empty and it has no component which is a 2-sphere. Moreover the application of Haken's Lemma above shows that $g(H_2) < g(H_1)$.

Since we are enlarging H_1, f is a homeomorphism outside H_2, and clearly H_2 still meets the the defining condition (3) of $\mathcal{H}$. Moreover $c(H_2) \leq c(H_1)$. Hence the transformation from H_1 to H_2 preserves properties (b) and (c) in the conclusion of Lemma 9. Since it strictly reduces $g(H_1)$, after a finite number of such transformations we will reach a 3-submanifolds $H_* \subset N$ such that H_* meets conditions (b) and (c) in the conclusion of Lemma 9, but does not contain any essential 2-sphere. This proves Lemma 9. □

Lemma 8 and Lemma 9 imply Lemma 6. Hence we have proved Proposition 2 under condition $(*)$. □

Proof of Proposition 2 Let M and N be two closed, small 3-manifolds which are not homeomorphic. Suppose there is degree one map $f : M \to N$ which is a homeomorphism outside an irreducible submanifold $H \subset N$ such that: for each component U of H, either $g(U) < g(N)$ or U does not carry $\pi_1 N$.

Condition $(*)$ in the above proof of Proposition 2 is only used in the proof of Lemma 5, when H contains a 3-ball component B^3 and that $M - \mathrm{int} f^{-1}(B^3) = B^3_*$ and $f^{-1}(B^3) \neq B^3_*$. Indeed we can now prove that this case cannot occur.

If this case happens then $\pi_1 N = \{1\}$ and thus $mg(H) < g(N)$, since every component of H carries $\pi_1 N$. By replacing $f^{-1}(B^3)$ by a 3-ball $B^3_{\#}$, we obtain a degree one map $\bar{f} : S^3 = B^3_* \cup B^3_{\#} \to N$ defined by $\bar{f}|B_* = f|B_*$ and $\bar{f}| : B^3_{\#} \to B_3$ is a homeomorphism. Then $\bar{f} : S^3 \to N$ is a map which is a homeomorphism outside a submanifold $H' = H - B^3$. Clearly $mg(H') = mg(H) < g(N)$. Furthermore condition $(*)$ now holds.

Since Proposition 2 has been proved under condition $(*)$, we have that N must be homeomorphic to S^3, since S^3 does not contain any incompressible surface. It would follow that $mg(H) < 0$, which is impossible.

The proof of Proposition 2, and hence of Theorem 1 is now complete. □

4 Heegaard genus of small 3-manifolds

This section is devoted to the proof of Theorem 2.

Let M be a closed orientable irreducible 3-manifold. Let $F \subset M$ be a closed orientable surface (not necessary connected) which splits M into finitely many compact connected 3-manifolds $U_1, \ldots, U_n$.

Let $M \backslash \mathcal{N}(F)$ be the manifold M split along the surface F. We define the complexity of the pair (M, F) as

$$c(M, F) = \{\sigma(F), \pi_0(M \backslash \mathcal{N}(F))\},$$

where $\sigma(F)$ is the sum of the squares of the genera of the components of F and $\pi_0(M \backslash \mathcal{N}(F))$ is the number of components of $M \backslash \mathcal{N}(F)$.

Let $\mathcal{F}$ be the set of all closed surfaces F such that for each component U_i of $M \setminus F$, either $g(U_i) < g(M)$ or U_i does not carry $\pi_1 M$.

Remark 5 This condition implies that the surface $F \neq \emptyset$ for every $F \in \mathcal{F}$.

With the hypothesis of Theorem 2, the set $\mathcal{F}$ is not empty. Let $F \in \mathcal{F}$ be a surface realizing the minimal complexity. Then the following Lemma implies Theorem 2.

Lemma 11 *A surface $F \in \mathcal{F}$ realizing the minimal complexity contains no 2-sphere component and is incompressible.*

Proof The arguments are analogous to those used in the proof of Propositions 2. We argue by contradiction.

Suppose that F contains a 2-sphere component S^2. It bounds a 3-ball $B^3 \subset M$, since M is irreducible. Let U_1 and U_2 be the closures of the components of $M \setminus \mathcal{N}(F)$ which contain S^2. Then by Lemma 7 (i), either:

- $U_2 \subset B^3$ and $U_1 \cap B^3 = S^2$, or
- $U_1 \subset B^3$ and $U_2 \cap B^3 = S^2$.

Since those two cases are symmetric, we may assume that we are in the first case. We consider the surface F' corresponding to the decomposition $\{U'_1, \dots, U'_k\}$ of M with $U'_1 = U_1 \cup B^3$, after forgetting all $U_i \subset B^3$ and then re-indexing the remaining U_i's to be $U'_2, \dots, U'_k$. This operation does not increase the Heegaard genus of any one of the components of the new decomposition. Moreover if U_1 does not carry $\pi_1 M$, the same holds for U'_1. Hence F' still belongs to $\mathcal{F}$. However, this operation strictly decreases the number of components of F, hence $c(F') < c(F)$, in contradiction with our choice of F.

Suppose that the surface F is compressible. Then some essential simple closed curve γ on F bounds an embedded disk in M. Let D' be a such a compression disk with the minimum number of circles of intersection in $\text{int}D' \cap F$. Then a subdisk of D' bounded by an innermost circle of intersection is contained inside one of the U_i, say U_1.

Let $(D, \partial D) \subset (U_1, F \cap \partial U_1)$ be such an innermost disk. Let U_2 be adjacent to U_1 along F, such that $\partial D \subset \partial U_2$. By surgery along D, we get a new surface F' which gives a new decomposition $\{U'_1, \dots, U'_n\}$ of M as follows:

$$U'_1 = U_1 \backslash \mathcal{N}(D), \qquad U'_2 = U_2 \cup \mathcal{N}(D), \qquad U'_i = U_i, \text{ for } i \geq 3.$$

Then $g(U'_i) \leq g(U_i)$, for $i = 1, \dots, n$. Moreover if U_i does not carry $\pi_1 M$, the same holds for U'_i. Hence $F' \in \mathcal{F}$. However, $\sigma(F') < \sigma(F)$ since ∂D is an essential circle on F. Therefore $c(F') < c(F)$ and we reach a contradiction. □

5 Null-homotopic knot with small unknotting number

In this section we prove Theorem 3.

Suppose M is a closed, small 3-manifold and $k \subset M$ is a null-homotopic knot with $u(k) < g(M)$. Then clearly M is not the 3-sphere.

If k is a non-trivial knot in a 3-ball $B^3 \subset M$. Then the compact 3-manifold $B^3(k, \lambda)$ obtained by any non-trivial surgery of slope λ on k will not be a 3-ball by [GL]. Therefore $M(k, \lambda)$ contains an essential 2-sphere.

Hence below we assume that k is not contained in a 3-ball.

Since the knot $k \subset M$ is null-homotopic with unknotting number $u(k)$, k can be obtained from a trivial knot $k' \subset B^3 \subset M$ by $u(k)$ self-crossing changes. Let $D' \subset M$ be an embedded disk bounded by k'. If we let D' move following the self-crossing changes from k' to k, then each self-crossing change corresponds to an identification of pairs of arcs in D'. Hence one obtains a singular disk Δ in M with $\partial\Delta = k$ and with $u(k)$ clasp singularities. Since Δ has the homotopy type of a graph, its regular neighborhood $\mathcal{N}(\Delta)$ is a handlebody of genus $g(\mathcal{N}(\Delta)) = u(k) < g(M)$.

First we prove the following lemma which is a particular case of a more general result about Dehn surgeries on null-homotopic knots, obtained in [BBDM]. Since this paper is not yet available, we give here a simpler proof in this particular case.

Lemma 12 *With the hypothesis above, if the slope α is not the meridian slope of k, then $M(k, \alpha)$ is not homeomorphic to M.*

Proof Since M is irreducible and $k \subset M$ is not contained in a 3-ball, $M - \mathrm{int}\mathcal{N}(k)$ is irreducible and ∂-irreducible. Hence $1 \leq u(k) < g(M)$ and M cannot be a lens space.

Let consider the set $\mathcal{W}$ of compact, connected, orientable, 3-submanifolds $W \subset M$ such that:

(1) $k \subset W$ is null-homotopic in W;

(2) there is no 2-sphere component in ∂W;

(3) $g(W) < g(M)$.

By hypothesis the set $\mathcal{W}$ is not empty since a regular neighborhood $\mathcal{N}(\Delta)$ of a singular unknotting disk for k is a handlebody of genus ≥ 1.

Claim 1 *For a 3-submanifold $W_0 \in \mathcal{W}$ with a minimal complexity $c(W_0) = \sigma(\partial W_0)$, the surface ∂W_0 is incompressible in the exterior $M - \mathrm{int}\mathcal{N}(k)$.*

Proof If ∂W_0 is compressible in $M - \mathrm{int}W_0$, let $(D, \partial D) \hookrightarrow (M - \mathrm{int}W_0, \partial W_0)$ be a compression disk for ∂W_0. The 3-manifold $W_1 = W_0 \cup \mathcal{N}(D)$, obtained by adding a 2-andle to W_0, is a compact, connected submanifold of M containing k.

Any 2-sphere in ∂W_1 bounds a 3-ball in $M - \mathrm{int}\mathcal{N}(k)$ since it is irreductible. Hence after gluing some 3-ball along the boundary, we may assume that W_1 contains no 2-sphere component. Moreover $k \subset W_1$ is null-homotopic in W_1 and $g(W_1) \leq g(W_0) < g(M)$. It follows that $W_1 \in \mathcal{W}$. Since $c(W_1) < c(W_0)$ we get a contradiction.

If ∂W_0 is compressible in $W_0 - \mathrm{int}\mathcal{N}(k)$, let $(D, \partial D) \hookrightarrow, (W_0 - \mathrm{int}\mathcal{N}(k), \partial W_0)$ be a compression disk for ∂W_0. Let W_2 be the component of the 3-manifold $W_0 \backslash \mathcal{N}(D)$ which contains k. As above, after possibly gluing some 3-ball along the boundary, we may assume that ∂W_2 contains no 2-sphere component. The knot $k \subset W_2$ is null-homotopic in W_2, since it is null-homotopic in W_0 and $\pi_1 W_2$ is a factor of the free product decomposition of W_0 induced by the $\partial -$ *compression* disk D. Moreover by Lemma 1 $g(W_2) \leq g(W_0) < g(M)$. It follows that $W_2 \in \mathcal{W}$ and $c(W_2) < c(W_0)$. As above this contradicts the minimality of $c(W_0)$. □

To finish the proof of Lemma 12 we distinguish two cases:

(a) *The surface ∂W_0 is compressible in $W_0(k, \alpha)$* Then one can apply Scharlemann's theorem [Sch, Thm 6.1]. The fact that $k \subset W_0$ is null-homotopic rules out cases a) and b) of Scharlemann's theorem. Moreover by [BW, Prop.3.2] there is a degree one map $g : W_0(k, \alpha) \to W_0$, and thus there is a simple closed curve on ∂W_0 which is a compression curve both in $W_0(k, \alpha)$ and in W_0. Therefore case d) of Scharlemann's theorem cannot occure. The remaining case c) of Scharlemann's theorem shows that $k \subset W_0$ is a non-trivial cable of a knot $k_0 \subset W_0$ and that the surgery slope α corresponds to the slope of the cabling annulus. But then the manifold $M(k, \alpha)$ is the connected sum of a non-trivial Lens space with a manifold obtained by Dehn surgery along k_0. If $M(k, \alpha)$ is homeomorphic to the small 3-manifold M, then M and $M(k, \alpha)$ both would be homeomorphic to a Lens space, which is impossible since $1 \leq u(k) < g(M)$.

(b) *The surface ∂W_0 is incompressible in $W_0(k,\alpha)$* Since ∂W_0 is incompressible in $M-\mathcal{N}(k)$, it is incompressible in $M(k,\alpha)$. Therefore $M(k,\alpha)$ and M cannot be homeomorphic since M is a small manifold. □

It follows from [BW, Prop.3.2] that there is a degree one map $f: M(k,\alpha) \to M$ which is a homeomorphism outside $\mathcal{N}(\Delta)$. Since $g(\mathcal{N}(\Delta)) = u(k) < g(M)$, Theorem 3 is a consequence of Theorem 1 and Lemma 12. □

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