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Canonical Decompositions of 3{Manifolds

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Abstract

We describe a new approach to the canonical decompositions of 3{manifolds along tori and annuli due to Jaco{Shalen and Johannson (with ideas from Waldhausen) | the so-called JSJ{decomposition theorem. This approach gives an accessible proof of the decomposition theorem; in particular it does not use the annulus{torus theorems, and the theory of Seifert brations does not need to be developed in advance.

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1 Introduction

In this paper we describe a proof of the so-called JSJ–decomposition theorem for 3–manifolds. This proof was developed as an exercise for ourselves, to confirm an approach that we hope will be useful for JSJ–decomposition in the group-theoretic context. It seems to give a particularly accessible proof of JSJ for 3–manifolds, involving few prerequisites. For example, it does not use the annulus–torus theorems, and the theory of Seifert fibrations does not need to be developed in advance.

We do not give the simplest version of our proof. We prove JSJ for orientable Haken 3–manifolds with incompressible boundary. This involves splitting along tori and annuli. As we describe later, if one restricts to the case that all boundary components are tori, then one only needs to split along tori. With this restriction our proof becomes very much simpler, with many fewer case distinctions. The general JSJ–decomposition can then be deduced quite easily from this special case by doubling the 3–manifold along its boundary. We took the more direct but less simple approach because this was an exercise to test a concept: there is no clear analogue of the peripheral structure and that of a double in the group theoretic case and we preferred our approach to be closer to possible group theoretic analogues.

We do not prove all the properties of the JSJ–decomposition, although it seems that this could be done from our approach with a little more effort. The main results we do not prove — the enclosing theorem and Johannson deformation theorem — have very readable accounts in Jaco's book [4].

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2 Canonical Surfaces and W–Decomposition

We consider orientable Haken 3–manifolds with incompressible boundary and consider splittings along essential annuli and tori. Let S be an annulus or torus that is properly embedded in $(M, \partial M)$ and which is essential (incompressible and not boundary-parallel).

Definition 2.1 S will be called *canonical* if any other properly embedded essential annulus or torus T can be isotoped to be disjoint from S.

Take a disjoint collection $\{S_1, \dots, S_s\}$ of canonical surfaces in M such that no two of the S_i are parallel;

the collection is maximal among disjoint collections of canonical surfaces with no two parallel.

A maximal system exists because of the Kneser–Haken finiteness theorem. The result of splitting M along such a system will be called a *Waldhausen decomposition* (or briefly *W–decomposition*) of M. It is fairly close to the usual JSJ–decomposition which will be described later. The maximal system of pairwise non-parallel canonical surfaces will be called a *W–system*.

The following lemma shows that the W–system $\{S_1, \dots, S_s\}$ is unique up to isotopy.

Lemma 2.2 *Let $S_1, \dots, S_k$ be pairwise disjoint and non-parallel canonical surfaces in $(M, \partial M)$. Then any incompressible annulus or torus T in $(M, \partial M)$ can be isotoped to be disjoint from $S_1 \cup \dots \cup S_k$. Moreover, if T is not parallel to any S_i then the final position of T in $M - (S_1 \cup \dots \cup S_k)$ is determined up to isotopy.*

Proof By assumption we can isotop T off each S_i individually. Writing $T = S_0$, the lemma is thus a special case of the following stronger result:

Lemma 2.3 *Suppose $\{S_0, S_1, \dots, S_k\}$ are incompressible surfaces in an irreducible manifold M such that each pair can be isotoped to be disjoint. Then they can be isotoped to be pairwise disjoint and the resulting embedded surface $S_0 \cup \dots \cup S_k$ in M is determined up to isotopy.*

Without the uniqueness statement a quick conceptual proof of this uses minimal surfaces (one can also use a traditional cut-and-paste type proof; see below). Choose a metric on M which is product metric near ∂M. Freedman, Hass and Scott show in [2] that the least area representatives of the S_i are pairwise disjoint or identical. When two least area representatives are identical, we isotop them off each other in a normal direction (no two of the S_i embed to identical one-sided least area surfaces since an embedded one-sided surface can never be isotoped off itself: the transverse intersection with an isotopic copy is a 1–manifold dual to the first Stiefel–Whitney class of the normal bundle of the surface). There is a slight complication in that the least area representative $S_i \to S_i' \subset M$ may not be an embedding; it can be a double cover onto an embedded one-sided surface S_i'. But in this case we use a nearby embedding onto the boundary of a tubular neighbourhood of S_i'.

In [2] surfaces are considered up to homotopy rather than isotopy, so the above argument uses the fact that two homotopic embeddings of an incompressible surface are isotopic (see [9], Corollary 5.5).

For the uniqueness statement assume we have $S_1, \dots, S_k$ disjointly embedded and then have two different embeddings of $S = S_0$ disjoint from $T = S_1 \cup \dots \cup S_k$. Let $f\colon S \times I \to M$ be a homotopy between these two embeddings and make it transverse to T. The inverse image of T is either empty or a system of closed surfaces in the interior of $S \times I$. Now use Dehn's Lemma and Loop Theorem to make these incompressible and, of course, at the same time modify the homotopy (this procedure is described in Lemma 1.1 of [10] for example). We eliminate 2–spheres in the inverse image of T similarly. If we end up with nothing in the inverse image of T we are done. Otherwise each component T' in the inverse image is a parallel copy of S in $S \times I$ whose fundamental group maps injectively into that of some component S_i of T. This implies that S can be homotoped into S_i and its fundamental group $\pi_1(S)$ is conjugate into some $\pi_1(S_i)$. It is a standard fact (see eg [8]) in this situation of two incompressible surfaces having comparable fundamental groups that, up to conjugation, either $\pi_1(S) = \pi_1(S_j)$ or S_j is one-sided and $\pi_1(S)$ is the fundamental group of the boundary of a regular neighbourhood of T and thus of index 2 in $\pi_1(S_j)$. We thus see that either S is parallel to S_j and is being isotoped across S_j or it is a neighbourhood boundary of a one-sided S_j and is being isotoped across S_j. The uniqueness statement thus follows.

One can take a similar approach to prove the existence of the isotopy using Waldhausen's classification [9] of proper incompressible surfaces in $S \times I$ to show that S_0 can be isotoped off all of $S_1, \dots, S_k$ if it can be isotoped off each of them. □

The thing that makes decomposition along incompressible annuli and tori special is the fact that they have particularly simple intersection with other incompressible surfaces.

Lemma 2.4 *If a properly embedded incompressible torus or annulus T in an irreducible manifold M has been isotoped to intersect another properly embedded incompressible surface F with as few components in the intersection as possible, then the intersection consists of a family of parallel essential simple closed curves on T or possibly a family of parallel transverse intervals if T is an annulus.*

Proof We just prove the torus case. Suppose the intersection is non-empty. If we cut T along the intersection curves then the conclusion to be proved is that

T is cut into annuli. Since the Euler characteristics of the pieces of T must add to the Euler characteristic of T, which is zero, if not all the pieces are annuli then there must be at least one disk. The boundary curve of this disk bounds a disk in F by incompressibility of F, and these two disks bound a ball in M by irreducibility of M. We can isotop over this ball to reduce the number of intersection components, contradicting minimality. □

3 Properties of the W–decomposition

Let $M_1, \ldots, M_m$ be the result of performing the W–decomposition of M along the W–system $\{S_1 \cup \cdots \cup S_s\}$.

We denote by $\partial_1 M_i$ the part of ∂M_i coming from $S_1 \cup \cdots \cup S_s$ and by $\partial_0 M_i$ the part coming from ∂M. Thus $\partial_0 M_i$ and $\partial_1 M_i$ are complementary in ∂M_i except for meeting along circles. The components of $\partial_1 M_i$ are annuli and tori. Both $\partial_1 M_i$ and $\partial_0 M_i$ are incompressible since we started with an M with ∂M incompressible. However, it is possible that ∂M_i is not incompressible.

Lemma 2.2 shows that any incompressible annulus or torus S in M can be isotoped into the interior of one of the M_i. If this S is an annulus its boundary ∂S is necessarily in $\partial_0 M_i$. By the maximality of $\{S_1, \ldots, S_s\}$, if S is canonical it must be parallel to one of the S_j. However, there may be such S which are not canonical and not parallel to an S_j.

Definition 3.1 We call M_i *simple* if any essential annulus or torus $(S, \partial S) \subset (M_i, \partial_0 M_i)$ is parallel to $\partial_1 M_i$. We call M_i *special simple* if, in addition, it admits an essential annulus in $(M_i, \partial_1 M_i)$ (allowing the possibility that the annulus may be parallel to $\partial_0 M_i$). This will, in particular, be true if some component of $\partial_0 M_i$ is an annulus.

Proposition 3.2 *If M_i is non-simple then $(M_i, \partial_0 M_i)$ is either Seifert fibred or an I–bundle.*

If M_i is special simple then either $(M_i, \partial_0 M_i)$ is Seifert fibred of one of the types illustrated in Figure 1 or $(M_i, \partial_0 M_i) = (T^2 \times I, \emptyset)$ (in which case M is a T^2–bundle over S^1 with holonomy of trace $\neq \pm 2$).

A consequence of this proposition is that an irreducible manifold M that has an essential annulus or torus but no canonical one is an I–bundle or Seifert fibered.

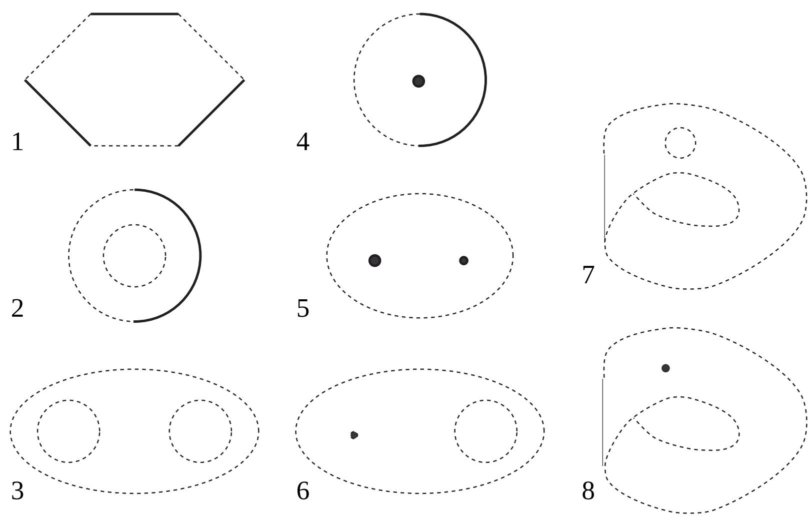

Figure 1: The eight Seifert fibred building blocks. In each case we have drawn the base surface with the portion of the boundary in ∂_0 resp. ∂_1 drawn solid resp. dashed. The dots in cases 4,5,6 represent singular fibres and in case 8 it represents a fibre that may or may not be singular. These cases thus represent infinite families.

Proof We first consider the non-simple case. There are a few cases that will be exceptional from the point of view of our argument and that we will need to treat specially as they come up. They are the cases

- M is an I–bundle over a torus or Klein bottle — then M also fibres as an S^1–bundle over annulus or Möbius band respectively;
- M is an S^1–bundle over torus or Klein bottle.

These are, in fact, among the non-simple manifolds that admit more than one fibred structure. In these cases it is not hard to see that no essential annulus or torus is canonical, so the W–decomposition of M is trivial, so $M = M_1$.

We drop the index and denote $(N, \partial_0 N, \partial_1 N) = (M_i, \partial_0 M_i, \partial_1 M_i)$.

Consider a maximal disjoint collection of pairwise non-parallel essential annuli and tori $\{T_1, \dots, T_r\}$ in $(N, \partial_0 N)$. Split N along this collection into pieces $N_1, \dots, N_n$. We still denote by $\partial_0 N_i$ the part coming from ∂M, that is, $\partial_0 N_i = \partial_0 N \cap N_i = \partial M \cap N_i$, and we denote by $\partial_1 N_i$ the rest of ∂N_i.

We shall show that if M is not one of the exceptions listed above, then the pieces N_i are fibred either as I–bundles or Seifert fibrations, and, moreover, these fibrations match up when we glue the N_i together to form N. In fact:

Claim 1 *Each N_i is either an I–bundle over a twice punctured disk, a Möbius band, or a punctured Möbius band or is Seifert fibred of one of the types in Figure 1.*

$\partial_1 N_i$ consists of annuli and tori, some of which may come from the S_j, but at least one of which comes from a T_j. For this T_j we let T_j' be an essential annulus or torus that intersects T_j essentially and we assume the intersection $T_j \cap T_j'$ is minimal.

The proof is a case by case analysis of this situation. Since the proofs in the different cases are fairly similar, we give a complete argument in only a couple of typical cases.

Case A $\partial T_j \cap \partial T_j' \neq \emptyset$. We shall see that $(N_i, \partial_0 N_i)$ is an $(I, \partial I)$–bundle over a surface and $\partial_1 N_i$ is the part of the bundle lying over the boundary of the surface. The surface in question is either a twice punctured disk, a Möbius band, or a punctured Möbius band (except when $N = M$ was itself an I–bundle over a torus or Klein bottle).

This will be proved case by case. Denote a component of $T_j' \cap N_i$ which intersects T_j by P and let s be a component of the intersection $P \cap T_j$. Since $T_j' \cap (T_1 \cup \cdots \cup T_t)$ will be a finite number of segments crossing T_j', there will be another segment s' in $\partial P \cap (T_1 \cup \cdots \cup T_t)$ and the rest of ∂P will consist of two segments in $\partial_0 N_i$. Let s' be in T_k. We distinguish two subcases.

Case A1 $T_j \neq T_k$. Note that P is an I–bundle over an interval with s and s' the fibres over the ends of this interval, and T_j and T_k are I–bundles over circles. Cutting T_j and T_k along s and s', pushing them just inside N_i, and then pasting parallel copies of P gives an annulus (see Figure 2). It cannot be parallel into $\partial_0 N_i$ because then T_j would be inessential (note

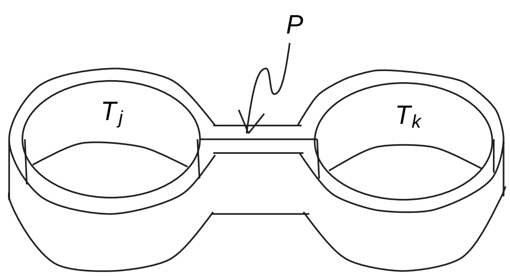

Figure 2: Case A1

that our assumption of boundary-incompressibility implies that an annulus is inessential if an arc across the annulus is isotopic into the boundary). It can not be isotopically trivial since then T_j and T_k would be parallel. It is thus parallel into $\partial_1 N_i$. Thus N_i is an I–bundle over a twice punctured disk.

Case A2 $T_j = T_k$. Since P is an I–bundle over an interval with s and s' the fibres over the ends of this interval, and T_j is an I–bundle over a circle, we may orient the fibres of these bundles so that s is oriented the same in each. Then s' may or may not be oriented the same in each. Moreover, P may meet T_j at s' at the same side of T_j as it meets it at s or at the opposite side.

Case A2.1.1 P meets T_j both times from the same side and s' has the same orientation in both I–bundles. Then if we cut T_j along $s \cup s'$ and glue two parallel copies P' and P'' of P we get two annuli in N_i (see Figure 3). If either

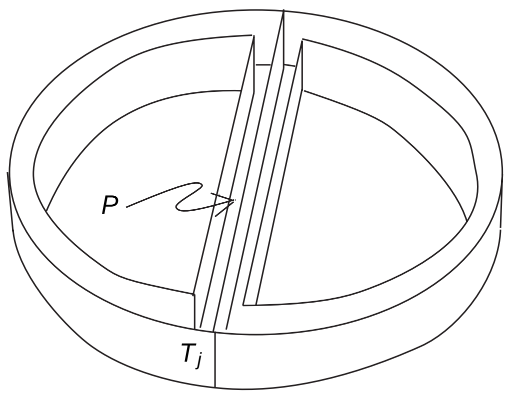

Figure 3: Case A2.1.1

of these annuli is trivial in N_i we could have removed the intersection $s \cup s'$ of T_j and T_j' by an isotopy. Thus they are both non-trivial and must be parallel to components of $\partial_1 N_i$ or $\partial_0 N_i$. If either is parallel into $\partial_0 N_i$ then T_j would have been inessential. They are thus both parallel into $\partial_1 N_i$ and we see that N_i is an I–bundle over a twice punctured disk.

Case A2.1.2 P meets T_j both times from the same side and s' has opposite orientations in the two I–bundles. Then cutting T_j as above and gluing two copies P' and P'' of P yields a single annulus (see Figure 4) which may be isotopically trivial or parallel into $\partial_1 N_i$ (as before, it cannot be parallel into $\partial_0 N_i$). This gives an I–bundle over a Möbius band or punctured Möbius band.

Case A2.2 P meets T_j from opposite sides (and s' has the same or opposite orientations in the two I–bundles). After cutting open along T_j we have two copies $T_j^{(1)}$ and $T_j^{(2)}$ of T_j and this becomes similar to case A1: cutting $T_j^{(1)}$ and $T_j^{(2)}$ along s and s' and pasting parallel copies of P gives an annulus. As

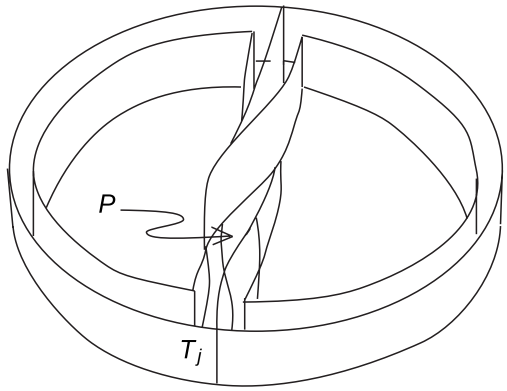

Figure 4: Case A2.1.2

in Case A1, this annulus cannot be parallel into $\partial_0 N_i$. It may be parallel into $\partial_1 N_i$, and then N_i is an I–bundle over a twice punctured disk. Unlike case A1, it may also be isotopically trivial, in which case N_i is an I–bundle over an annulus, so $N = M$ is an I–bundle over the torus or Klein bottle, giving two of the exceptional cases mentioned at the start of the proof.

Case B $\partial T_j \cap \partial T_j' = \emptyset$. In this case we will see that N_i is Seifert fibred of one of eight basic types of Figure 1 (in the exceptional cases mentioned at the start of the proof — where the W–decomposition is trivial and $N = M$ is itself an S^1–bundle over an annulus, Möbius band, torus, or Klein bottle — there is just one piece N_1 after cutting and it does not occur among the eight types).

Denote a component of $T_j' \cap N_i$ which intersects T_j by P and let s be a component of the intersection $P \cap T_j$. This P is an annulus and s is one of its boundary components. Denote the other boundary component by s'. It either lies on $\partial_0 N_i$ or on some T_k.

Case B1 s' lies on $\partial_0 N_i$. There are two subcases according as T_j is an annulus or torus.

Case B1.1 T_j is an annulus (and s' lies on $\partial_0 N_i$; here, and in the following, subcases inherit the assumptions of their parent case). Cutting T_j along s and pasting parallel copies of P to the resulting annuli gives a pair of annuli which must both be parallel into $\partial_1 N_i$, since if either were parallel into $\partial_0 N_i$ we could have removed the intersection s of T_j and T_j' by an isotopy. This gives Figure 1.1 (ie case 1 of Figure 1).

Case B1.2 T_j is a torus. Cutting T_j along s and pasting parallel copies of P to the resulting annulus gives an annulus which must be parallel into $\partial_1 N_i$, since if it were parallel into $\partial_0 N_i$ then T_j would be boundary parallel. This gives Figure 1.2.

Case B2 s' lies on T_k and $T_k \neq T_j$. There are three subcases according as both, one, or neither of T_j and T_k are annuli.

Case B2.1 T_j and T_k both annuli. Cutting T_j and T_k along s and s' and pasting parallel copies of P to the resulting annuli gives a pair of annuli. They cannot both be parallel into $\partial_1 N_i$, for then we would get a picture like Figure 1.1 but based on an octagon rather than a hexagon and an annulus joining opposite ∂_0 sides of this would contradict the fact that the T_i's formed a maximal family. They cannot both be parallel into $\partial_0 N_i$ for then T_i and T_j would be parallel. Thus one is parallel into $\partial_1 N_i$ and the other into $\partial_0 N_i$. This gives Figure 1.1. again.

Case B2.2 One of T_j and T_k an annulus and the other a torus. Cut and paste as before gives a single annulus which may be parallel into ∂_0, giving Figure 1.2. It cannot be parallel into ∂_1, since then we would get a picture like Figure 1.2 but with two ∂_0 annuli in the outer boundary rather than just one. One could span an essential annulus from one of these components of ∂_0 to itself around the torus, contradicting maximality of the family of T_j's.

Case B2.3 Both of T_j and T_k tori. Cut and paste as before gives a single torus. It cannot be parallel into ∂_0 since then the annulus that can be spanned across P from this torus to itself would contradict the fact that the T_j's form a maximal family. It may be parallel into ∂_1 leading to Figure 1.3 or it may also bound a solid torus which gives 1.6.

Case B3 s' lies on T_j. There two subcases according as T_j is an annulus or torus, and each subcase splits into three subcases according to whether P meets T_j from the same or opposite sides at s and s', and if the same side, then whether fibre orientations match at s and s'. We shall describe them briefly and leave details to the reader.

Case B3.1 T_j an annulus.

Case B3.1.1 P meets the annulus T_j both times from the same side.

Case B3.1.1.1 The orientations of s and s' match. Cut and paste as before gives an annulus and a torus. The annulus cannot be parallel into ∂_1 since otherwise one can find an annulus that contradicts maximality of the family of T_j's, so it is parallel into ∂_0. Similarly, the torus cannot be parallel into ∂_0. This leads to the cases of Figure 1.2 or 1.4 according to whether the torus is parallel into ∂_1 or bounds a solid torus.

Case B3.1.1.2 Orientations of s and s' do not match. Cut and paste as before gives an annulus. Whether it is parallel into ∂_0 or ∂_1, N_i is a circle bundle over a Möbius band with part of its boundary in ∂_0 and an annulus from ∂_0 to ∂_0 running once around the Möbius band contradicts the maximality of the family of T_j's. Thus this case cannot occur.

Case B3.1.2 Assumptions of Case B3.1 and P meets the annulus T_j from opposite sides. This gives the same as Case B2.1, except for one extra possibility, since both annuli of Case B2.1 being parallel into ∂_0 is not ruled out. This extra case leads to $N = M$ being an S^1–bundle over the annulus or Möbius band.

Case B3.2 Assumptions of Case B3 with T_j a torus.

Case B3.2.1 P meets the torus T_j both times from the same side.

Case B3.2.1.1 Orientations of s and s' match. Cut and paste gives two tori, neither of which is parallel to ∂_0. This leads to Figure 1.3, 1.5, or 1.6.

Case B3.2.1.2 Orientations of s and s' do not match. Cut and paste gives one torus which may be parallel to ∂_1 or bound a solid torus. This gives Figure 1.7 or 1.8.

Case B3.2.2 Assumptions of Case B3.2 and P meets the torus T_j from opposite sides. This gives the same as Case B2.3, except that in the situation of Figure 1.6 the "singular" fibre need not actually be singular, in which case $N = M$ is an S^1–bundle over the torus or Klein bottle (these are among the special manifolds listed at the start of the proof).

This completes the analysis of the pieces N_i and thus proves Claim 1. We must now verify that the fibred structures on the pieces N_i match up in N.

Suppose two pieces N_{i_1} and N_{i_2} meet across T_j and let T_j' be as above. The above argument showed that a fibred structure can be chosen on these two pieces to match the fibred structure of the T_j' and hence to match each other across T_j. Thus if every piece N_i has a unique fibred structure then the fibred structures must match across every T_j.

Claim 2 *The only pieces N_i for which the fibration is not unique up to isotopy are:*

1) *I–bundle over Möbius band, which also admits a fibration as*

2) *the Seifert fibration of Figure 1.4 with degree 2 exceptional fibre;*

3) *the Seifert fibration of Figure 1.5 with both exceptional fibres of degree 2, which also admits the structure of*

4) *the circle bundle of Figure 1.8 with no exceptional fibre.*

If we grant this claim then the matching of fibrations on the various N_i follows: since in the above non-unique cases N_i has only one boundary component T_j, we simply choose the fibration on this N_i that is forced by T_j' and it will then match the neighbouring piece.

Claim 2 follows by showing that the fibration is unique in all other cases. It is clear that the only N_i that is both an I–bundle and Seifert fibred is the one of cases 1 and 2, so we need only consider non-uniqueness of Seifert fibration. All we really need for the above proof is that the Seifert fibration is unique up to isotopy when restricted to the boundary, which is clear for cases 1, 2, 4 of Figure 1 and follows from the fact that a circle fibre generates an infinite cyclic normal subgroup of the fundamental group in the other cases. The uniqueness of the fibration on the whole of N_i follows, if desired, by a standard argument once at least one fibre has been determined.

To complete the proof of the proposition we must consider the case that M_i is special simple, so it is simple but has an essential annulus in $(M_i, \partial_1 M_i)$. If we call this annulus P we can repeat word for word the arguments of Case B above to see that M_i is of one of the eight types listed in Figure 1 or is an S^1–bundle over an annulus or Möbius band with boundary belonging to $\partial_1 M_i$. Since S^1–bundle over Möbius band is included in the eight types and S^1–bundle over annulus is $T^2 \times I$, the statement of the proposition follows. Note that the $T^2 \times I$ case can only occur if the two boundary tori are the same in M, in which case M is a torus bundle over the circle. The holonomy of this torus bundle cannot have trace ± 2 since then the torus would not be canonical (in fact M would be an S^1–bundle over torus or Klein bottle). □

It is worth noting that the above proof shows also that the only non-simple Seifert fibred manifolds with non-unique Seifert fibration are those with trivial W–decomposition mentioned at the beginning of the above proof (circle bundles over annulus, Möbius band, torus, or Klein bottle) and the one that comes from matching the non-unique cases 3 and 4 of the above Claim 2 together, giving the Seifert fibration over the projective plane with unnormalized Seifert invariant $\{-1; (2,1), (2,-1)\}$ (two exceptional fibres of degree 2 and rational Euler number of the fibration equal to zero). This manifold has two distinct fibrations of this type. Note that matching two of case 3 or two of case 4 together gives the tangent circle bundle of the Klein bottle, which is one of the examples with trivial W–decomposition already mentioned, but we see this way its Seifert fibration over S^2 with four degree two exceptional fibres (unnormalized invariant $\{0; (2,1), (2,1), (2,-1), (2,-1)\}$).

We shall classify the pieces M_i with $\partial_1 \neq \emptyset$ into three types:

M_i is *strongly simple*, by which we mean simple and not special simple;

M_i is an I-bundle;

M_i is Seifert fibred.

We first discuss which M_i belong to more than one type.

Proposition 3.3 *The only cases of an M_i having more than one type are:*

The I-bundle over Möbius band is also Seifert fibered.

The I-bundles over twice punctured disk and once punctured Möbius band are also strongly simple.

Proof If M_i admits both a Seifert fibration and an I-bundle structure, it is an I-bundle over an annulus or Möbius band (since we are assuming $\partial_1 \neq \emptyset$). If M_i is I-bundle over annulus then $\partial_1 M_i$ consists of two annuli. Since they are parallel in M_i they must have been equal in M, so M is obtained by pasting these two annuli together, that is, M is an I-bundle over torus or Klein bottle. But the annulus is then not canonical (in fact, such M has trivial W-decomposition), so this cannot occur. Thus the only case is that M is I-bundle over Möbius band.

If M_i is both strongly simple and an I-bundle then we have already pointed out that M_i cannot be I-bundle over annulus. The I-bundle over Möbius band is Seifert fibred and is therefore special simple (see below). An I-bundle over a bounded surface of Euler number less than -1 will not be simple. Thus the only cases are those of the proposition.

Suppose M_i is Seifert fibred and simple. If some component of $\partial_0 M_i$ is an annulus then M_i is special simple. Otherwise, $\partial_1 M_i$ consists of tori and M_i is Seifert fibred with at least two singular fibres if the base is a disk and at least one if it is an annulus. It is thus easy to see that for each component of $\partial_1 M_i$ there will be an essential annulus with both boundaries in this component, so M_i is again special simple. Thus M_i cannot be both Seifert fibred and strongly simple. □

We now want to investigate the possibility of adjacent fibred pieces in the W-decomposition having matching fibres where they meet. We therefore assume that the W-decomposition is non-trivial, so M_i has non-empty ∂_1.

Suppose the pieces M_i and M_k on the two sides of a canonical annulus S_j are both I-bundles. If $i \neq k$ then in each of M_i and M_k we can find an essential

$I \times I$ from S_j to itself. Gluing these gives an essential annulus crossing S_j essentially. Thus S_j was not canonical. If $i = k$ the argument is similar. Thus I–bundles cannot be adjacent in the W–decomposition.

It remains to consider the case that both pieces on each side of a canonical annulus or torus S_j are Seifert fibred. Suppose the fibrations on both sides match along S_j.

If a piece M_i adjacent to S_j is not simple we shall decompose it as in the proof of Proposition 3.2 and, for the moment, just consider the simple piece of M_i adjacent to S_j. We will call it N_i for convenience. We first assume that N_i is not pasted to itself across S_j.

This N_i has an embedded essential annulus compatible with its fibration of one of the following types:

- an essential annulus from S_j to a component of $\partial_0 N_i$ (“essential” here means incompressible and not parallel into $\partial_1 N_i$ by an isotopy that keeps the one boundary component in ∂_0 and the other in ∂_1, but of course allows the first to isotop to $\partial_0 \cap \partial_1$);
- an essential annulus from S_j to S_j in N_i (incompressible and not parallel to $\partial_1 N_i$).

This can be seen on a case by case basis by considering the eight cases of Figure 1.

If we had an essential annulus of the first type in N_i then, depending on the situation on the other side of S_j, we can either glue it to a similar annulus on the other side to obtain an essential annulus in $(M, \partial M)$ crossing S_j in a circle, or glue two parallel copies to an essential annulus from S_j to itself on the other side, to obtain an essential annulus crossing S_j in two circles. Either way we see that S_j was not canonical, so this cannot occur.

Thus the only possibility is that we have essential annuli from S_j to itself on both sides of S_j, which we can then glue together to get a torus or Klein bottle crossing S_j in two circles. If it were a Klein bottle, then the boundary of a regular neighbourhood would be an essential torus crossing S_j in four circles, contradicting that S_j is canonical. If it is a torus it will be essential unless it is parallel into ∂M. The only way this can happen is if the pieces on each side of S_j are of the type of case 2 or 4 of Figure 1 and S_j is the annular part of ∂_1 of these pieces. Moreover, we see that the M_i on each side of S_j is simple, for N_i would otherwise have to be case 2 of Figure 1 with another Seifert building block pasted along the inside torus component of $\partial_1 N_i$ and we could find an annulus in M_i from S_j to itself that goes through this additional building block and is therefore not parallel to ∂M.

We must finally consider the case that N_i is pasted to itself across S_j. An annulus connecting the two boundary components of N_i that are pasted becomes a torus or Klein bottle in M. If it were a Klein bottle, then the boundary of a regular neighbourhood would be an essential torus crossing S_j in two circles, contradicting that S_j is canonical. If it is a torus it is essential unless it is parallel into ∂M. By considering the cases in Figure 1 one sees that the only possible N_i is Figure 1.1 pasted as in the bottom picture of Figure 5. Moreover, as in the previous paragraph we see that there cannot be another Seifert building block pasted to the remaining free annulus.

Summarising, we have:

Lemma 3.4 *If two fibred pieces of the W–decomposition are adjacent in M with matching fibrations then they are each of type 2 or 5 of Figure 1 matched along the annular part of ∂_1. If a fibred piece is adjacent to itself then it is of type 1 of Figure 1. The possibilities are drawn in Figure 5.* □

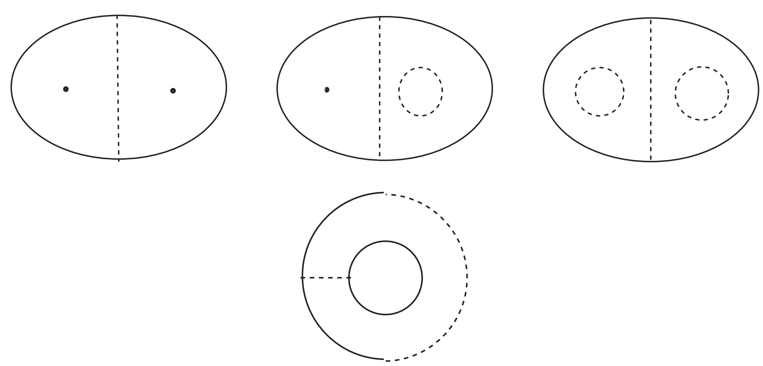

Figure 5: Matched annuli. In each case we have drawn the base surface of the Seifert fibration with the portion of the boundary in ∂_0 drawn solid. The matched annulus lies over the dashed line in the interior.

Definition 3.5 We shall call an annulus separating two matched fibred pieces or a fibred piece from itself as in Lemma 3.4 a *matched annulus*. If we delete all matched annuli from the W–system then the remaining surfaces will be called the *JSJ–system* and the decomposition of M along this JSJ–system will be called the *JSJ–decomposition of M*.

Let $\{S_1,\dots,S_t\}$ be the JSJ–system of canonical surfaces, so that splitting along this system gives the JSJ–decomposition. We recover the W–decomposition as follows: for each piece M_i of the JSJ–decomposition as in Figure 5 we add an annulus as in that figure.

4 Other versions of JSJ–decomposition

The JSJ–decomposition is the same as the canonical splitting as described by Jaco and Shalen in [3] Chapter V, section 4. That decomposition is characterised by the fact that it is a decomposition along a minimal family of essential annuli and tori that decompose M into fibred and simple non-fibred pieces. The JSJ–system of surfaces satisfies this minimality condition by construction.

Jaco and Shalen's "characteristic submanifold" is a fibred submanifold Σ of M which is essentially the union of the fibred parts of the JSJ–splitting except that:

- wherever two fibred parts of the JSJ–decomposition meet along an essential torus or annulus, thicken that torus or annulus and add the resulting $T^2 \times I$ or $A \times I$ to the complementary part $\overline{M - \Sigma}$;
- wherever two non-fibred pieces meet along an annulus or torus, thicken that surface and add the resulting $A \times I$ or $T^2 \times I$ to Σ.

The special case that M has Euler characteristic 0, so its boundary consists only of tori, deserves mention. If there is an annulus in the maximal system of canonical surfaces, then any adjacent piece M_i of the W–decomposition has an annular component in its ∂_0 and is therefore Seifert fibred by Proposition 3.2. Moreover, the fibrations on these adjacent pieces match along this annulus, so it is a matched annulus. Hence:

Proposition 4.1 *If M has Euler characteristic* 0 *(equivalently, ∂M consists only of tori) then the JSJ–system consists only of tori.* □

This proposition seems less well-known than it should be. It is contained in Proposition 10.6.2 of [5] and is used extensively without reference in [1].

There is another modification of the JSJ–decomposition that is often useful, called the *geometric decomposition*, since it is the decomposition that underlies Thurston's geometrisation conjecture (or rather "geometrisation theorem" since

Thurston proved it in the Haken case). The geometric decomposition may have incompressible Möbius bands and Klein bottles as well as annuli and tori in the splitting surface. It is obtained from the JSJ–decomposition by eliminating all pieces which are fibered over the Möbius band as follows:

> delete the canonical annulus that bounds any piece which is an I–bundle over a Möbius band and replace it by the I–bundle over the core circle of the Möbius band (which is itself a Möbius band), and
>
> delete the canonical torus that bounds any piece which is an S^1–bundle over a Möbius band and replace it by the S^1–bundle over the core circle of the Möbius band (which is a Klein bottle).

One advantage of the geometric decomposition is that it lifts correctly in finite covering spaces.

We may also consider only essential annuli or only essential tori and restrict our definition of “canonical” to the one type of surface. Thus we define an essential annulus to be “annulus-canonical” if it can be isotoped off any other essential annulus, and an essential torus is “torus-canonical” if it can be isotoped off any other essential torus.

Proposition 4.2 *An essential annulus is annulus-canonical if and only if it is canonical or is as in Figure 6. An essential torus is torus-canonical if and only if it is canonical or is parallel to a torus formed from a canonical annulus and an annulus in* ∂M *(Figure 7).*

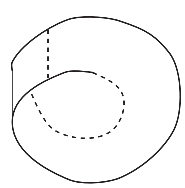

Figure 6: The annulus over the dashed interval is annulus-canonical although it has an essential transverse torus. The dashed portion of the Möbius band boundary represents a canonical annulus.

Proof The “if” is clear. For the only if, note that any annulus-canonical annulus that is not canonical will intersect an essential torus and must therefore occur inside a Seifert fibered piece. It will thus play a similar role to the matched canonical annuli discussed above. But the argument we used to classify matched canonical annuli shows that any matched annulus-canonical annulus is as in

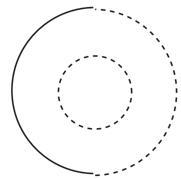

Figure 7: The torus over the inner circle is always torus-canonical, unless it bounds a solid torus. The dashed portion of the outer boundary represents a canonical annulus.

Lemma 3.4 and is thus canonical, except that the argument using the boundary of a regular neighbourhood of a Klein bottle no longer eliminates the case of Figure 6. The argument in the torus case is similar and left to the reader. □

5 Only torus boundary components

If M only has torus boundary components then we can do W{decomposition from the start only using torus-canonical tori. This leads to a much simpler proof. There are many fewer cases to consider and the issue of \matched annuli" disappears | Seifert brations never match across a canonical torus, so the W{decomposition one gets by this approach is exactly the JSJ{decomposition. The general case as described by Jaco and Shalen can then be deduced from this case by a doubling argument. If one is only interested in 3{manifold splittings this is therefore probably the best approach.

We sketch the argument. If M is not an S^1{bundle over torus or Klein bottle we cut a non-simple piece $N = M_i$ of this \toral" W{decomposition into pieces N_j along a maximal system of disjoint non-parallel essential tori, analogously to the proof of Proposition 3.2. These pieces turn out to be of nine basic types, namely what one gets by taking the examples of Figure 1 that have no annuli in $@_0$ (cases 3,5,6,7,8), and then allowing some but not all of the boundary components to be in $@_0$ (so, for example, Case 3 splits into three types according to whether 0, 1, or 2 of the boundary components are in $@_0$.) One then shows, as before, that the brations match up to give a Seifert bration of N.

The classi cation of pieces M_i that are both simple and Seifert bered is well known and can also easily be extracted from our discussion. The ones with boundary are precisely cases 3,5,6,7,8 of Figure 1 but with any number of boundary components in $@_0 M = @M$. The closed ones are manifolds with nite $H_1(M)$ that ber over S^2 with at most three exceptional bers or over $\mathbb{R}P^2$ with at most one exceptional ber.

6 Bibliographical Remarks

The theory of characteristic submanifolds of 3{manifolds for Haken manifolds with toral boundaries was rst outlined by Waldhausen in [11]; see also [12] for his later account of the topic. It was worked out in this form by Johannson [5]. The description of the decomposition in terms of annuli and tori was given in Jaco{Shalen's memoir [3] (see also Scott's paper [7] for this description as well as a proof of the Enclosing Theorem). The idea of canonical surfaces is suggested by the work of Sela and Thurston; a similar idea was used independently by Leeb and Scott in [6] in the context of non-positively curved manifolds. Generalizations using submanifolds of the boundary are discussed in both Jaco{Shalen and Johannson; the characteristic submanifold that we described here is with respect to the whole boundary (T = @M, in the terminology of [4]).

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