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Embeddings from the point of view of immersion theory : Part I

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Abstract

Let M and N be smooth manifolds without boundary. Immersion theory suggests that an understanding of the space of smooth embeddings $\mathrm{emb}(M, N)$ should come from an analysis of the cofunctor $V \mapsto \mathrm{emb}(V, N)$ from the poset $\mathcal{O}$ of open subsets of M to spaces. We therefore abstract some of the properties of this cofunctor, and develop a suitable calculus of such cofunctors, Goodwillie style, with Taylor series and so on. The terms of the Taylor series for the cofunctor $V \mapsto \mathrm{emb}(V, N)$ are explicitly determined. In a sequel to this paper, we introduce the concept of an analytic cofunctor from $\mathcal{O}$ to spaces, and show that the Taylor series of an analytic cofunctor F converges to F. Deep excision theorems due to Goodwillie and Goodwillie–Klein imply that the cofunctor $V \mapsto \mathrm{emb}(V, N)$ is analytic when $\dim(N) - \dim(M) \geq 3$.

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0 Introduction

Recently Goodwillie [9], [10], [11] and Goodwillie–Klein [12] proved higher excision theorems of Blakers–Massey type for spaces of smooth embeddings. In conjunction with a calculus framework, these lead to a calculation of such spaces when the codimension is at least 3. Here the goal is to set up the calculus framework. This is very similar to Goodwillie's calculus of homotopy functors [6], [7], [8], but it is not a special case. Much of it has been known to Goodwillie for a long time. For some history and a slow introduction, see [23]. If a reckless introduction is required, read on — but be prepared for Grothendieck topologies [18] and homotopy limits [1], [23, section 1].

Let M and N be smooth manifolds without boundary. Write $\mathrm{imm}(M,N)$ for the space of smooth immersions from M to N. Let $\mathcal{O}$ be the poset of open subsets of M, ordered by inclusion. One of the basic ideas of immersion theory since Gromov [14], [16], [19] is that $\mathrm{imm}(M,N)$ should be regarded as just one value of the cofunctor $V \mapsto \mathrm{imm}(V,N)$ from $\mathcal{O}$ to spaces. Here $\mathcal{O}$ is treated as a category, with exactly one morphism $V \to W$ if $V \subset W$, and no morphism if $V \not\subset W$; a *cofunctor* is a contravariant functor.

The poset or category $\mathcal{O}$ has a Grothendieck topology [18, III.2.2] which we denote by J_1. Namely, a family of morphisms $\{V_i \to W \mid i \in S\}$ qualifies as a *covering* in J_1 if every point of W is contained in some V_i. More generally, for each $k > 0$ there is a Grothendieck topology J_k on $\mathcal{O}$ in which a family of morphisms $\{V_i \to W \mid i \in S\}$ qualifies as a *covering* if every finite subset of W with at most k elements is contained in some V_i. We will say that a cofunctor F from $\mathcal{O}$ to spaces is a *homotopy sheaf* with respect to the Grothendieck topology J_k if for any covering $\{V_i \to W \mid i \in S\}$ in J_k the canonical map

$$F(W) \longrightarrow \operatorname*{holim}_{\emptyset \neq R \subset S} F(\cap_{i \in R} V_i)$$

is a homotopy equivalence. Here R runs through the finite nonempty subsets of S. In view of the homotopy invariance properties of homotopy inverse limits, the condition means that the values of F on large open sets are sufficiently determined for the homotopy theorist by the behavior of F on certain small open sets; however, it depends on k how much smallness we can afford. — The main theorem of immersion theory is that the cofunctor $V \mapsto \mathrm{imm}(V,N)$ from $\mathcal{O}$ to spaces is a homotopy sheaf with respect to J_1, provided $\dim(N)$ is greater than $\dim(M)$ or $\dim(M) = \dim(N)$ and M has no compact component.

In this form, the theorem may not be very recognizable. It can be decoded as follows. Let Z be the space of all triples (x, y, f) where $x \in M$, $y \in N$ and $f\colon T_xM \to T_yN$ is a linear monomorphism. Let $p\colon Z \to M$ be the projection to the first coordinate. For $V \in \mathcal{O}$ we denote by $\Gamma(p;V)$ the space of partial sections of p defined over V. It is not hard to see that $V \mapsto \Gamma(p;V)$ is a homotopy sheaf with respect to J_1. (Briefly: if $\{V_i \to W\}$ is a covering in J_1, then the canonical map q: $\mathrm{hocolim}_R \bigcap_{i \in R} V_i \to W$ is a homotopy equivalence according to [24], so that $\Gamma(p;W) \simeq \Gamma(q^*p) = \mathrm{holim}_R\, \Gamma(p; \bigcap_{i\in R} V_i)$.) There is an obvious inclusion

$$(*) \qquad \mathrm{imm}(V;N) \hookrightarrow \Gamma(p;V)$$

which we regard as a natural transformation between cofunctors in the variable V. We want to show that $(*)$ is a homotopy equivalence for every V, in particular for $V = M$; this is the decoded version of the main theorem of immersion theory, as stated in Haefliger–Poenaru [15] for example (in the PL setting). By inspection, $(*)$ is indeed a homotopy equivalence when V is diffeomorphic to $\mathbb{R}^m$. An arbitrary V has a smooth triangulation and can then be covered by the open stars V_i of the triangulation. Since $(*)$ is a homotopy equivalence for the V_i and their finite intersections, it is a homotopy equivalence for V by the homotopy sheaf property.

Let us now take a look at the space of smooth embeddings $\mathrm{emb}(M;N)$ from the same point of view. As before, we think of $\mathrm{emb}(M;N)$ as just one value of the cofunctor $V \mapsto \mathrm{emb}(V;N)$ from $\mathcal{O}$ to spaces. The cofunctor is clearly not a homotopy sheaf with respect to the Grothendieck topology J_1, except in some very trivial cases. For if it were, the inclusion

$$(**) \qquad \mathrm{emb}(V;N) \longrightarrow \mathrm{imm}(V;N)$$

would have to be a homotopy equivalence for every $V \in \mathcal{O}$, since it is clearly a homotopy equivalence when V is diffeomorphic to $\mathbb{R}^m$. In fact it is quite appropriate to think of the cofunctor $V \mapsto \mathrm{imm}(V;N)$ as the *homotopy sheafification* of $V \mapsto \mathrm{emb}(V;N)$, again with respect to J_1. The natural transformation $(**)$ has a suitable universal property which justifies the terminology.

Clearly now is the time to try out the smaller Grothendieck topologies J_k on $\mathcal{O}$. For each $k > 0$ the cofunctor $V \mapsto \mathrm{emb}(V;N)$ has a homotopy sheafification with respect to J_k. Denote this by $V \mapsto T_k\,\mathrm{emb}(V;N)$. Thus $V \mapsto T_k\,\mathrm{emb}(V;N)$ is a homotopy sheaf on $\mathcal{O}$ with respect to J_k and there is a natural transformation

$$(***) \qquad \mathrm{emb}(V;N) \longrightarrow T_k\,\mathrm{emb}(V;N)$$

which should be regarded as the *best approximation* of $V \mapsto \mathrm{emb}(V;N)$ by a cofunctor which is a homotopy sheaf with respect to J_k. I do not know of any convincing geometric interpretations of $T_k\,\mathrm{emb}(V;N)$ except of course in the case $k = 1$, which we have already discussed. As Goodwillie explained to me, his excision theorem for diffeomorphisms [9], [10], [11] and improvements due to Goodwillie–Klein [12] imply that $(*)$ is $(k(n-m-2)+1-m)$–connected where $m = \dim(M)$ and $n = \dim(N)$. In particular, if the codimension $n-m$ is greater than 2, then the connectivity of $(*)$ tends to infinity with k. The suggested interpretation of this result is that, if $n-m > 2$, then $V \mapsto \mathrm{emb}(V, N)$ behaves more and more like a homotopy sheaf on $\mathcal{O}$, with respect to $\mathcal{J}_k$, as k tends to infinity.

Suppose now that $M \subset N$, so that $\mathrm{emb}(V, N)$ is a based space for each open $V \subset M$. Then the following general method for calculating or partially calculating $\mathrm{emb}(M, N)$ is second to none. Try to determine the cofunctors

$$V \mapsto \text{homotopy fiber of } [T_k \,\mathrm{emb}(V, N) \to T_{k-1} \,\mathrm{emb}(V, N)]$$

for the first few $k > 0$. These cofunctors admit a surprisingly simple description in terms of configuration spaces; see Theorem 8.5, and [23]. Try to determine the extensions (this tends to be very hard) and finally specialize, letting $V = M$. This program is already outlined in Goodwillie's expanded thesis [9, section Intro.C] for spaces of concordance embeddings (a special case of a relative case), with a pessimistic note added in revision: "$\ldots$ it might never be [written up] ...". It is also carried out to some extent in a simple case in [23]. More details on the same case can be found in Goodwillie–Weiss [13].

Organization Part I (this paper) is about the series of cofunctors $V \mapsto T_k \,\mathrm{emb}(V, N)$, called the *Taylor series* of the cofunctor $V \mapsto \mathrm{emb}(V, N)$. It is also about Taylor series of other cofunctors of a similar type, but it does not address convergence questions. These will be the subject of Part II ([13], joint work with Goodwillie).

Convention Since homotopy limits are so ubiquitous in this paper, we need a "convenient" category of topological spaces with good homotopy limits. The category of fibrant simplicial sets is such a category. In the sequel, "Space" with a capital S means *fibrant simplicial set*. As a rule, we work with (co–)functors whose values are Spaces and whose arguments are spaces (say, manifolds). However, there are some situations, for example in section 9, where it is a good idea to replace the manifolds by their singular simplicial sets. Such a replacement is often understood.

1 Good Cofunctors

1.1 Definition A smooth codimension zero embedding $i_1\colon V \to W$ between smooth manifolds without boundary is an *isotopy equivalence* if there exists a smooth embedding $i_2\colon W \to V$ such that $i_1 i_2$ and $i_2 i_1$ are smoothly isotopic to id_W and id_V, respectively.

In the sequel M is a smooth manifold without boundary, and $\mathcal{O}$ is the poset of open subsets of M, ordered by inclusion. Usually we think of $\mathcal{O}$ as a category, with exactly one morphism $V \to W$ if $V \subset W$, and no morphism if $V \not\subset W$. A cofunctor (=contravariant functor) F from $\mathcal{O}$ to Spaces is *good* if it satisfies the following conditions.

(a) F takes isotopy equivalences to homotopy equivalences.

(b) For any sequence $\{V_i \mid i \ge 0\}$ of objects in $\mathcal{O}$ with $V_i \subset V_{i+1}$ for all $i \ge 0$, the following canonical map is a weak homotopy equivalence:

$$F\Big(\bigcup_i V_i\Big) \longrightarrow \operatorname*{holim}_i F(V_i).$$

1.2 Notation $\mathcal{F}$ is the category of all good cofunctors from $\mathcal{O}$ to Spaces. The morphisms in $\mathcal{F}$ are the natural transformations. A morphism $g\colon F_1 \to F_2$ is an *equivalence* if $g_V\colon F_1(V) \longrightarrow F_2(V)$ is a homotopy equivalence for all V in $\mathcal{O}$. Two objects in $\mathcal{F}$ are *equivalent* if they can be related by a chain of equivalences.

1.3 Examples For any smooth manifold N without boundary, there are cofunctors from $\mathcal{O}$ to Spaces given by $V \mapsto \mathrm{emb}(V,N)$ (Space of smooth embeddings) and $V \mapsto \mathrm{imm}(V,N)$ (Space of smooth immersions). To be more precise, we think of $\mathrm{emb}(V,N)$ and $\mathrm{imm}(V,N)$ as geometric realizations of simplicial sets: for example, a 0–simplex of $\mathrm{imm}(V,N)$ is a smooth immersion $V \to N$, and a 1–simplex in $\mathrm{imm}(V,N)$ is a smooth immersion $V \times \Delta^1 \to N \times \Delta^1$ respecting the projection to Δ^1.

1.4 Proposition *The cofunctors* $\mathrm{imm}(-,N)$ *and* $\mathrm{emb}(-,N)$ *are good.*

Part (a) of goodness is easily verified for both $\mathrm{imm}(-;N)$ and $\mathrm{emb}(-;N)$. Namely, suppose that $i_1\colon V \to W$ is an isotopy equivalence between smooth manifolds, with isotopy inverse $i_2\colon W \to V$ and isotopies $\{h_t\colon V \to V\}$, $\{k_t\colon W \to W\}$ from i_2i_1 to id_V and from i_1i_2 to id_W, respectively. Then $\{h_t\colon V \to V\}$ gives rise to a map of simplicial sets

$$\mathrm{imm}(V;N) \times \Delta^1 \to \mathrm{imm}(V;N)$$

which is a homotopy from $(i_2i_1)^*$ to the identity. Similarly $\{k_t\colon W \to W\}$ gives rise to a homotopy connecting $(i_1i_2)^*$ and the identity on $\mathrm{imm}(W;N)$. Therefore $\mathrm{imm}(-;N)$ is isotopy invariant. The same reasoning applies to $\mathrm{emb}(-;N)$.

To establish part (b) of goodness, we note that it is enough to consider the case where M is connected. Then a sequence $\{V_i\}$ as in part (b) will either be stationary, in which case we are done, or almost all the V_i are open manifolds (no compact components).

1.5 Lemma *Suppose that $V \in \mathcal{O}$ has no compact components. Suppose also that $V = \bigcup_i K_i$ where each K_i is a smooth compact manifold with boundary, contained in the interior of K_{i+1}, for $i \ge 0$. Then the canonical maps*

$$\mathrm{imm}(V;N) \to \operatorname*{holim}_i \mathrm{imm}(K_i;N), \qquad \mathrm{emb}(V;N) \to \operatorname*{holim}_i \mathrm{emb}(K_i;N)$$

are homotopy equivalences.

Proof By the isotopy extension theorem, the restriction from $\mathrm{emb}(K_{i+1};N)$ to $\mathrm{emb}(K_i;N)$ is a Kan fibration of simplicial sets. It is a standard result of immersion theory, much more difficult to establish than the isotopy extension theorem, that the restriction map from $\mathrm{imm}(K_{i+1};N)$ to $\mathrm{imm}(K_i;N)$ is a Kan fibration. See especially Haefliger–Poenaru [15]; although this is written in PL language, it is one of the clearest references.

Let $\mathrm{emb}_!(V;N)$ be the Space of *thick* embeddings $V \to N$, that is, embeddings $f\colon V \to N$ together with a *sober* extension of f to an embedding $D(\nu_f) \to N$, where $D(\nu_f)$ is the total space of the normal disk bundle of f. (The word *sober* means that the resulting bundle isomorphism between the normal bundle of M in $D(\nu_f)$ and ν_f itself is the canonical one.) Define $\mathrm{emb}_!(K_i;N)$ similarly. In the commutative diagram

$$\begin{array}{ccc} \mathrm{emb}_!(V;N) & \xrightarrow{\cong} & \lim_i \mathrm{emb}_!(K_i;N) \\ \big\downarrow{\scriptstyle \text{forget}} & & \big\downarrow{\scriptstyle \text{forget}} \\ \mathrm{emb}(V;N) & \longrightarrow & \lim_i \mathrm{emb}(K_i;N) \end{array}$$

the left–hand vertical arrow is a homotopy equivalence by inspection, and the right–hand vertical arrow is a homotopy equivalence because, according to Bousfield–Kan [1], the canonical map from the limit to the homotopy limit of a tower of Kan fibrations is a homotopy equivalence of simplicial sets. (Hence the limits in the right–hand column could be replaced by homotopy limits.) Hence the lower horizontal arrow is a homotopy equivalence. *Note:* the lower horizontal arrow is not always an isomorphism of simplicial sets — injective immersions are not always embeddings. □

Suppose now that $V = \bigcup_i V_i$ as in part (b) of goodness, and that V has no compact components. Each V_i can be written as a union $\bigcup_j K_{ij}$ where each K_{ij} is smooth compact with boundary, and K_{ij} is contained in the interior of $K_{i(j+1)}$. Moreover we can arrange that K_{ij} is also contained in the interior of $K_{(i+1)j}$. Writing $F(-)$ to mean $\mathrm{imm}(-;N)$, we have a commutative diagram of restriction maps

$$\begin{array}{ccc} F(V) & \longrightarrow & \mathrm{holim}_i\, F(V_i) \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{holim}_i\, F(K_{ii}) & \longleftarrow & \mathrm{holim}_i\, \mathrm{holim}_j\, F(K_{ij}) \end{array}$$

where the vertical arrows are homotopy equivalences by 1.5 and the lower horizontal arrow is a homotopy equivalence by [4, 9.3]. (Here we identify $\mathrm{holim}_i\,\mathrm{holim}_j$ with holim_{ij}.) This shows that the cofunctor $\mathrm{imm}(-;N)$ has property (b). The same argument applies to the cofunctor $\mathrm{emb}(-;N)$. Hence 1.4 is proved. □

2 Polynomial Cofunctors

The following, up to and including Definition 2.2, is a quotation from [23]. Suppose that F belongs to $\mathcal{F}$ and that V belongs to $\mathcal{O}$, and let $A_0, A_1, \dots, A_k$ be pairwise disjoint closed subsets of V. Let $\mathcal{P}_{k+1}$ be the power set of $[k] = \{0, 1, \dots, k\}$. This is a poset, ordered by inclusion. We make a functor from $\mathcal{P}_{k+1}$ to Spaces by

$$S \mapsto F\left(V \setminus \bigcup_{i\in S} A_i\right) \qquad (*)$$

for S in $\mathcal{P}_{k+1}$. Recall that, in general, a functor $\mathcal{X}$ from $\mathcal{P}_{k+1}$ to Spaces is called a $(k+1)$–cube of Spaces.

2.1 Definition ([6], [7]) The *total homotopy fiber* of the cube $\mathcal{X}$ is the homotopy fiber of the canonical map

$$\mathcal{X}(\emptyset) \longrightarrow \operatorname*{holim}_{S \neq \emptyset} \mathcal{X}(S).$$

If the canonical map $\mathcal{X}(\emptyset) \to \operatorname{holim}_{S\neq\emptyset} \mathcal{X}(S)$ is a homotopy equivalence, then $\mathcal{X}$ is *homotopy Cartesian* or just *Cartesian*.

A *cofunctor* $\mathcal{Y}$ from P_{k+1} to spaces will also be called a *cube* of spaces, since P_{k+1} is isomorphic to its own opposite. The *total homotopy fiber* of $\mathcal{Y}$ is the homotopy fiber of $\mathcal{Y}([k]) \to \operatorname{holim}_{S\neq[k]} \mathcal{Y}(S)$.

Inspired by [7, 3.1] we decree:

2.2 Definition The cofunctor F is *polynomial of degree* $\le k$ if the $(k+1)$–cube ( ) is Cartesian for arbitrary V in $\mathcal{O}$ and pairwise disjoint closed subsets $A_0, \dots, A_k$ of V.

Remark In Goodwillie's calculus of functors, a functor from spaces to spaces is *of degree* $\le k$ if it takes strongly cocartesian $(k+1)$–cubes to Cartesian $(k+1)$–cubes. The pairwise disjointness condition in 2.2 is there precisely to ensure that the cube given by $S \mapsto V \smallsetminus \bigcup_{i\in S} A_i$ is strongly cocartesian.

2.3 Example The cofunctor $V \mapsto \operatorname{imm}(V, N)$ is polynomial of degree ≤ 1 if either $\dim(N) > \dim(M)$ or the dimensions are equal and M has no compact component. This amounts to saying that for open subsets V_1 and V_2 of M, the following square of restriction maps is a homotopy pullback square:

$$\begin{array}{ccc} \operatorname{imm}(V_1 \cup V_2, N) & \longrightarrow & \operatorname{imm}(V_1, N) \\ \big\downarrow & & \big\downarrow \\ \operatorname{imm}(V_2, N) & \longrightarrow & \operatorname{imm}(V_1 \cap V_2, N). \end{array}$$

To prove this we use lemma 1.5. Accordingly it is enough to prove that

$$(\quad) \qquad \begin{array}{ccc} \operatorname{imm}(K_1 \cup K_2, N) & \longrightarrow & \operatorname{imm}(K_1, N) \\ \big\downarrow & & \big\downarrow \\ \operatorname{imm}(K_2, N) & \longrightarrow & \operatorname{imm}(K_1 \cap K_2, N) \end{array}$$

is a homotopy pullback square whenever $K_1, K_2 \subset M$ are smooth compact codimension zero submanifolds of M whose boundaries intersect transversely. (Then $K_1 \cap K_2$ is smooth "with corners".) But ( ) is a strict pullback square of Spaces in which all arrows are (Kan) fibrations, by [15]. □

2.4 Example Fix a space X, and for $V \in \mathcal{O}$ let $\binom{V}{k}$ be the configuration space of unordered k–tuples in V. This is the complement of the fat diagonal in the k–fold symmetric product $(V \times V \times \cdots \times V)/\Sigma_k$. The cofunctor

$$V \mapsto \text{map}\left(\binom{V}{k}, X\right)$$

where map denotes a simplicial set of maps, is polynomial of degree $\le k$. Here is a sketch proof: Let $A_0, A_1, \ldots, A_k$ be pairwise disjoint closed subsets of V. Any unordered k–tuple in V must have empty intersection with one of the A_i. Therefore

$$\binom{V}{k} = \bigcup_i \binom{V \smallsetminus A_i}{k}$$

and it is not hard to deduce that the canonical map

$$\operatorname*{hocolim}_{\substack{S \subset \{0,1,\ldots,k\} \\ S \neq \emptyset}} \binom{V \smallsetminus \cup_{i\in S} A_i}{k} \longrightarrow \binom{V}{k}$$

is a homotopy equivalence. Compare [24]. Applying $\text{map}(\text{—}; X)$ turns the homotopy colimit into a homotopy limit and the proof is complete. □

2.5 Example Let $\mathcal{A}$ be a small category and let $\Phi\colon \mathcal{A} \to \mathcal{F}$ be a functor, which we will write in the form $a \mapsto \Phi_a$. Suppose that each Φ_a is polynomial of degree $\le k$. Then

$$V \mapsto \operatorname*{holim}_a \Phi_a(V)$$

is in $\mathcal{F}$, and is polynomial of degree $\le k$. Special case: For $\mathcal{A}$ take the poset of nonempty subsets of $\{0, 1\}$, and conclude that $\mathcal{F}$ is closed under homotopy pullbacks.

3 Special Open Sets

Let $\mathcal{O}k$ consist of all open subsets of M which are diffeomorphic to a disjoint union of at most k copies of $\mathbb{R}^m$, where $m = \dim(M)$. We think of $\mathcal{O}k$ as a full subcategory of $\mathcal{O}$. There is an important relationship between $\mathcal{O}k$ and definition 2.2 which we will work out later, and which is roughly as follows. A good cofunctor from $\mathcal{O}$ to Spaces which is polynomial of degree $\le k$ is *determined* by its restriction to $\mathcal{O}k$, and moreover the restriction to $\mathcal{O}k$ can be *arbitrarily prescribed*. — In this section, however, we merely examine the homotopy type of $|\mathcal{O}k|$ and use the results to study the process of *inflation* (right Kan extension) of a cofunctor along the inclusion $\mathcal{O}k \hookrightarrow \mathcal{O}$.

For the proof of lemma 3.9 below, we need *double categories* [17]. Recall first that a category C consists of two classes, $ob(C)$ and $mor(C)$, as well as maps $s,t\colon mor(C) \to ob(C)$ (*source* and *target*) and $1\colon ob(C) \to mor(C)$ and

$$\circ\colon mor(C)_t \times_s mor(C) \longrightarrow mor(C)$$

(*composition*), where ${}_t\times_s$ denotes the fibered product (or pullback) over $ob(C)$. The maps $s,t,1$ and $\circ$ satisfy certain relations. A *double category* is a category object in the category of categories. Thus a double category C consists of two categories, $ob(C)$ and $mor(C)$, as well as functors $s,t\colon mor(C) \to ob(C)$ (*source* and *target*) and $1\colon ob(C) \to mor(C)$ and

$$\circ\colon mor(C)_t \times_s mor(C) \longrightarrow mor(C)$$

(*composition*) where ${}_t\times_s$ denotes the fibered product (or pullback) over $ob(C)$. These functors $s,t,1$ and $\circ$ satisfy the expected relations. Alternative definition: A double category consists of four classes, $ob(ob(C))$, $mor(ob(C))$, $ob(mor(C))$ and $mor(mor(C))$, and certain maps relating them $\ldots$ This definition has the advantage of being more symmetric. In particular, we see that a double category C determines two ordinary categories, the *horizontal* category C_h and the *vertical* category C_v, both with object class $ob(ob(C))$. The morphism class of C_h is $ob(mor(C))$, that of C_v is $mor(ob(C))$.

The *nerve* of a double category C is a bisimplicial set, denoted by $|C|$.

3.1 Example Suppose that two groups H and V act on the same set S (both on the left). Make a double category C with $ob(ob(C)) = S$, $ob(mor(C)) = S \times H$, $mor(ob(C)) = S \times V$, and

$$mor(mor(C)) := \{(s,h_1,h_2,v_1,v_2) \mid v_2h_1s = h_2v_1s\}.$$

Thus an element in $mor(mor(C))$ is a "commutative diagram"

$$\begin{array}{ccc} v_1s & \xrightarrow{h_2} & h_2v_1s = v_2h_1s \\ \Big\uparrow{\scriptstyle v_1} & & \Big\uparrow{\scriptstyle v_2} \\ s & \xrightarrow{h_1} & h_1s \end{array}$$

where the vertices are in S and the labelled arrows indicate left multiplication by suitable elements of H or V.

3.2 Example An ordinary category A gives rise to a double category denoted AA with $(AA)_h = A = (AA)_v$ and with $mor(mor(AA))$ equal to the class of commutative squares in A. More generally, if A is a subcategory of another category B containing all objects of B, then we can form a double category AB

such that $(\mathcal{AB})_h = \mathcal{B}$, $(\mathcal{AB})_v = \mathcal{A}$, and such that $mor(mor(\mathcal{AB}))$ is the class of commutative squares in $\mathcal{B}$ whose vertical arrows belong to the subcategory $\mathcal{A}$:

$$\begin{array}{ccc} C & \longrightarrow & D \\ \Big\uparrow & & \Big\uparrow \\ A & \longrightarrow & B . \end{array}$$

3.3 Lemma [22, Lemma 1.6.5] *The inclusion of nerves $|\mathcal{B}| \to |\mathcal{AB}|$ is a homotopy equivalence.*

Recall that the homotopy limit of a cofunctor F from a small (ordinary) category $\mathcal{C}$ to $\mathcal{T}$, the category of Spaces, is the totalization of the cosimplicial Space

$$p \mapsto \prod_{G:[p]\to\mathcal{C}} F(G(0))$$

where the product is taken over all functors G from $[p] = \{0, 1, \dots, p\}$ to $\mathcal{C}$. What can we do if $\mathcal{C}$ is a double category and F is a (double) cofunctor from $\mathcal{C}$ to $\mathcal{TT}$? Then we define the homotopy limit as the totalization of the bi–cosimplicial Space

$$(p, q) \mapsto \prod_{H:[p]\times[q]\to\mathcal{C}} F(G(0, 0)).$$

Note that $[p] \times [q]$ is a double category, horizontal arrows being those which do not change the second coordinate and vertical arrows being those which do not change the first coordinate.

We need a variation on 3.3 involving homotopy limits. In the situation of 3.3, assume that F is a cofunctor from $\mathcal{B}$ to Spaces ($=\mathcal{T}$) taking all morphisms in $\mathcal{A}$ to homotopy equivalences. We can think of F as a double cofunctor from $\mathcal{AB}$ to $\mathcal{TT}$.

3.4 Lemma *The projection*

$$\operatorname*{holim}_{\mathcal{AB}} F \to \operatorname*{holim}_{\mathcal{B}} F$$

is a homotopy equivalence.

Proof Let $\mathcal{A}_p\mathcal{B}$ be the ordinary category whose objects are diagrams of the form $A_0 \to \cdots \to A_p$ in $\mathcal{A}$, with natural transformations in $\mathcal{B}$ between such diagrams as morphisms. It is enough to show that the face functor

$$d\colon (A_0 \to \cdots \to A_p) \mapsto A_0$$

induces a homotopy equivalence

$$d^* : \operatorname*{holim}_{\mathcal{B}} F \longrightarrow \operatorname*{holim}_{\mathcal{A}_p\mathcal{B}} Fd .$$

The face functor d has an obvious left adjoint, say e. Thus there is a natural transformation ε from ed to the identity on $\mathcal{A}_p\mathcal{B}$. The natural transformation is a functor

$$\varepsilon : [1] \times \mathcal{A}_p\mathcal{B} \longrightarrow \mathcal{A}_p\mathcal{B} .$$

Now the key observation is that $Fd\varepsilon$ equals the composition

$$[1] \times \mathcal{A}_p\mathcal{B} \xrightarrow{\text{projection}} \mathcal{A}_p\mathcal{B} \xrightarrow{Fd} \mathcal{T} .$$

Hence ε^* can be defined as a map from $\operatorname{holim} Fd$ to $\operatorname{holim}(Fd \circ \text{proj})$. Now $i_0^*\varepsilon^* = (ed)^*$ and $i_1^*\varepsilon^* = \text{id}$, where i_0 and i_1 are the standard injections of $\mathcal{A}_p\mathcal{B}$ in $[1] \times \mathcal{A}_p\mathcal{B}$. Therefore $(ed)^*$ is homotopic to the identity. Also, de is an identity functor. □

To be more specific now, let $\mathcal{I}k \subset \mathcal{O}k$ be the subcategory consisting of all morphisms which are isotopy equivalences. Eventually we will be interested in the double category $\mathcal{I}k\mathcal{O}k$. Right now we need a lemma concerning $\mathcal{I}k$ itself.

3.5 Lemma

$$|\mathcal{I}k| \simeq \coprod_{0 \le j \le k} \binom{M}{j} .$$

Proof Observe that $\mathcal{I}k$ is a coproduct $\coprod \mathcal{I}^{(j)}$ where $0 \le j \le k$ and the objects of $\mathcal{I}^{(j)}$ are the open subsets of M diffeomorphic to a union of j copies of $\mathbb{R}^m$. We have to show

$$|\mathcal{I}^{(j)}| \simeq \binom{M}{j} .$$

For $j = 0$ this is obvious. Here is a proof for $j = 1$, following [5, 3.1]. Let $E \subset |\mathcal{I}^{(1)}| \times M$ consist of all pairs (x, y) such that the (open) cell of $|\mathcal{I}^{(1)}|$ containing x corresponds to a nondegenerate simplex (diagram in $\mathcal{I}^{(1)}$)

$$V_0 \to V_1 \to \cdots \to V_r$$

where $y \in V_r$. The projection maps

$$|I^{(1)}| \longleftarrow E \longrightarrow M$$

are *almost locally trivial* in the sense of [20, A.1], since E is open in $|I^{(1)}| \times M$. By [20, A.2] it is enough to verify that both have contractible fibers. Each fiber of $E \longrightarrow |I^{(1)}|$ is homeomorphic to euclidean space $\mathbb{R}^n$.

Let E_y be the fiber of $E \to M$ over $y \in M$. This embeds in $|I^{(1)}|$ under the projection, and we can describe it as the union of all open cells corresponding to nondegenerate simplices $(U_0 \to \cdots \to U_k)$ where U_k contains y. There is a subspace $D_y \subset E_y$ defined as the union of all open cells corresponding to nondegenerate simplices $(U_0 \to \cdots \to U_k)$ where U_0 contains y. Note the following:

- D_y is a deformation retract of E_y. Namely, suppose that x in E_y belongs to a cell corresponding to a simplex $(U_0, \dots, U_k)$ with $y \in U_k$. Let $(x_0, x_1, \dots, x_k)$ be the barycentric coordinates of x in that simplex, all $x_i > 0$, and let $j \le k$ be the least integer such that $y \in U_j$. Define a deformation retraction by

$$h_{1-t}(x) := (tx_{\text{no}} + x_{\text{yes}})^{-1}(tx_0, \dots, tx_{j-1}, x_j, \dots, x_k)$$
$$x_{\text{no}} := \sum_{i<j} x_i \qquad x_{\text{yes}} := \sum_{i \ge j} x_i$$

  for $t \in [0, 1]$, using the barycentric coordinates in the same simplex.

- D_y is homeomorphic to the classifying space of the poset of all $U \in I^{(1)}$ containing y. The opposite poset is directed, so D_y is contractible.

Hence E_y is contractible, and the proof for $j = 1$ is complete. In the general case $j \ge 1$ let

$$E \subset |I^{(j)}| \times \binom{M}{j}$$

consist of all pairs (x, S) such that the (open) cell of $|I^{(j)}|$ containing x corresponds to a nondegenerate simplex

$$V_0 \to V_1 \to \cdots \to V_r$$

(diagram in $I^{(j)}$) where each component of V_r contains exactly one point from S. Again the projections from E to $|I^{(j)}|$ and to $\binom{M}{j}$ are homotopy equivalences. □

For $p \geq 0$ let $\mathcal{I}k\mathcal{O}k_p$ be the category whose objects are functors $G\colon [p] \to \mathcal{O}k$ and whose morphisms are double functors

$$[1] \times [p] \longrightarrow \mathcal{I}k\mathcal{O}k.$$

(Note that the nerve of the simplicial category $p \mapsto \mathcal{I}k\mathcal{O}k_p$ is isomorphic to the nerve of the double category $\mathcal{I}k\mathcal{O}k$.) The rule $G \mapsto G(p)$ is a functor from $\mathcal{I}k\mathcal{O}k_p$ to $\mathcal{I}k$. In the next lemma we have to make explicit reference to M and another manifold V, so we write $\mathcal{O}k(M)$, $\mathcal{I}k(M)$ and so on.

3.6 Lemma *For any object V in $\mathcal{I}k(M)$, the homotopy fiber over the 0–simplex V of the map*

$$|\mathcal{I}k\mathcal{O}k_p(M)| \longrightarrow |\mathcal{I}k(M)|$$

induced by $G \mapsto G(p)$ is homotopy equivalent to $|\mathcal{I}k\mathcal{O}k_{p-1}(V)|$.

3.7 Remark Combining 3.6 and 3.5, and induction on p, we can get a very good idea of the homotopy type of $|\mathcal{I}k\mathcal{O}k_p(M)|$. In particular, the functor

$$V \mapsto |\mathcal{I}k\mathcal{O}k_p(V)|$$

from $\mathcal{O} = \mathcal{O}(M)$ to Spaces takes isotopy equivalences to homotopy equivalences because the functors $V \mapsto \square^V_j$ have this property.

Proof of 3.6 Using Thomason's homotopy colimit theorem [21] we can make the identification

$$|\mathcal{I}k\mathcal{O}k_p(M)| \simeq \operatorname*{hocolim}_{V \in \mathcal{I}k(M)} |\mathcal{I}k\mathcal{O}k_{p-1}(V)|.$$

Then the map under investigation corresponds to the projection from the homotopy colimit to the nerve of $\mathcal{I}k(M)$. This map is already a quasifibration of simplicial sets. Namely, all morphisms $V_1 \to V_2$ in $\mathcal{I}k(M)$ are isotopy equivalences by definition, and inductively we may assume that the functor $V \mapsto |\mathcal{I}k\mathcal{O}k_{p-1}(V)|$ takes isotopy equivalences to homotopy equivalences (see remark 3.7). Therefore the homotopy fiber that we are interested in has the same homotopy type as the honest fiber. □

Let E be a cofunctor from $\mathcal{O}k = \mathcal{O}_k(M)$ to Spaces taking morphisms in $\mathcal{O}k$ which are isotopy equivalences to homotopy equivalences. Use this to define a cofunctor $E^!$ from $\mathcal{O}$ to Spaces by the formula

$$E^!(V) = \operatorname*{holim}_{U \in \mathcal{O}k(V)} E(U).$$

In categorical patois: $E^!$ is the homotopy right Kan extension of E along the inclusion functor $\mathcal{O}k \to \mathcal{O}$.

3.8 Lemma *$E^{!}$ is good.*

Proof From 3.4 we know that the projection

$$\operatorname*{holim}_{U \in IkOk(V)} E(U) \longrightarrow \operatorname*{holim}_{U \in Ok(V)} E(U)$$

is a homotopy equivalence. The domain of this projection can be thought of as the totalization of the cosimplicial Space

$$p \mapsto \operatorname*{holim}_{U_0 \to \cdots \to U_p} E(U_0)$$

where the homotopy limit, $\operatorname{holim} E(U_0)$, is taken over $IkOk_p(V)$ as defined just before 3.6. Note that the cofunctor $(U_0 \to \cdots \to U_p) \mapsto E(U_0)$ takes all morphisms to homotopy equivalences. Hence its homotopy colimit is quasifibered over the nerve of the indexing category, and its homotopy limit may be identified (up to homotopy equivalence) with the section Space of the associated fibration. Using 3.6 and 3.7 now we see that

$$V \mapsto \operatorname*{holim}_{U_0 \to \cdots \to U_p} E(U_0)$$

is a good cofunctor $E^{!}_p$ for each p. Hence $E^{!}$ is good, too. □

We come to the main result of the section. It is similar to certain well–known statements about *small simplices*, for example [2, III.7.3], which are commonly used to prove excision theorems. Let $\mathcal{U}$ be an open cover of M. We say that $V \in Ok$ is $\mathcal{U}$–small if each connected component of V is contained in some open set of the cover $\mathcal{U}$. Let $\mathcal{U}Ok = \mathcal{U}Ok(M)$ be the full sub–poset of Ok consisting of the $\mathcal{U}$–small objects. For $V \in O$ let

$$\mathcal{U}E^{!}(V) := \operatorname*{holim}_{U \in \mathcal{U}Ok(V)} E(U).$$

3.9 Theorem *The projection $E^{!}(V) \to \mathcal{U}E^{!}(V)$ is a homotopy equivalence.*

Proof Using the notation from the proof of 3.8, and obvious $\mathcal{U}$–modifications, we see that it suffices to prove that the projection $E^{!}_p(V) \to \mathcal{U}E^{!}_p(V)$ is a homotopy equivalence, for all V and p. However, the analysis of $E^{!}_p(V)$ as a section Space (proof of 3.8) works equally well for $\mathcal{U}E^{!}_p(V)$, and gives the same result up to homotopy equivalence. In particular 3.5 and 3.6 go through in the $\mathcal{U}$–setting. □

4 Construction of Polynomial Cofunctors

We continue to assume that E is a cofunctor from $\mathcal{O}k$ to Spaces taking isotopy equivalences to homotopy equivalences.

4.1 Theorem *The cofunctor $E^!$ on $\mathcal{O}$ is polynomial of degree $\le k$.*

Proof We have to verify that the condition in 2.2 is satisfied. Without loss of generality, $V = M$. Then $A_0, A_1, \dots, A_k$ are pairwise disjoint closed subsets of M. Let $M_i = M \smallsetminus A_i$ and $M_S = \bigcap_{i \in S} M_i$ for $S \subset \{0, 1, \dots, k\}$. Using 3.9, all we have to show is that the $(k+1)$–cube of Spaces

$$S \mapsto \mathcal{U}E^!(M_S)$$

is homotopy Cartesian. Here $\mathcal{U}$ can be any open cover of M, and in the present circumstances we choose it so that none of the open sets in $\mathcal{U}$ meets more than one A_i. Then

$$\mathcal{U}\mathcal{O}k = \bigcup_i \mathcal{U}\mathcal{O}k(M_i).$$

(This is the pigeonhole principle again: Each component of an object U in $\mathcal{U}\mathcal{O}_k$ meets at most one of the A_i, but since U has at most k components, $U \cap A_i = \emptyset$ for some i.) With lemma 4.2 below, we conclude that the canonical map

$$\operatorname*{holim}_{\mathcal{U}\mathcal{O}k} E \longrightarrow \operatorname*{holim}_{S \neq \emptyset} \operatorname*{holim}_{\mathcal{U}\mathcal{O}k(M_S)} E$$

is a homotopy equivalence. But this is what we had to show. $\square$

In lemma 4.2 just below, an *ideal* in a poset Q is a subset R of Q such that for every $b \in R$, all $a \in Q$ with $a \le b$ belong to R.

4.2 Lemma *Suppose that the poset Q is a union of ideals Q_i, where $i \in T$. For finite nonempty $S \subset T$ let $Q_S = \bigcap_{i \in S} Q_i$. Let E be a cofunctor from Q to Spaces. Then the canonical map*

$$\operatorname*{holim}_{Q} E \longrightarrow \operatorname*{holim}_{S \neq \emptyset} \operatorname*{holim}_{Q_S} E$$

is a homotopy equivalence.

Proof Let $\int Q_S$ be the poset consisting of all pairs (S, x) where $S \subset T$ is finite, nonempty and where $(S, x) \le (S', y)$ if and only if $S \supset S'$ and $x \le y$ in Q. The forgetful map taking (S, x) to x is a functor $u\colon \int Q_S \to Q$. It is right cofinal, ie the *under* category $y{\downarrow}u$ is contractible for every y in Q. Therefore, by the *cofinality theorem for homotopy inverse limits* [1, ch. XI, 9.2], [4, 9.3] the obvious map

$$\operatorname*{holim}_{x \in Q} E(x) \longrightarrow \operatorname*{holim}_{(S,x) \in \int Q_S} E(x)$$

is a homotopy equivalence. (Note that it has to be right cofinal instead of the usual left cofinal because we are dealing with a *cofunctor* E.) By inspection, the codomain of this map is homeomorphic to

$$\operatorname*{holim}_{S \neq \emptyset} \operatorname*{holim}_{Q_S} E. \qquad \square$$

Remark Note that the obvious map $E(U) \to E^!(U)$ is a homotopy equivalence for every U in $\mathcal{O}k$. This is again an application of the cofinality theorem for homotopy inverse limits, although a much more obvious one. In this sense $E^!$ extends E.

5 Characterizations of Polynomial Cofunctors

5.1 Theorem *Let $g\colon F_1 \to F_2$ be a morphism in $\mathcal{F}$. Suppose that both F_1 and F_2 are polynomial of degree $\le k$. If $g\colon F_1(V) \to F_2(V)$ is a homotopy equivalence for all $V \in \mathcal{O}k$, then it is a homotopy equivalence for all $V \in \mathcal{O}$.*

Proof Suppose that $g\colon F_1(V) \to F_2(V)$ is a homotopy equivalence for all $V \in \mathcal{O}k$. Suppose also that $W \in \mathcal{O}r$, where $r > k$. Let $A_0, A_1, \ldots, A_k$ be distinct components of W and let $W_S = \bigcap_{i \in S}(W \smallsetminus A_i)$ for $S \subset \{0, 1, \ldots, k\}$. Then

$$F_i(W) \simeq \operatorname*{holim}_{S \neq 0} F_i(W_S)$$

for $i = 1, 2$ and therefore $g\colon F_1(W) \to F_2(W)$ is a homotopy equivalence provided g from $F_1(W_S) \to F_2(W_S)$ is a homotopy equivalence for all nonempty $S \subset \{0, 1, \ldots, k\}$. But W_S for $S \neq \emptyset$ has fewer components than W, so by induction the proviso is correct. This takes care of all $W \in \bigcup_r \mathcal{O}r$.

Next, suppose that $W = \operatorname{int}(L)$ where L is a smooth compact codimension zero submanifold of M. Choose a handle decomposition for L, let s be the maximum of the indices of the handles, and let t be the number of handles of index s that occur. If $s = 0$ we have $W \in \mathcal{O}r$ for some r and this case has been dealt with. If $s > 0$, let $e\colon \mathbb{D}^{m-s} \times \mathbb{D}^s \to L$ be one of the s–handles. We assume that $e^{-1}(\partial L)$ is $\partial\mathbb{D}^{m-s} \times \mathbb{D}^s$. Since $s > 0$ we can find pairwise disjoint small closed disks $C_0, \dots, C_k$ in $\mathbb{D}^s$ and we let

$$A_i := e(\mathbb{D}^{m-s} \times C_i) \cap W$$

for $0 \le i \le k$. Then each A_i is closed in W and $W \smallsetminus A_i$ is the interior of a smooth handlebody in M which has a handle decomposition with no handles of index $> s$, and fewer than t handles of index s. The same is true for $W_S := \bigcap_{i \in S}(W \smallsetminus A_i)$ provided $S \neq \emptyset$. Therefore, by induction,

$$\gamma\colon F_1(W_S) \longrightarrow F_2(W_S)$$

is a homotopy equivalence for $\emptyset \neq S \subset \{0, 1, \dots, k\}$ and consequently the right–hand vertical arrow in

$$\begin{array}{ccc} F_1(W) & \longrightarrow & \operatorname{holim}_{S \neq \emptyset} F_1(W_S) \\ \Big\downarrow \gamma & & \Big\downarrow \gamma \\ F_2(W) & \longrightarrow & \operatorname{holim}_{S \neq \emptyset} F_2(W_S) \end{array}$$

is a homotopy equivalence. But the two horizontal arrows are also homotopy equivalences, because F_1 and F_2 are polynomial of degree $\le k$. Therefore the left–hand vertical arrow is a homotopy equivalence. This takes care of every $W \in \mathcal{O}$ which is the interior of a compact smooth handlebody in M.

The general case follows because F_1 and F_2 are good cofunctors; see especially property (b) in the definition of goodness, just after 1.1. □

For F in $\mathcal{F}$ let T_kF be the homotopy right Kan extension of the restriction of F to $\mathcal{O}k$. The explicit formula is

$$T_kF(V) := \operatorname*{holim}_{U \in \mathcal{O}k(V)} F(U).$$

From section 3 and section 4 we know that T_kF is good and polynomial of degree $\le k$. There is an obvious forgetful morphism $\eta_k\colon F \to T_kF$. Clearly the natural map $\eta_k\colon F(U) \to T_kF(U)$ is a homotopy equivalence for every $U \in \mathcal{O}k$. Hence, by 5.1, if F is already polynomial of degree $\le k$, then η_k from $F(V)$ to $T_kF(V)$ is a homotopy equivalence for every $V \in \mathcal{O}$. In this sense an F which is polynomial of degree $\le k$ is determined by its restriction E to $\mathcal{O}k$. The restriction does of course take isotopy equivalences in $\mathcal{O}k$ to homotopy equivalences. We saw in section 4 that that is essentially the only condition it must satisfy.

The polynomial objects in $\mathcal{F}$ can also be characterized in sheaf theoretic terms. Recall the Grothendieck topologies J_k on $\mathcal{O}$, from the introduction.

5.2 Theorem *A good cofunctor F from $\mathcal{O}$ to Spaces is polynomial of degree $\le k$ if and only if it is a homotopy sheaf with respect to the Grothendieck topology $\mathcal{J}_k$.*

Proof Suppose that F is a homotopy sheaf with respect to $\mathcal{J}_k$. Let $V \in \mathcal{O}$ and pairwise disjoint closed subsets $A_0, \dots, A_k$ of V be given. Let $V_i = V \smallsetminus A_i$. Then the inclusions V_i for $0 \le i \le k$ form a covering of V in the Grothendieck topology $\mathcal{J}_k$. Hence

$$F(V) \longrightarrow \operatorname*{holim}_R F(\cap_{i \in R} V_i)$$

is a homotopy equivalence; the homotopy limit is taken over the nonempty subsets R of $\{0, \dots, k\}$. This shows that F is polynomial of degree $\le k$.

Conversely, suppose that F is polynomial of degree $\le k$. Let $W \in \mathcal{O}$ be given and let $\{V_i \to W \mid i \in S\}$ be a covering of W in the Grothendieck topology $\mathcal{J}_k$. Let E be the restriction of F to $\mathcal{O}k$. Define $\varepsilon E^!$ as in section 3, just before 3.9, where ε is the covering $\{V_i\}$. Up to equivalence, F and $\varepsilon E^!$ are the same. By 4.2, the canonical map

$$\varepsilon E^!(W) \longrightarrow \operatorname*{holim}_R \varepsilon E^!(\cap_{i \in R} V_i)$$

is a homotopy equivalence. Here again, R runs through the finite nonempty subsets R of S. □

6 Approximation by Polynomial Cofunctors

From section 5, we have for every $k \ge 0$ an endofunctor $T_k\colon \mathcal{F} \to \mathcal{F}$ given by the rule $F \mapsto T_kF$, and a natural transformation from the identity $\mathcal{F} \to \mathcal{F}$ to T_k given by $\eta_k\colon F \to T_kF$ for all F. It is sometimes convenient to define T_{-1} as well, by $T_{-1}F(V) := *$. The following theorem is mostly a summary of results from section 5. It tries to say that T_k is essentially *left adjoint* to the inclusion functor $\mathcal{F}_k \to \mathcal{F}$. Here $\mathcal{F}_k$ is the full subcategory of $\mathcal{F}$ consisting of the objects which are polynomial of degree $\le k$. Compare [25, Thm.6.1].

6.1 Theorem *The following holds for every F in $\mathcal{F}$ and every $k \ge 0$.*

(1) *T_kF is polynomial of degree $\le k$.*

(2) *If F is already polynomial of degree $\le k$, then $\eta_k\colon F \to T_kF$ is an equivalence.*

(3) *$T_k(\eta_k)\colon T_kF \to T_k(T_kF)$ is an equivalence.*

Proof Properties (1) and (2) have been established in section 5. As for (3), we can use 5.1 and we then only have to verify that

$$T_k(\eta_k)\colon T_kF(W) \to T_k(T_kF(W))$$

is a homotopy equivalence for every $W \in \mathcal{O}k$. Written out in detail the map takes the form

$$\begin{aligned}\operatorname*{holim}_{V\in\mathcal{O}k(W)} F(V) &\longrightarrow \operatorname*{holim}_{V\in\mathcal{O}k(W)} T_kF(V)\\ &= \operatorname*{holim}_{V\in\mathcal{O}k(W)} \operatorname*{holim}_{U\in\mathcal{O}k(V)} F(U)\end{aligned}$$

and it is induced by the maps $F(V) \to \operatorname{holim}_U F(U)$ for V in $\mathcal{O}k(V)$. These maps are clearly homotopy equivalences, since the identity morphism $V \to V$ is a terminal object in $\mathcal{O}k(V)$. □

Remark One way of saying that the inclusion of a full subcategory, say $\mathcal{A} \to \mathcal{B}$, has a left adjoint is to say that there exists a functor $T\colon \mathcal{B} \to \mathcal{B}$ and a natural transformation $\eta\colon \mathrm{id}_{\mathcal{B}} \to T$ with the following properties.

(1) $T(b)$ belongs to $\mathcal{A}$ for every b in $\mathcal{B}$.

(2) For a in $\mathcal{A}$, the morphism $\eta\colon a \to T(a)$ is an isomorphism.

(3) For b in $\mathcal{B}$, the morphism $T(\eta)\colon T(b) \to T(T(b))$ is an isomorphism.

From the definitions, there are forgetful transformations $r_k\colon T_kF \to T_{k-1}F$ for any F and any $k > 0$. They satisfy the relations $r_k\eta_k = \eta_{k-1}\colon F \to T_{k-1}F$. Therefore

$$(*) \qquad \{\eta_k\}\colon F \longrightarrow \operatorname*{holim}_k T_kF$$

is defined. The codomain, with its inverse filtration, may be called the *Taylor tower* of F. Usually one wants to know whether $(*)$ is a homotopy equivalence. More precisely one can ask two questions:

- Does the Taylor tower of F converge?
- If it does converge, does it converge to F?

Regarding the first question: although $\operatorname{holim}_k T_kF$ is always defined, we would not speak of convergence unless the connectivity of $r_k\colon T_kF(V) \to T_{k-1}F(V)$ tends to infinity with k, independently of V.

7 More Examples of Polynomial Cofunctors

7.1 Example Let $p\colon Z \to \binom{M}{k}$ be a fibration. For $U \subset \binom{M}{k}$ let $\Gamma(p\,;U)$ be the Space of partial sections of p defined over U. The cofunctor F on $\mathcal{O}$ defined by $F(V) := \Gamma(p\,;\binom{V}{k})$ is good and, moreover, it is polynomial of degree $\le k$. This can be proved like 2.4.

Keep the notation of 7.1. Let $\Delta_k V$ be the complement of $\binom{V}{k}$ in the k–fold symmetric power $\mathrm{sp}_k V := (V \times V \times \cdots \times V)/\Sigma_k$. The homotopy colimit in the next lemma is taken over the poset of all neighborhoods Q of $\Delta_k V$ in $\mathrm{sp}_k V$.

7.2 Lemma *The cofunctor G on $\mathcal{O}$ given by*

$$G(V) := \operatorname*{hocolim}_Q \Gamma\Big(p\,;\binom{V}{k} \cap Q\Big)$$

is good.

Proof We concentrate on part (b) of goodness to begin with. Fix V and choose a smooth triangulation on the k–fold product $(V)^k$, equivariant with respect to the symmetric group Σ_k. Then $\mathrm{sp}_k V$ has a preferred PL structure and $\Delta_k V$ is a PL subspace, so we can speak of *regular neighborhoods* of $\Delta_k V$. It is clear that all regular neighborhoods of $\Delta_k V$ have the same homotopy type, and that each neighborhood of $\Delta_k V$ contains a regular one. Therefore, if L is a regular neighborhood of $\Delta_k V$, then the canonical inclusion

$$\Gamma\Big(p\,;\binom{V}{k} \cap \mathrm{int}(L)\Big) \longrightarrow \operatorname*{hocolim}_Q \Gamma\Big(p\,;\binom{V}{k} \cap Q\Big)$$

is a homotopy equivalence. This observation tends to simplify matters. Another observation which tends to complicate matters is that for an open subset U of V and a regular neighborhood L as above, the intersection of L with $\mathrm{sp}_k U$ will usually not be a regular neighborhood of $\Delta_k U$. However, we can establish goodness as follows. Suppose that

$$V = \bigcup_i K_i$$

where each K_i is a smooth compact codimension zero submanifold of V, and $K_i \subset \mathrm{int}(K_{i+1})$. As in the proof of 1.4, it is enough to show that the canonical map

$$G(V) \longrightarrow \operatorname*{holim}_i G(\mathrm{int}(K_i))$$

is a homotopy equivalence. Abbreviate $\mathrm{int}(K_i) = V_i$. Choose a regular neighborhood L of $\Delta_k V$ in $\mathrm{sp}_k V$ such that $L \cap \mathrm{sp}_k(K_i)$ is a regular neighborhood of $\Delta_k(K_i)$ in $\mathrm{sp}_k(K_i)$ for each i. Then it is not hard to see that the inclusion

$$\Gamma\Big(p\,;\binom{V_i}{k} \cap \mathrm{int}(L)\Big) \longrightarrow \operatorname*{hocolim}_R \Gamma\Big(p\,;\binom{V_i}{k} \cap R\Big)$$

is a homotopy equivalence, for each i. Therefore, in the commutative diagram

$$\begin{array}{ccc}
\Gamma\big(p\,;\binom{V}{k}\cap\operatorname{int}(L)\big) & \longrightarrow & \operatorname{holim}_i\Gamma\big(p\,;\binom{V_i}{k}\cap\operatorname{int}(L)\big)\\
\Big\downarrow & & \Big\downarrow\\
\operatorname{hocolim}_Q\Gamma\big(p\,;\binom{V}{k}\cap Q\big) & \longrightarrow & \operatorname{holim}_i\operatorname{hocolim}_R\Gamma\big(p\,;\binom{V_i}{k}\cap R\big)
\end{array}$$

the two vertical arrows are homotopy equivalences. The upper horizontal arrow is also a homotopy equivalence by inspection. Hence the lower horizontal arrow is a homotopy equivalence. This completes the proof of part (b) of goodness.

Proof of part (a) of goodness: Suppose that $W\hookrightarrow V$ in $\mathcal{O}$ is an isotopy equivalence. Let $\{j_t\colon V\to V\}$ be a smooth isotopy of embeddings, with $j_0=\mathrm{id}_V$ and $\mathrm{im}(j_1)=W$. Let

$$X:=\operatorname*{hocolim}_R\Gamma\big(j^*p\,;\big(\tbinom{V}{k}\times I\big)\cap R\big)$$

where $I=[0,1]$ and j^*p is the pullback of p under the map

$$\binom{V}{k}\times I\longrightarrow\binom{V}{k}\;;\qquad (S,t)\mapsto j_t(S)$$

and R runs over the neighborhoods of $\Delta_k V\times I$ in $\mathrm{sp}_k V\times I$. *Key observation:* Every R contains a neighborhood of the form $Q\times I$, where $Q\subset\mathrm{sp}_k V$. This implies that the restriction maps

$$G(W)\xleftarrow{\;\rho_W\;}X\xrightarrow{\;\rho_V\;}G(V)$$

(induced by the bundle maps $j_1^*p\to j^*p\leftarrow j_0^*p$) are homotopy equivalences. The restriction map $G(V)\to G(W)$ that we are interested in can be written as a composition

$$G(V)\xrightarrow{\;j^*\;}X\xrightarrow{\;\rho_W\;}G(W)$$

where the arrow labelled j^* is right inverse to ρ_V. Therefore the restriction map $G(V)\to G(W)$ is a homotopy equivalence. $\square$

7.3 Lemma *The cofunctor G in 7.2. is polynomial of degree $\le k$.*

Proof Fix $W\in\mathcal{O}$ and let $A_0,\dots,A_k$ be closed and pairwise disjoint in W. Let $W_i:=W\smallsetminus A_i$ and choose neighborhoods Q_i of $\Delta_k W_i$ in $\mathrm{sp}_k W_i$. Let

$$W_S=\cap_{i\in S}W_i$$
$$Q_S=\cap_{i\in S}Q_i$$

for nonempty $S \subset \{0,1,\ldots,k\}$, and $W_\emptyset = W$, $Q_\emptyset = \cup_i Q_i$. Then

$$\binom{W}{k} \cap Q_\emptyset = \bigcup_i \binom{W_i}{k} \cap Q_i \simeq \operatorname*{hocolim}_{S\neq\emptyset} \binom{W_S}{k} \cap Q_S$$

which shows, much as in the proof of 2.4, that the obvious map

$$\Gamma\left(p;\binom{W}{k} \cap Q_\emptyset\right) \longrightarrow \operatorname*{holim}_{S\neq\emptyset} \Gamma\left(p;\binom{W_S}{k} \cap Q_S\right)$$

is a homotopy equivalence. We can now complete the proof with two observations. Firstly, the neighborhoods of $\Delta_k W_S$ of the form Q_S, as above, form an *initial* subset [17] in the poset of all neighborhoods. Secondly, there are situations in which homotopy inverse limits commute (up to homotopy equivalence) with homotopy direct limits, and this is one of them. Here we are interested in a double homotopy limit/colimit of the form

$$\operatorname*{holim}_{S\neq\emptyset}\ \operatorname*{hocolim}_{Q_0,\ldots,Q_k}\ (\ —\ )$$

where the blank indicates an expression depending on S and the Q_i (actually only on the Q_i for $i \in S$). Clearly sublemma 7.4 below applies. □

7.4 Sublemma *Let X be a functor from a product $A \times B$ to Spaces, where A and B are posets. Suppose that A is finite and that B is directed. Then*

$$\operatorname*{hocolim}_{b\in B}\ \operatorname*{holim}_{a\in A} X(a,b) \simeq \operatorname*{holim}_{a\in A}\ \operatorname*{hocolim}_{b\in B} X(a,b).$$

Proof Since B is a directed poset, the homotopy colimits may be replaced by honest colimits [1]. The universal property of colimits yields a map

$$\operatorname*{colim}_{b\in B}\ \operatorname*{holim}_{a\in A} X(a,b) \simeq \operatorname*{holim}_{a\in A}\ \operatorname*{colim}_{b\in B} X(a,b)$$

which is an isomorphism, by inspection. □

7.5 Proposition *The cofunctor G in 7.2 and 7.3 is in fact polynomial of degree* $\le k-1$.

Proof We must show that $\eta_k\colon G \to T_{k-1}G$ is an equivalence. Since G and $T_{k-1}G$ are both polynomial of degree $\le k$, it is enough to check that

$$\eta_k\colon G(V) \longrightarrow T_{k-1}G(V)$$

is an equivalence for every $V \in \mathcal{O}k$. See 5.1. If V belongs to $\mathcal{O}r$ for some $r < k$, this is obvious. So we may assume that V has exactly k connected components, each diffeomorphic to $\mathbb{R}^m$. Denote these components by $A_0, \dots, A_{k-1}$. If we can show that the upper horizontal arrow in

$$\begin{array}{ccc} G(V) & \longrightarrow & \operatorname{holim}_{S\neq\emptyset} G(\bigcup_{i\in S} A_i) \\ \big\downarrow & & \big\downarrow \\ T_{k-1}G(V) & \longrightarrow & \operatorname{holim}_{S\neq\emptyset} T_{k-1}G(\bigcup_{i\in S} A_i) \end{array}$$

is a homotopy equivalence, then we are done because the lower horizontal and the right–hand vertical arrows are homotopy equivalences. However, this follows in the usual manner (compare proof of 2.4 and of 7.3) from the observation that

$$\binom{V}{k} \cap Q = \bigcup_i \binom{V \smallsetminus A_i}{k} \cap Q$$

for sufficiently small neighborhoods Q of $\Delta_k V$ in $\mathrm{sp}_k V$. Notice that the observation as such is new because this time the closed subsets A_i are k in number, not $k+1$. □

We are now in a position to understand the relationship between F in 7.1 and G in 7.2. There is an obvious inclusion $e\colon F(V) \to G(V)$, natural in V.

7.6 Proposition *The morphism $T_{k-1}e\colon T_{k-1}F \to T_{k-1}G$ is an equivalence.*

Proof By 5.1, it suffices to show that $e\colon F(V) \to G(V)$ is a homotopy equivalence for any V which is diffeomorphic to a disjoint union of ℓ copies of $\mathbb{R}^m$, where $\ell < k$. For such a V choose open subsets

$$V = V_0 \supset V_1 \supset V_2 \supset V_3 \supset \cdots$$

such that the inclusions $V_{i+1} \to V_i$ are isotopy equivalences, such that the closure of V_{i+1} in V_i is compact, and such that $\bigcap_i V_i$ is a discrete set consisting (necessarily) of ℓ points, one in each component of V. In the commutative square

$$(*) \qquad \begin{array}{ccc} F(V) & \longrightarrow & \operatorname{hocolim}_i F(V_i) \\ \big\downarrow & & \big\downarrow \\ G(V) & \longrightarrow & \operatorname{hocolim}_i G(V_i) \end{array}$$

the horizontal arrows are now homotopy equivalences because F and G take isotopy equivalences to homotopy equivalences. On the other hand, suppose that Q is a neighborhood of $\Delta_k V_i$ in $\mathrm{sp}_k V_i$ for some i. Then clearly there exists an integer $j > i$ such that all of $\mathrm{sp}_k V_j$ is contained in Q. It follows that the inclusion of $\mathrm{hocolim}_i F(V_i)$ in

$$\operatorname*{hocolim}_i G(V_i) = \operatorname*{hocolim}_i \operatorname*{hocolim}_Q \Gamma\left(p ; \binom{V_i}{k} \cap Q\right)$$

is a homotopy equivalence. Hence all arrows in ( ) are homotopy equivalences. □

8 Homogeneous Cofunctors

8.1 Definition A cofunctor E in $\mathcal{F}$ is *homogeneous of degree* k, where $k \ge 0$, if it is polynomial of degree $\le k$ and if $T_{k-1}E(V)$ is contractible for each $V \in \mathcal{O}$.

Remark The cofunctor given by $E(V) = *$ for all V is homogeneous of degree k for any $k \ge 0$. Conversely, if E is homogeneous of degree k and homogeneous of degree ℓ, where $\ell < k$, then clearly $E(V) \simeq T_{k-1}E(V) \simeq *$.

8.2 Example Let F in $\mathcal{F}$ be arbitrary, and select a point in $F(M)$, if one exists. Then $T_kF(V)$ is pointed for all V and k. Therefore a new cofunctor L_kF can be defined by

$$L_kF(V) := \mathrm{hofiber}[T_kF(V) \longrightarrow T_{k-1}F(V)].$$

It follows from 6.1 that E is homogeneous of degree k.

8.3 Example Starting with a fibration $p\colon Z \to \binom{M}{k}$, define F as in 7.1 and define G as in 7.2. Select a point in $G(M)$. Then

$$E(V) := \mathrm{hofiber}[F(V) \longrightarrow G(V)]$$

is defined. It follows from 7.6 that E is homogeneous of degree k.

Example 8.3 deserves to be studied more. Ultimately E has been constructed in terms of the fibration p, and a partial section of p defined near the fat diagonal $\Delta_k M$. Is it possible to recover p from E? In particular, for $S \in \binom{M}{k}$, can we describe the fiber $p^{-1}(S)$ in terms of E?

Note that S is a subset of M with k elements. Let V be a tubular neighborhood of $S \subset M$, so that V is diffeomorphic to a disjoint union of k copies of $\mathbb{R}^m$. Then S belongs to $\binom{V}{k} \subset \binom{M}{k}$ and therefore we have maps

$$E(V) \longrightarrow F(V) = \Gamma\left(p ; \binom{V}{k}\right) \xrightarrow{\text{evaluation}} p^{-1}(S).$$

8.4 Proposition *The composite map $E(V) \to p^{-1}(S)$ is a homotopy equivalence.*

Hence we can indeed describe $p^{-1}(S)$ in terms of E, up to homotopy equivalence: namely, as $E(V)$ for a tubular neighborhood V os S in M.

Proof of 8.4 Much as in the proof of 7.6 we choose a sequence of open subsets

$$V = V_0 \supset V_1 \supset V_2 \supset V_3 \supset \ldots$$

such that the inclusions $V_{i+1} \to V_i$ are isotopy equivalences, such that the closure of V_{i+1} in V_i is compact, and such that $\bigcap_i V_i = S$. We note that

$$F(V) = \prod_j \Gamma(p\,; U_j)$$

where the U_j are the connected components of $\binom{V}{k}$. Among these components we single out U_0, the component containing S. It is the only component whose closure in $\mathrm{sp}_k V$ does not meet $\Delta_k V$. For the remaining components we can use an idea as in the proof of 7.6, and find

$$G(V) \simeq \prod_{j \neq 0} \Gamma(p\,; U_j).$$

Therefore $F(V) \simeq E(V) \times G(V)$ and the composition

$$E(V) \to F(V) \to \Gamma(p\,; U_0) \to p^{-1}(S)$$

is a homotopy equivalence. □

Digression Knowing all the fibers of a fibration is not the same as knowing the fibration. However, in the present case we can also “describe” the entire fibration p in 8.3 in terms of the cofunctor E. Recall from the proof of 3.5 the poset $I^{(k)}$. Its elements are the open subsets of M which are diffeomorphic to a disjoint union of k copies of $\mathbb{R}^m$, and for $V, W \in I^{(k)}$ we decree $V \le W$ if and only if $V \subset W$ and the inclusion is an isotopy equivalence. We saw that

$$|I^{(k)}| \simeq \binom{M}{k}.$$

Since $I^{(k)} \subset \mathcal{O}$, we can restrict E to $I^{(k)}$. The restricted cofunctor takes all morphisms to homotopy equivalences, so that the projection

$$\operatorname*{hocolim}_{I^{(k)}} E \longrightarrow |I^{(k)}|$$

is a quasifibration. The associated fibration is the one we are looking for. This motivates the following classification theorem for homogeneous cofunctors.

8.5 Theorem *Up to equivalence, all objects in $\mathcal{F}$ which are homogeneous of degree k are of the type discussed in 8.3.*

Outline of proof Of course, the digression just above already gives the idea of the proof, but we have to proceed a little more cautiously. The plan is: Given E, homogeneous of degree but not necessarily defined in terms of some fibration, construct the appropriate F, polynomial of degree $\le k$, and a morphism $E \to F$. Then show that F is equivalent to a cofunctor of type $V \mapsto \Gamma(p ; {}^{V}_{k})$. as in 7.1. This step requires a lemma, 8.6 below. Finally identify E with the homotopy fiber of the canonical morphism from F to $T_{k-1}F$.

8.6 Lemma [3, 3.12] *Suppose that Y is a functor from a small category $\mathcal{A}$ to the category of Spaces. If Y takes all morphisms in $\mathcal{A}$ to homotopy equivalences, then the canonical projection* $\operatorname{hocolim}_{\mathcal{A}} Y \to |\mathcal{A}|$ *is a quasifibration. The section Space of the associated fibration is homotopy equivalent to* $\operatorname{holim}_{\mathcal{A}} X$.

Sketch proof of 8.6 The quasifibration statement is obvious. We denote the total Space of the associated fibration by T, so that $\operatorname{hocolim}_{\mathcal{A}} Y \simeq T$ by a homotopy equivalence. For the statement about the section Space, recall that $\operatorname{holim} Y$ can be defined as the Space of natural transformations $\tilde{*}_{\mathcal{A}} \to Y$, where $*_{\mathcal{A}}$ is the constant functor $a \mapsto *$ on $\mathcal{A}$, and $\tilde{*}_{\mathcal{A}}$ is a *CW–functor* weakly equivalent to it (some explanations below). The standard choice is

$$\tilde{*}_{\mathcal{A}}(a) := |\mathcal{A}\downarrow a|.$$

CW–functor refers to a functor with a *CW–decomposition* where the cells are of the form $\mathbb{R}^i \times \operatorname{mor}(b, -)$ for some $b \in \mathcal{A}$ and some i. *Weakly equivalent to* $*_{\mathcal{A}}$ means here that there is an augmentation $\tilde{*}_{\mathcal{A}}(a) \to *_{\mathcal{A}}(a)$, natural in a, which is a homotopy equivalence for each a. In other words, $\tilde{*}_{\mathcal{A}}(a)$ is always contractible. — Suppose now that X is *any* CW–functor from $\mathcal{A}$ to spaces. There are obvious embeddings

$$\operatorname{nat}(X, Y) \longrightarrow \operatorname{map}_{|\mathcal{A}|}(\operatorname{hocolim} X, \operatorname{hocolim} Y) \longrightarrow \operatorname{map}_{|\mathcal{A}|}(\operatorname{hocolim} X, T)$$

where $\operatorname{map}_{|\mathcal{A}|}$ is for Spaces of maps over $|\mathcal{A}|$. One shows by induction over the skeletons of X that the composite embedding is a homotopy equivalence. In particular, this holds for $X = \tilde{*}_{\mathcal{A}}$. □

Proof of 8.5 Suppose that E in $\mathcal{F}$ is homogeneous of degree k. Define a cofunctor F_0 from $\mathcal{O}$ to Spaces by

$$F_0(V) := \operatorname*{holim}_{U \in \mathcal{I}^{(k)}(V)} E(U).$$

Here $I^{(k)}(V) \subset I^{(k)}$ is the full sub–poset consisting of all $U \in I^{(k)}$ which are contained in V. For the meaning of $I^{(k)}$, see the digression preceding 8.5. By 8.6, the cofunctor F_0 is equivalent to another cofunctor F_1 given by a formula of type

$$F_1(V) = \Gamma(q_V)$$

where q_V is a certain fibration on $|I^{(k)}(V)|$. The fibration q_V is natural in W, in the sense that a morphism $V \to W$ in $\mathcal{O}$ induces a map from the total Space of q_V to that of q_W, covering the inclusion

$$|I^{(k)}(V)| \hookrightarrow |I^{(k)}(W)|.$$

By inspection, this map of total Spaces maps each fiber of q_V to the corresponding fiber of q_W by a homotopy equivalence. Hence F_1 is equivalent to the cofunctor F_2 given by

$$F_2(V) := \Gamma\big(q_M; |I^{(k)}(V)|\big).$$

Finally we know from 3.5 (and proof) that $|I^{(k)}(V)| \simeq \binom{V}{k}$, and this can be understood as a chain of natural homotopy equivalences (natural in $V \in \mathcal{O}$). It follows easily that F_2 is equivalent to a cofunctor F_3 given by a formula of type

$$F_3(V) := \Gamma\Big(p; \binom{V}{k}\Big)$$

where p is a fibration on $\binom{M}{k}$. This is exactly the kind of cofunctor introduced in section 7, so we now write $F := F_3$. From the definition, F belongs to $\mathcal{F}$. Replacing E by an equivalent cofunctor if necessary, we can assume that E maps directly to F instead of F_0. If $S \in \binom{M}{k}$ and V is a tubular neighborhood of $S \subset M$, then the composition

$$E(V) \longrightarrow F(V) = \Gamma\Big(p; \binom{V}{k}\Big) \xrightarrow{\text{eval.}} p^{-1}(S)$$

is a homotopy equivalence, by construction and inspection. This is of course reminiscent of 8.4. Now form the commutative square

$$(*) \qquad \begin{array}{ccc} E & \longrightarrow & F \\ \Big\downarrow{\scriptstyle \eta_{k-1}} & & \Big\downarrow{\scriptstyle \eta_{k-1}} \\ T_{k-1}E & \longrightarrow & T_{k-1}F \end{array}$$

and recall that $T_{k-1}E(V)$ is contractible for all $V \in \mathcal{O}$. Given our analysis of $T_{k-1}F$ in section 7, we can complete the proof of 8.5 by showing that $(*)$ is homotopy Cartesian. By 2.5 and 5.1, it suffices to check that

$$(*)\qquad \begin{array}{ccc} E(V) & \longrightarrow & F(V) \\ \big\downarrow \eta_{k-1} & & \big\downarrow \eta_{k-1} \\ T_{k-1}E(V) & \longrightarrow & T_{k-1}F(V) \end{array}$$

is homotopy Cartesian for all $V \in \mathcal{O}k$. If it happens that $V \in \mathcal{O}r \subset \mathcal{O}k$ for some $r < k$, then we have $E(V) \simeq *$ by homogeneity and $F(V) \to T_{k-1}F(V)$ is a homotopy equivalence, by section 5 and section 6. If not, then V has k connected components and is a tubular neighborhood of some $S \subset M$, where $S \in \binom{M}{k}$. Using 8.4 now (and 7.6), and our observation above which seemed so reminiscent of 8.4, we find that $(*)$ is again homotopy Cartesian. □

9 The Homogeneous Layers of a Good Cofunctor

In this section we work with a fixed F in $\mathcal{F}$ and a distinguished element $* \in F(M)$, which we call the base point. Since M is the terminal object in $\mathcal{O}$, we may then regard F as a cofunctor from $\mathcal{O}$ to pointed Spaces. Define L_kF as in 8.2, and call it *the k–th homogeneous layer of F*. According to 8.5, the homogeneous cofunctor L_kF can be classified by some fibration $p\colon Z \to \binom{M}{k}$, and a partial section of it defined near the fat diagonal $\Delta_k M$. What does p look like? The answer is implicit in the last section. Recall that

$$\binom{M}{k} \simeq |I^{(k)}|,$$

in the notation of 3.5 and sequel. For any $V \in I^{(k)}$ with components V_s, where $s \in \pi_0(V)$, the rule taking a subset R of $\pi_0(V)$ to the Space $F(\bigcup_{s\in R} V_s)$ is a k–cube of Spaces:

$$(**)\qquad R \mapsto F(\textstyle\bigcup_{s\in R} V_s) \qquad (R \subset \pi_0(V)).$$

As such it has a total homotopy fiber (see 2.1) which we denote by $\Phi(V)$. Note that $V \mapsto \Phi(V)$ is a cofunctor from $I^{(k)}$ to Spaces taking all morphisms to homotopy equivalences.

9.1 Proposition *The fibration which classifies L_kF is the one associated with the quasifibration*

$$\operatorname*{hocolim}_{V\in I^{(k)}} \Phi(V) \longrightarrow |I^{(k)}|.$$

Remark Our classifying fibrations on $\binom{M}{k}$ should always come with partial sections defined near the fat diagonal. Note that Φ is a cofunctor from $\mathcal{I}^{(k)}$ to *pointed* Spaces, so that the (quasi)–fibration in 9.1 does in fact have a preferred global section.

Proof of 9.1 Write $j = k-1$ (for typographic reasons). By section 8, it is enough to show that $L_kF(V) \simeq \Phi(V)$ for $V \in \mathcal{I}^{(k)}$, by a chain of natural pointed homotopy equivalences. Since $V \in \mathcal{I}^{(k)} \subset \mathcal{O}k$, we have

$$\eta_k\colon F(V) \xrightarrow{\simeq} T_kF(V)$$

so that $L_kF(V)$ is homotopy equivalent to the homotopy fiber of the map $\eta_j\colon F(V) \longrightarrow T_jF(V)$. Recall that $T_jF(V)$ is defined as

$$\operatorname*{holim}_{U \in \mathcal{O}j(V)} F(U).$$

Now observe that the inclusion of posets

$$\{\textstyle\bigcup_{s\in R} V_s \mid R \subset \pi_0(V),\ R \neq \pi_0(V)\} \hookrightarrow \mathcal{O}j(V)$$

is right cofinal. Complete the proof by applying the cofinality theorem for homotopy inverse limits. □

In the case of an embedding cofunctor, $F(V) = \mathrm{emb}(V, N)$ as in 1.3, proposition 9.1 can be made much more explicit. We need a base point in $\mathrm{emb}(M, N)$, so we may as well assume that M is a smooth submanifold of N. For $S \in \binom{M}{k}$ let $\Psi(S)$ be the total homotopy fiber of the k–cube of pointed Spaces

$$(*) \qquad \{\mathrm{emb}(R, N) \mid R \subset S\}.$$

These Spaces are pointed because $R \subset S \subset M \subset N$.

9.2 Theorem *For $k \geq 2$, the homogeneous cofunctor $L_k\,\mathrm{emb}(-, N)$ is classified by the fibration $p\colon Z \to \binom{M}{k}$ with fibers $p^{-1}(S) = \Psi(S)$.*

Proof The first and most important observation here is that, for every V in $\mathcal{I}^{(k)}$ and every $S \in \binom{M}{k}$ which has exactly one point in each component of V, the obvious restriction map

$$\rho\colon \Phi(V) \longrightarrow \Psi(S)$$

is a homotopy equivalence. This can be seen as follows. For each $R \subset S$, there is a homotopy pullback square

$$\begin{array}{ccc} \mathrm{emb}(V_R, N) & \longrightarrow & \mathrm{mono}(TM|_R, TN) \\ \downarrow & & \downarrow \\ \mathrm{emb}(R, N) & \longrightarrow & N^R \end{array}$$

where mono denotes a Space of bundle monomorphisms (vector bundle morphisms which are mono in each fiber of the domain). Allowing R to be a variable subset of S we may think of it as a square in which each vertex is a k–cube of Spaces. The total homotopy fibers of these k–cubes will then again form a homotopy pullback square. But the two k–cubes in the right–hand column are fibrant, so their total homotopy fibers agree with their total fibers, which reduce to a single point if $k \geq 2$ (but not if $k = 1$). Therefore the total homotopy fibers of the k–cubes in the left–hand column are homotopy equivalent, which amounts to saying that $\phi\colon \Phi(V) \to \Psi(V)$ is a homotopy equivalence.

Now let $E_\flat$ be the homotopy colimit of the cofunctor taking $V \in I^{(k)}$ to the space of sections of $V \to \Phi_0(V)$. There are obvious forgetful maps

$$|I^{(k)}| \longleftarrow E_\flat \longrightarrow \binom{M}{k} .$$

The first of these is a homotopy equivalence by inspection. Comparison with the space E in the proof of 3.5 (towards the end) shows that the second map is also a homotopy equivalence. In more detail, there is a commutative diagram

$$\begin{array}{ccccc}
|I^{(k)}| & \xleftarrow{\ \simeq\ } & E_\flat & \longrightarrow & \binom{M}{k} \\
\big\downarrow{=} & & \big\downarrow & & \big\downarrow{=} \\
|I^{(k)}| & \xleftarrow{\ \simeq\ } & E & \xrightarrow{\ \simeq\ } & \binom{M}{k} .
\end{array}$$

Let p_1 be the pullback of the quasifibration in 9.1 to $E_\flat$, and let p_2 be the pullback of the fibration in 9.2 to $E_\flat$. From the observation made at the beginning of this proof, it is clear that there is a map over $E_\flat$ from p_1 to p_2 which maps each fiber of p_1 to the corresponding fiber of p_2 by a homotopy equivalence. □

In proposition 9.2, the case $k = 1$ has been excluded because it is different. However, it is also well understood: We have

$$T_1 \operatorname{emb}(-;N) = L_1 \operatorname{emb}(-;N) \simeq \operatorname{imm}(-;N).$$

This follows easily from 5.1 and the observation that all arrows in the commutative square

$$\begin{array}{ccc}
\operatorname{emb}(V;N) & \longrightarrow & \operatorname{imm}(V;N) \\
\big\downarrow{\eta_1} & & \big\downarrow{\eta_1} \\
T_1 \operatorname{emb}(V;N) & \longrightarrow & T_1 \operatorname{imm}(V;N)
\end{array}$$

are homotopy equivalences if $V \in \mathcal{O}_1$.

10 Boundary Conditions

So far all manifolds considered were without boundary. When there are boundaries, the theory looks slightly different. The following is an outline.

Suppose that M^m is smooth, possibly with boundary. Let $\mathcal{O}$ be the poset of all open subsets of M which contain ∂M. A cofunctor from $\mathcal{O}$ to Spaces is *good* if it satisfies conditions (a) and (b) just before 1.2, literally. In (a) we use a definition of *isotopy equivalence* which is appropriate for manifolds with boundary: a smooth codimension embedding $(V;\partial V) \to (W;\partial W)$ is an isotopy equivalence if, and so on.

10.1 Example Suppose that M is a neat smooth submanifold of another smooth manifold N with boundary. That is, M meets ∂N transversely, and $\partial M = M \cap \partial N$. For V in $\mathcal{O}$ let $F(V)$ be the Space of smooth embeddings $V \to N$ which agree with the inclusion near $\partial M \subset V$. Then F is good.

10.2 Example Suppose that M is a smooth submanifold with boundary of another smooth manifold N *without* boundary. For V in $\mathcal{O}$ let $F(V)$ be the Space of smooth embeddings $V \to N$ which agree with the inclusion near $\partial M \subset V$. Then F is good.

In practice example 10.1 is more important because it cannot be reduced to simpler cases, whereas 10.2 can often be so reduced. For example, with F as in 10.2 there is a fibration sequence up to homotopy

$$F(M) \longrightarrow \mathrm{emb}(M \setminus \partial M; N) \longrightarrow \mathrm{emb}(\partial M; N)$$

provided ∂M is compact. This follows from the isotopy extension theorem. It is a mistake to think that a similar reduction is possible in the case of 10.1. (Unfortunately I made that mistake in [23, section 5], trying to avoid further definitions; the calculations done there are nevertheless correct.)

In both examples, 10.1 and 10.2, the values $F(V)$ are contractible for collar neighborhoods V of ∂M. For general F, this may not be the case.

The definition of a *polynomial* cofunctor of some degree $\le k$ is again literally the same as before (2.2); we must insist that the closed subsets $A_0, \dots, A_k$ of $V \in \mathcal{O}$ have empty intersection with ∂M, since otherwise $F(V \setminus \cup_{i \in S} A_i)$ is not defined.

The definition of the full subcategory $\mathcal{O}k$ is more complicated. An element $V \in \mathcal{O}$ belongs to $\mathcal{O}k$ if it is a union of two disjoint open subsets V_1 and

V_2, where V_1 is a collar about ∂M (diffeomorphic to $\partial M \times [0,1)$) and V_2 is diffeomorphic to a disjoint union of $\leq k$ copies of $\mathbb{R}^m$.

Later we will need a certain subcategory $I^{(k)}$ of $\mathcal{O}k$. An object of $\mathcal{O}k$ belongs to $I^{(k)}$ if it has exactly k components not meeting ∂M; the morphisms in $I^{(k)}$ are the inclusions which are isotopy equivalences.

As before, T_kF can be defined as the homotopy right Kan extension along $\mathcal{O}k \to \mathcal{O}$ of $F|\mathcal{O}k$. It turns out to be polynomial of degree $\leq k$, and it turns out that $\eta_k\colon F \to T_kF$ has the properties listed in 6.1.

If $F(M)$ comes with a selected base point, then we can define $L_kF(V)$ as the homotopy fiber of $T_kF(V) \to T_{k-1}F(V)$. The cofunctor L_kF is homogeneous of degree k (definition like 8.1).

A general procedure for making homogeneous cofunctors of degree k on $\mathcal{O}$ is as follows. *Notation:* $\not\partial$ is the "delete boundary" command. Let $p\colon Z \to \binom{\not\partial M}{k}$ be a fibration. Suppose that it has a distinguished partial section defined near K, where K consists of all the points in the symmetric product $\mathrm{sp}_k M$ having at least two identical coordinates, or having at least one coordinate in ∂M. For V in $\mathcal{O}$ let $E(V)$ be the Space of (partial) sections of p defined over $\binom{\not\partial V}{k}$ which agree with the distinguished (zero) section near K. Then E is homogeneous of degree k.

There is a classification theorem for homogeneous cofunctors of degree k on $\mathcal{O}$, to the effect that up to equivalence they can all be obtained in the way just described. The classifying fibration p for a homogeneous E of degree k can be found/recovered as follows. Suppose that $S \subset M$ has k elements. Choose $V \in \mathcal{O}k$ so that V contains $S \cup \partial M$ as a deformation retract. For $R \subset S$, let V_R be the union of the components of V which meet $\partial M \cup R$. Let $\Phi(V)$ be the total homotopy fiber of the k-cube

$$R \mapsto F(V_R).$$

Then $\Phi(V) \simeq p^{-1}(S)$. If more detailed information is needed, one has to resort to quasifibrations: the rule $V \mapsto \Phi(V)$ can be regarded as a cofunctor on $I^{(k)}$ and it gives rise to a quasifibration on $|I^{(k)}| \simeq \binom{\not\partial M}{k}$. The associated fibration is p.

10.3 Example In the situation of 10.1, the classifying fibration p_k for L_kF has $p_k^{-1}(S)$ equal to the total homotopy fiber of the k-cube

$$R \mapsto \mathrm{emb}(R, N)$$

for $R \subset S$, provided $k \geq 2$. The case $k = 0$ is uninteresting (fiber contractible, base a single point). The case $k = 1$ is different as usual; for $s \in M$, the fiber $p_1^{-1}(\{s\})$ is the space of linear monomorphisms $T_sM \to T(\mathbb{R}^N)$. All this is exactly as in 9.2. For example, suppose that M is compact (with boundary). Then $L_kF(M)$ is homotopy equivalent to the space of sections of p_k with *compact support*. In other words, we are dealing with sections defined on all of the configuration space $\binom{M}{k}$ and equal to the zero section outside a compact set.

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References

[1] A K Bousfield, D M Kan, *Homotopy limits, completions and localizations*, Lect. Notes in Math. vol. 304, Springer (1973)

[2] A Dold, *Lectures on Algebraic Topology*, Grundlehren series, Springer (1972)

[3] W Dwyer, *The centralizer decomposition of BG*, from: "Algebraic topology: new trends in localization and periodicity (San Feliu de Guixols, 1994)", Progr. in Math. 136 Birkhäuser (1996) 167–184

[4] W Dwyer, D M Kan, *A classification theorem for diagrams of simplicial sets*, Topology, 23 (1984) 139–155

[5] W Dwyer, M Weiss, B Williams, *A parametrized index theorem for the algebraic K–theory Euler class, Parts I + II*, preprint (1998)

[6] T Goodwillie, *Calculus I: The first derivative of pseudoisotopy theory*, K–Theory, 4 (1990) 1–27

[7] T Goodwillie, *Calculus II: Analytic Functors*, K–Theory, 5 (1992) 295–332

[8] T Goodwillie, *Calculus III: The Taylor series of a homotopy functor*, in preparation

[9] T Goodwillie, *A multiple disjunction lemma for smooth concordance embeddings*, Memoirs Amer. Math. Soc. 86 no. 431 (1990)

[10] T Goodwillie, *Excision estimates for spaces of homotopy equivalences*, preprint, Brown University (1995)

[11] T Goodwillie, *Excision estimates for spaces of diffeomorphisms*, preprint, Brown University (1998)

[12] T Goodwillie, J Klein, *Excision estimates for spaces of Poincare embeddings*, in preparation

[13] T Goodwillie, M Weiss, *Embeddings from the point of view of immersion theory: Part II*, Geometry and Topology, 3 (1999) 103{118

[14] M L Gromov, *Stable mappings of foliations into manifolds*, Izv. Akad. Nauk. SSSR Mat. 33 (1969) 707{734, Trans. Math. USSR (Izvestia) 3 (1969) 671{693

[15] A Haefliger, V Poenaru, *Classi cation des immersions combinatoires*, Publ. Math. I.H.E.S. 23 (1964) 75{91

[16] R Kirby, L Siebenmann, *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, Ann. of Math. Studies vol. 88, Princeton Univ. Press (1977)

[17] S Mac Lane, *Categories for the Working Mathematician*, Springer (1971)

[18] S Mac Lane, I Moerdijk, *Sheaves in Geometry and Logic*, Springer{Verlag (1992)

[19] V Poenaru, *Homotopy and di erentiable singularities*, from: \Manifolds{Amsterdam 1970", Springer Lect. Notes in Math. vol. 197, 106{132

[20] G Segal, *Classifying spaces related to foliations*, Topology, 13 (1974) 293{312

[21] R Thomason, *Homotopy colimits in the category of small categories*, Math. Proc. Camb. Phil. Soc. 85 (1979) 91{109

[22] F Waldhausen, *Algebraic K{theory of Spaces*, from: \Proc. of 1983 Rutgers Conf. on Algebraic Topology", Lect. Notes in Math. 1126 (1985) 318{419

[23] M Weiss, *Calculus of Embeddings*, Bull. Amer. Math. Soc. 33 (1996) 177{187

[24] M Weiss, *Curvature and nite domination*, Proc. Amer. Math. Soc. 124 (1996) 615{622

[25] M Weiss, *Orthogonal Calculus*, Trans. Amer. Math. Soc. 347 (1995) 3743{3796, *Erratum*, Trans. Amer. Math. Soc. 350 (1998) 851{855