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4{manifolds as covers of the 4{sphere branched over non-singular surfaces

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Abstract

We prove the long-standing Montesinos conjecture that any closed oriented PL 4{manifold M is a simple covering of S^4 branched over a locally flat surface (cf [12]). In fact, we show how to eliminate all the node singularities of the branching set of any simple 4{fold branched covering M ! S^4 arising from the representation theorem given in [13]. Namely, we construct a suitable cobordism between the 5{fold stabilization of such a covering (obtained by adding a fth trivial sheet) and a new 5{fold covering M ! S^4 whose branching set is locally flat. It is still an open question whether the fth sheet is really needed or not.

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1 Introduction

The idea of representing manifolds as branched covers of spheres, extending the classical theory of ramified surfaces introduced by Riemann, is due to Alexander [1] and dates back to 1920. He proved that for any orientable closed PL manifold M of dimension m there is a branched covering of $M \to S^m$.

We recall that a non-degenerate PL map $p: M \to N$ between compact PL manifolds is called a *branched covering* if there exists an $(m-2)$–subcomplex $B_p \subset N$, the *branching set* of p, such that the restriction $p_|: M - p^{-1}(B_p) \to N - B_p$ is an ordinary covering of finite degree d. If B_p is minimal with respect to such property, then we have $B_p = p(S_p)$, where S_p is the *singular set* of p, that is the set of points at which p is not locally injective. In this case, both B_p and S_p, as well as the *pseudo-singular set* $S'_p = \mathrm{Cl}(p^{-1}(B_p) - S_p)$, are (possibly empty) homogeneously $(m-2)$–dimensional complexes.

Since p is completely determined (up to PL homeomorphism) by the ordinary covering $p_|$ (cf [3]), we can describe it in terms of its branching set B_p and its monodromy $\omega_p: \pi_1(N - B_p) \to \Sigma_d$ (uniquely defined up to conjugation in Σ_d, depending on the numbering of the sheets).

If $N = S^m$ then a convenient description of p can be given by labelling each $(m-2)$–simplex of B_p by the monodromy of the corresponding meridian loop, since such loops generate the fundamental group $\pi_1(S^m - B_p)$.

Therefore, we can reformulate the Alexander's result as follows: any orientable closed PL manifold M of dimension m can be represented by a labelled $(m-2)$–subcomplex of S^m.

Of course, in order to make such representation method effective, some control is needed on the degree d and on the complexity of the local structure of B_p and ω_p. Unfortunately, there is no such control in the original Alexander's proof, being d dependent on the number of simplices of a triangulation of M and B_p equal to the $(m-2)$–skeleton of an m–simplex. Even at the present, as far as we know, the only general (for any m) results in this direction are the negative ones obtained by Berstein and Edmonds [2]: for representing all the m–manifolds at least m sheets are necessary (for example this happens of the m–torus T^m) and in general we cannot require B_p to be non-singular (the counterexamples they give have dimension $m \geq 8$). On the contrary, the situation is much better for $m \leq 4$.

The case of surfaces is trivial: the closed (connected) orientable surface T_g of genus g is a 2–fold cover of S^2 branched over $2g+2$ points. For $m = 3$, Hilden [4], Hirsch [6] and Montesinos [11] independently proved that any orientable closed (connected) 3–manifold is a simple 3–fold cover of S^3 branched over a knot.

For $m = 4$, the representation theorem proved by Piergallini [13] asserts that any orientable closed (connected) PL 4–manifold is a simple 4–fold cover of S^4 branched over a transversally immersed PL surface. Simple means that the monodromy of each meridian loop is a transposition. On the other hand, a transversally immersed PL surface is a subcomplex which is a locally flat PL surface at all its points but a finite number of *nodes* (transversal double points). So, the local models (up to PL equivalence) for the labelled branching set are the ones depicted in Figure 1, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$ (the monodromies of the meridian loops corresponding to sheets of the branching set meeting at a node must be disjoint). We remark that in general the branching surface cannot be required to be orientable (cf [13], [14]).

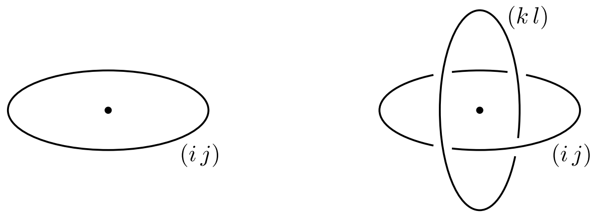

Figure 1

The question whether the nodes can be eliminated in order to get non-singular branching surfaces, as proposed by Montesinos in [12], was left open in [13].

In the next section we show how elimination of nodes can be performed up to cobordism of coverings, after the original 4–fold covering has been stabilized by adding a fifth trivial sheet. This proves the following representation theorem.

Theorem *Any orientable closed (connected) PL 4–manifold is a simple 5–fold cover of S^4 branched over a locally flat PL surface.*

2 Elimination of nodes

Let M be an orientable closed (connected) PL 4–manifold and let $p\colon M \to S^4$ be a 4–fold covering branched over a transversally immersed PL surface $F \subset S^4$ given by Theorem B of [13]. We denote by $q\colon M \to S^4$ the 5–fold branched covering obtained by stabilizing p with an extra trivial sheet. In terms of labelled branching set this means adding to the surface F, labelled

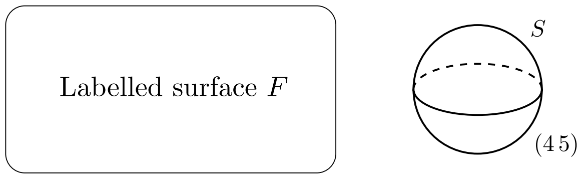

Figure 2

with transpositions in Σ_4, a separate unknotted 2–sphere S labelled with the transposition $(4\,5)$, as schematically shown in Figure 2.

Looking at the proof of Theorem B of [13], we see that nodes of the branching set of p come in pairs, in such a way that each pair consists of the end points of a simple arc contained in F and all these arcs are disjoint from each other.

Let $\alpha_1, \ldots, \alpha_n \subset F$ be such arcs and let ν_i and ν'_i be the nodes joined by α_i. The intersection of $F \cup S$ with a sufficiently small regular neighborhood $N(\alpha_i)$ of α_i in S^4 consists of a disk A_i containing α_i and two other disks B_i and B'_i transversally meeting A_i respectively at ν_i and ν'_i. Up to labelled isotopy, we can assume B_i and B'_i labelled with $(1\,2)$ and A_i labelled with $(3\,4)$, as in Figure 3 (remember that the monodromy of p is transitive, since M is connected). We also assume the $N(\alpha_i)$'s disjoint from each other.

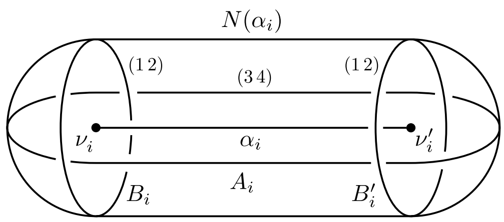

Figure 3

For future use, we modify the branching surface $F \cup S$ by "finger move" labelled isotopies, in order to introduce inside each $N(\alpha_i)$ two more small trivial disks C_i and C'_i respectively labelled by $(2\,4)$ and $(4\,5)$, as shown in Figure 4. This modification has the effect of connecting $q^{-1}(N(\alpha_i))$ making it PL equivalent to $S^1 \times B^3$.

Now, we consider the orientable 5–manifold $T = S^4 \times [0,1] \cup H_1 \cup \cdots \cup H_n$ obtained by attaching to $S^4 \times [0,1]$ a 1–handle H_i for each pair of nodes ν_i, ν'_i. The attaching cells of each H_i are $N(\nu_i) \times \{1\}$ and $N(\nu'_i) \times \{1\}$, where $N(\nu_i)$ and $N(\nu'_i)$ are regular neighborhoods ν_i and ν'_i in $N(\alpha_i) - (C_i \cup C'_i)$, such that all the intersections $D_i = N(\nu_i) \cap A_i$, $E_i = N(\nu_i) \cap B_i$, $D'_i = N(\nu'_i) \cap A_i$ and $E'_i = N(\nu'_i) \cap B'_i$ are again disks.

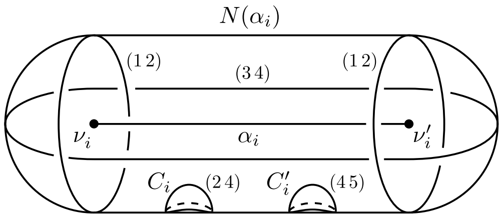

Figure 4

The product covering $q \times \mathrm{id}_{[0,1]}\colon M \times [0,1] \to S^4 \times [0,1]$ can be extended to a new 5–fold simple branched covering $r\colon W \to T$, where W is the result of adding appropriate 1–handles to $M \times [0,1]$ over the H_i's. In fact, the restrictions of $q \times \{1\}$ over $N(\nu_i) \times \{1\}$ and $N(\nu_i') \times \{1\}$ are equivalent, hence, by a suitable choice of the attaching map of H_i, we can define r over $H_i = B^4 \times [0,1]$ just by crossing the first restriction with the identity of $[0,1]$. Namely, the pair $(H_i, B_r \cap H_i)$ is equivalent to $(N(\nu_i), D_i \cup E_i) \times [0,1]$, with the monodromy of the meridian loops around $D_i \times [0,1]$ and $E_i \times [0,1]$ respectively equal to (1 2) and (3 4). Then, $r^{-1}(H_i)$ consists of three 1–handles attached to $M \times [0,1]$ at the three pairs of 4–cells making up the pair $(q^{-1}(N(\nu_i)), q^{-1}(N(\nu_i'))) \times \{1\}$. We denote by H_i', H_i'' and H_i''' these 1–handles in such a way that they involve respectively the sheets 1 and 2, the sheets 3 and 4, and the sheet 5 (see Figure 5, where the lighter lines represent the pseudo-singular set). We remark that the branching set B_r is a locally flat PL 3–manifold at all points but one transversal double arc inside each H_i between $\nu_i \times \{1\}$ and $\nu_i' \times \{1\}$.

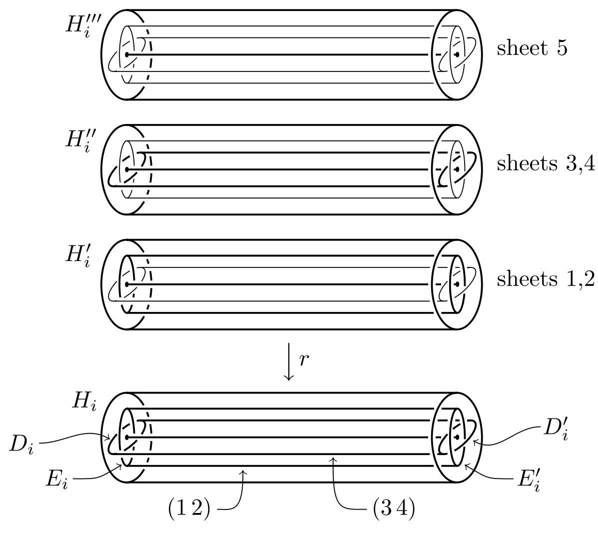

Figure 5

At this point, we want to simultaneously attach to T and W some 2–handles in order to kill the 1–handles $H_1, \dots, H_n$ attached to $S^4 \times [0,1]$ and the 1–handles $H_1', H_1'', H_1''', \dots, H_n', H_n'', H_n'''$ attached to $M \times [0,1]$, taking care that the branched covering r can be extended to these 2–handles.

For each $i = 1, \dots, n$, we consider a simple loop λ_i inside $\mathrm{Bd}\, T \cap (N(\alpha_i) \times \{1\} \cup H_i) - B_r$ running through H_i once and linking both the disks $C_i \times \{1\}$ and $C_i' \times \{1\}$ once, as shown in Figure 6. We observe that $r^{-1}(\lambda_i)$ consists of three loops $\lambda_i', \lambda_i'', \lambda_i''' \subset \mathrm{Bd}\, W - (S_r \cup S_r')$, such that: λ_i' runs through H_i' once and avoids $H_i'' \cup H_i'''$, λ_i'' runs through H_i'' once and avoids $H_i' \cup H_i'''$, while λ_i''' runs through each of H_i', H_i'' and H_i''' once.

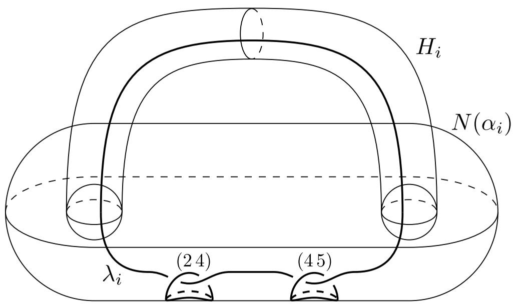

Figure 6

Then, the 5–manifold $T \cup L_1 \cup \dots \cup L_n$ obtained by attaching to T the 2–handle L_i along each loop λ_i (with arbitrary framing), is PL homeomorphic to $S^4 \times [0,1]$, since each L_i kills the corresponding H_i.

Analogously, the 5–manifold $W \cup (L_1' \cup L_1'' \cup L_1''') \cup \dots \cup (L_n' \cup L_n'' \cup L_n''')$ obtained by attaching to W the 2–handles L_i', L_i'' and L_i''' along the loops λ_i', λ_i'' and λ_i''' (with arbitrary framings), is PL homeomorphic to $M \times [0,1]$. In fact, we can cancel first each L_i''' with the corresponding H_i''' and then each L_i' and L_i'' respectively with H_i' and H_i''.

By choosing the attaching framings of the 2–handles L_i', L_i'' and L_i''' accordingly with the ones of the 2–handle L_i, we can extend the covering r to such 2–handles as suggested by Figure 7, where the branching set consists of the labelled 3–cells F_i and G_i transversal to the 2–handle L_i. Namely, we can glue the covering represented in the figure with r, since they coincide over the attaching tube around λ_i. Then, we can identify L_i' and L_i'' respectively with the trivial components over L_i corresponding to sheets 1 and 3, and L_i''' with the non-trivial component over L_i corresponding to sheets 2, 4 and 5.

In this way, we get an extension of r which is PL equivalent to a new branched covering $s\colon M \times [0,1] \to S^4 \times [0,1]$. Up to the natural identification between

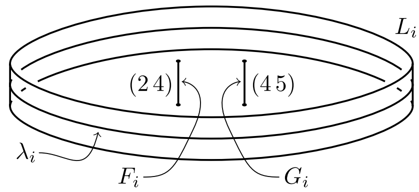

Figure 7

fibers and factors, the restriction of s over $S^4 - \{0\}$ coincides with q, while the restriction over $S^4 - \{1\}$ gives us a new 5–fold simple branched covering $q'\colon M \to S^4$.

The branching set $B_{q'}$ of q' is a locally flat PL surface in S^4. In fact, it is isotopically equivalent to the result of the following modifications performed on $B_q = F \cup S$, due to attaching handles: for each $i = 1, \dots, n$, the disks D_i, D'_i, E_i and E'_i are replaced by linked pipes respectively connecting $\mathrm{Bd}D_i$ with $\mathrm{Bd}D'_i$ and $\mathrm{Bd}E_i$ with $\mathrm{Bd}E'_i$; for each $i = 1, \dots, n$, the new trivial spheres $\mathrm{Bd}F_i$ and $\mathrm{Bd}G_i$ are added on.

3 Final remarks

The argument used in the previous section for eliminating nodes, with some minor variation, allows us to perform a variety of different modifications on branched coverings.

We can eliminate any pair of isolated singularities of the branching set, which are equivalent up to orientation reversing PL homeomorphisms, provided that the covering has at least one sheet more than the ones involved in them. For instance, this is a way, alternative with respect to the one of [13], to remove cusps from the branching set of a simple 4–fold covering of S^4.

On the other hand, by choosing the attaching balls of the 1–handle H_i centred at two non-singular points of the branching set with the same monodromy and letting the attaching loop of the 2–handle L_i have trivial monodromy, we get a new approach to surgery of simple branched coverings along symmetric knots (see [12]). In fact, in this case we have $d - 1$ handles over H_i and d handles over L_i, where d is the degree of the covering, and after cancellation we are left with one 2–handle attached to the covering manifold along the unique loop in the counterimage of the arc λ_i. Surgeries of greater indices (see [5]) can be realized similarly.

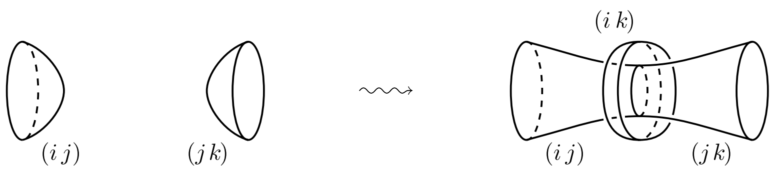

Figure 8

With a different choice of the monodromies, we can also perfom surgeries on the branching set without changing the covering manifold up to PL homeomorphisms. In particular, we get the move shown in Figure 8, which is the double of the move in Figure 12 of [13].

By using this move, we can connect all the non-trivial components of the branching surface, provided that the degree of the covering is at least 3, in such a way that the branching surface of the theorem can be assumed to have the following special form: $F = G \cup S_1 \cup \dots \cup S_k$, where $G \subset S^4$ is connected and $S_1, \dots, S_k$ is a family of separate trivial 2–spheres. Furthermore, we can perform hyperbolic transformations of G in order to make it unknotted (cf [7], [9]).

We observe that, in some sense, G represents the cobordism class of the covering manifold M, being $\sigma(M) = -F \cdot F/2 = -G \cdot G/2$ (cf [14]). On the other hand, the S_i's cannot be eliminated in general, that is the branching surface cannot be required to be connected. In fact, given any covering $M \to S^4$ branched over a locally flat PL surface F, we have $\chi(M) = 2d - \chi(F)$, where d is the degree of the covering. Then, by the Whitney inequality for the self-intersection of non-orientable surfaces in S^4 (cf [10]), F must have at least $d + |\sigma(M)|/2 - \chi(M)/2$ components.

Finally, we remark that our argument heavily depends on the fifth extra sheet for the elimination of nodes, hence it seems useless for solving the following question that remains still open (cf Problem 4.113 of Kirby's problem list [9]):

Question *Is any orientable closed (connected) PL 4–manifold a simple 4–fold cover of S^4 branched over a locally flat PL surface?*

References

[1] J W Alexander, *Note on Riemann spaces*, Bull. Amer. Math. Soc. 26 (1920) 370{373

[2] I Berstein, A L Edmonds, *The degree and branch set of a branched covering*, Invent. Math. 45 (1978) 213{220

[3] R H Fox, *Covering spaces with singularities*, Algebraic Geometry and Topology, A symposium in honour of S. Lefschetz, Princeton 1957, 243{257

[4] H M Hilden, *Every closed orientable 3{manifold is a 3{fold branched covering space of S^3*, Bull. Amer. Math. Soc. 80 (1974) 1243{1244

[5] H M Hilden, J M Montesinos, *Lifting surgeries to branched covering spaces*, Trans. Amer. Math. Soc. 259 (1980) 157{165

[6] U Hirsch, *Über o ene Abbildungen auf die 3{Sphäre*, Math. Z. 140 (1974) 203{230

[7] F Hosokawa, A Kawauchi, *Proposals for unknotted surfaces in four-spaces*, Osaka J. Math. 16 (1979) 233{248

[9] S Kamada, *Non-orientable surfaces in 4{space*, Osaka J. Math. 26 (1989) 367{385

[9] R Kirby, *Problems in low-dimensional topology*, Stud. Adv. Math. 2, part 2, Amer. Math. Soc. and International Press 1997

[10] W S Massey, *Proof of a conjecture of Whitney*, Paci c J. Math. 31 (1969) 143{156

[11] J M Montesinos, *A representation of closed, orientable 3{manifolds as 3{fold branched coverings of S^3*, Bull. Amer. Math. Soc. 80 (1974) 845{846

[12] J M Montesinos, *4{manifolds, 3{fold covering spaces and ribbons*, Trans. Amer. Math. Soc. 245 (1978) 453{467

[13] R Piergallini, *Four-manifolds as 4{fold branched covers of S^4*, Topology 34 (1995) 497{508

[14] O Ja Viro, *Signature of branched covering*, Trans. Mat. Zametki 36 (1984) 549{557