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Hyperbolic cone{manifolds with large cone{angles

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Abstract

We prove that every closed oriented 3{manifold admits a hyperbolic cone{manifold structure with cone{angle arbitrarily close to 2 .

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1 Introduction

Consider the hyperbolic 3-space in the upper half-space model $\mathbb{H}^3 \simeq \mathbb{C} \times \mathbb{R}_+$ and for $\alpha \in (0, 2\pi)$ set $S_\alpha = \{(e^{r+i\theta}, t) \mid \theta \in [0, \alpha], r \in [0, \infty), t \in \mathbb{R}_+\}$. The boundary of S_α is a union of two hyperbolic half-planes. Denote by $\mathbb{H}^3(\alpha)$ the space obtained from S_α by identifying both half-planes by a rotation around the vertical line $\{0\} \times \mathbb{R}_+$.

A distance on a 3-manifold M determines a hyperbolic cone-manifold structure with singular locus a link $L \subset M$ and cone-angle $\alpha \in (0, 2\pi)$, if every point $x \in M$ has a neighborhood which can be isometrically embedded either in $\mathbb{H}^3$ or in $\mathbb{H}^3(\alpha)$ depending on $x \in M \setminus L$ or $x \in L$.

Jean-Pierre Otal showed that the connected sum $\#^k(\mathbb{S}^2 \times \mathbb{S}^1)$ of k copies of $\mathbb{S}^2 \times \mathbb{S}^1$ admits a hyperbolic cone-manifold structure with cone-angle $2\pi - \epsilon$ for all $\epsilon > 0$ as follows: The manifold $\#^k(\mathbb{S}^2 \times \mathbb{S}^1)$ is the double of the genus k handlebody H. There is a convex-cocompact hyperbolic metric on the interior of H such that the boundary of the convex-core is bent along a simple closed curve γ with dihedral angle $\pi - \frac{1}{2}\epsilon$ [2]; the convex-core is homeomorphic to H and hence the double of the convex-core is homeomorphic to $\#^k(\mathbb{S}^2 \times \mathbb{S}^1)$. The induced distance determines a hyperbolic cone-manifold structure on $\#^k(\mathbb{S}^1 \times \mathbb{S}^2)$ with singular locus γ and cone-angle $2\pi - \epsilon$. The same argument applies for every manifold which is the double of a compact manifold whose interior admits a convex-cocompact hyperbolic metric. Michel Boileau asked whether every 3-manifold has this property. Our goal is to give a positive answer to this question. We prove:

Theorem 1 *Let M be a closed and orientable 3-manifold. For every ϵ there is a distance d_ϵ which determines a hyperbolic cone-manifold structure on M with cone-angle $2\pi - \epsilon$.*

Before going further, we remark that we do not claim that the singular locus is independent of ϵ.

We now sketch the proof of Theorem 1. First, we construct a compact manifold M^0, whose boundary consists of tori, and such that there is a sequence (M^0_n) of 3-manifolds obtained from M^0 by Dehn filling such that M^0_n is homeomorphic to M for all n. The especial structure of M^0 permits us to show that the interior $\operatorname{Int} M^0$ of the manifold M^0 admits, for every $\epsilon > 0$, a complete hyperbolic cone-manifold structure with cone-angle $2\pi - \epsilon$. Thus, it follows from the work of Hodgson and Kerckhoff [5] that for n sufficiently large there is a distance $d_{n_\epsilon} = d_\epsilon$ on the manifold $M^0_{n_\epsilon} = M$ which determines a hyperbolic cone-manifold structure with cone-angle $2\pi - \epsilon$.

Let (ϵ_i) be a non-increasing sequence of positive numbers tending to 0. If the corresponding sequence (n_{ϵ_i}) grows fast enough, then the pointed Gromov–Hausdorff limit of the sequence (M, d_{ϵ_i}) of metric spaces is a complete, smooth, hyperbolic manifold X with finite volume. Moreover, the volume of the (M, d_{ϵ_i}) converges to the volume of X when i tends to ∞; in particular the volume of (M, d_{ϵ_i}) is uniformly bounded.

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2 Preliminaries

2.1 Dehn filling

Let N be a compact manifold whose boundary consists of tori $T_1, \ldots, T_k$ and let $U_1, \ldots, U_k$ be solid tori. For any collection $\{\phi_i\}_{i=1,\ldots,k}$ of homeomorphisms $\phi_i : \partial U_i \to T_i$ let $N_{\phi_1,\ldots,\phi_k}$ be the manifolds obtained from N by attaching the solid torus U_i via ϕ_i to T_i for $i = 1, \ldots, k$.

Suppose that for all i we have a basis (m_i, l_i) of $H_1(T_i;\mathbb{Z})$ and let μ_i be the meridian of the solid torus U_i. There are coprime integers a_i, b_i with $\phi_i(\mu_i) = a_i m_i + b_i l_i$ in $H_1(T_i;\mathbb{Z})$ for all $i = 1, \ldots, k$. It is well known that the manifold $N_{\phi_1,\ldots,\phi_k}$ depends only on the set $\{a_1 m_1 + b_1 l_1, \ldots, a_k m_k + b_k l_k\}$ of homology classes. We denote this manifold by $N_{(a_1 m_1 + b_1 l_1),\ldots,(a_k m_k + b_k l_k)}$ and say that it has been obtained from N filling the curves $a_i m_i + b_i l_i$.

The following theorem, due to Hodgson and Kerckhoff [5] (see also [3]), generalizes Thurston's Dehn filling theorem:

Generalized Dehn filling theorem *Let N be a compact manifold whose boundary consists of tori $T_1, \ldots, T_k$ and let (m_i, l_i) be a basis of $H_1(T_i;\mathbb{Z})$ for $i = 1, \ldots, k$. Assume that the interior* Int N *of N admits a complete finite volume hyperbolic cone-manifold structure with cone-angle $\alpha \le 2\pi$. Then there exists $C > 0$ with the following property:*

The manifold $N_{(a_1 m_1 + b_1 l_1),\ldots,(a_k m_k + b_k l_k)}$ admits a hyperbolic cone-manifold structure with cone-angle α if $|a_i| + |b_i| \ge C$ for all $i = 1, \ldots, k$.

2.2 Geometrically finite manifolds

The *convex-core* of a complete hyperbolic manifold N with finitely generated fundamental group is the smallest closed convex set $CC(N)$ such that the inclusion $CC(N) \hookrightarrow N$ is a homotopy equivalence. The convex-core $CC(N)$ has empty interior if an only if N is Fuchsian; since we will not be interested in this case we assume from now on that the interior of the convex-core is not empty. We will only work with geometrically finite manifolds, i.e. the convex-core has finite volume. If N is geometrically finite then it is homeomorphic to the interior of a compact manifold N and the convex-core $CC(N)$ is homeomorphic to $N \setminus P$ where $P \subset \partial N$ is the union of all toroidal components of ∂N and of a collection of disjoint, non-parallel, essential simple closed curves. The pair $(N; P)$ is said to be the pared manifold associated to N and P is its parabolic locus ([6]).

A theorem of Thurston [13] states that the induced distance on the boundary $\partial CC(N)$ of the convex-core $CC(N)$ is a complete smooth hyperbolic metric with finite volume. The boundary components are in general not smoothly embedded, they are pleated surfaces bent along the so-called bending lamination. We will only consider geometrically finite manifolds for which the bending lamination is a weighted curve $\theta\gamma$. Here γ is the simple closed geodesic of N along which $\partial CC(N)$ is bent and $\pi - \theta$ is the dihedral angle.

The following theorem, due to Bonahon and Otal, is an especial case of [2, Theoreme 1].

Realization theorem *Let N be a compact 3-manifold with incompressible boundary whose interior* Int N *admits a complete hyperbolic metric with parabolic locus P. If $\gamma \subset \partial N \setminus P$ is an essential simple closed curve such that $\partial N \setminus (\gamma \cup P)$ is acylindrical then for every $\theta > 0$ there is a unique geometrically finite hyperbolic metric on* Int N *with parabolic locus P and bending lamination $\theta\gamma$.*

We refer to [4] and to [6] for more about the geometry of the convex-core of geometrically finite manifolds.

3 Proof of Theorem 1

Let $S \subset M$ be a closed embedded surface which determines a Heegaard splitting $M = H_1 \cup_\phi H_2$ of M. Here H_1 and H_2 are handlebodies and $\phi : \partial H_1 \to \partial H_2$

is the attaching homeomorphism. Without loss of generality we may assume that S has genus $g \geq 2$.

Lemma 2 *There is a pant decomposition P of ∂H_1 such that both pared manifolds (H_1, P) and $(H_2, \phi(P))$ have incompressible and acylindrical boundary.*

Proof The Masur domain $O(H_i)$ of the handlebody H_i is an open subset of $PML(\partial H_i)$, the space of projective measured laminations on ∂H_i. If γ is a weighted multicurve in the Masur domain then the pared manifold $(H_i, \mathrm{supp}(\gamma))$ has incompressible and acylindrical boundary, where $\mathrm{supp}(\gamma)$ is the support of γ (see [9, 10] for the properties of the Masur domain). Kerckhoff [7] proved that the Masur domain has full measure with respect to the measure class induced by the PL–structure of $PML(\partial H_i)$. The map $\phi : \partial H_1 \to \partial H_2$ induces a homeomorphism $\phi : PML(\partial H_1) \to PML(\partial H_2)$ which preserves the canonical measure class. In particular, the intersection of $O(H_1)$ and $\phi^{-1}(O(H_2))$ is not empty and open in $PML(\partial H_1)$. Since weighted multicurves are dense in $PML(\partial H_1)$ the result follows. □

Now, choose a pant decomposition $P = \{p_1, \ldots, p_{3g-3}\}$ of ∂H_1 as in Lemma 2 and identify it with a pant decomposition P of S. Let $S \times [-2, 2]$ be a regular neighborhood of S in M and U a regular neighborhood of $P \times \{-1, 1\}$ in $S \times [-2, 2]$; U is a union of disjoint open solid tori $U_1^+, \ldots, U_{3g-3}^+, U_1^-, \ldots, U_{3g-3}^-$ with $p_j^{\pm} = p_j \times \{\pm 1\} \subset U_j^{\pm}$ for all j. The boundary of the manifold $M' = M \setminus U$ is a collection of tori

$$\partial M' = T_1^+ \cup \cdots \cup T_{3g-3}^+ \cup T_1^- \cup \cdots \cup T_{3g-3}^-$$

where $T_j^{\pm}$ bounds $U_j^{\pm}$. We choose a basis $(l_j^{\pm}, m_j^{\pm})$ of $H_1(T_j^{\pm}; \mathbb{Z})$ for $j = 1, \ldots, 3g-3$ as follows:

$l_j^{\pm}$: For all j there is a properly embedded annulus

$$A_j : (\mathbb{S}^1 \times [-1, 1], \mathbb{S}^1 \times \{\pm 1\}) \to (M' \cap S \times [-2, 2], T_j^{\pm});$$

set $l_j^{\pm} = A_j|_{\mathbb{S}^1 \times \{\pm 1\}}$.

$m_j^{\pm}$: The curve $m_j^{\pm}$ is the meridian of the solid torus $U_j^{\pm}$ with the orientation chosen such that the algebraic intersection number of $m_j^{\pm}$ and $l_j^{\pm}$ is equal to 1.

For $n \in \mathbb{Z}$ let M^0_n be the manifold

$$M^0_n \stackrel{\text{def}}{=} M^0_{(nl_1^+ + m_1^+),\ldots,(nl_{3g-3}^+ + m_{3g-3}^+),(-nl_1^- + m_1^-),\ldots,(-nl_{3g-3}^- + m_{3g-3}^-)}$$

obtained by filling the curve $\pm nl_j^\pm + m_j^\pm$ for all j.

Let V_j be a regular neighborhood of the image of A_j in M^0; we may assume that $V_i \cap V_j = \emptyset$ for all $i \neq j$. The interior of the manifold $M^0 \setminus \cup_j V_j$ is homeomorphic to $M \setminus P$ and its boundary is a collection $T_1, \ldots, T_{3g-3}$ of tori. The complement of $M^0 \setminus \cup_j V_j$ in M^0_n is a union of $3g-3$ solid tori whose meridians do not depend on n. In particular, M^0_n is homeomorphic to M^0_0 for all n. Since M^0_0 is, by construction, homeomorphic to M, we obtain

Lemma 3 *The manifold M^0_n is homeomorphic to M for all $n \in \mathbb{Z}$.* □

In order to complete the proof of Theorem 1 we make use of the following result which we will show later on.

Proposition 4 *There is a link $L \subset$ Int M^0 such that for all $\epsilon > 0$ the manifold* Int M^0 *admits a complete, finite volume hyperbolic cone-manifold structure with singular locus L and cone-angle $2\pi - \epsilon$.*

We continue with the proof of Theorem 1. Since the manifold Int M^0 admits a complete finite volume hyperbolic cone-manifold structure with cone-angle $2\pi - \epsilon$ it follows from the Generalized Dehn filling theorem that there is some n such that M^0_n admits a hyperbolic cone-manifold structure with cone-angle $2\pi - \epsilon$, too. This concludes the proof of Theorem 1 since M and M^0_n are homeomorphic by Lemma 3.

We now prove Proposition 4. The surface S separates M^0 in two manifolds M^0_- and M^0_+. The boundary ∂M^0 is the union of a copy of S and the collection $P_\pm = \cup_{j=1,\ldots,3g-3} T_j^{\pm 1}$ of tori. It follows from the choice of P that the manifold M^0 is irreducible, atoroidal and has incompressible boundary. In particular, Thurston's Hyperbolization theorem [11] implies that the interior of M^0 admits a complete hyperbolic metric with parabolic locus $P_\pm$.

If $L \subset S$ is a simple closed curve such that $P \cup L$ fills S, then the pared manifold (M^0, L) is acylindrical. Bonahon and Otal's Realization theorem implies that for all $\epsilon > 0$ there is a geometrically finite hyperbolic metric g_ϵ on the interior of M^0 with parabolic locus $P_\pm$ and with bending lamination $\lambda_\epsilon = \epsilon/2\pi \cdot L$. The convex-core $CC(M^0, g_\epsilon)$ can be identified with $M^0 \setminus P_\pm$ and hence the boundary of the convex-core consists of a copy S of the surface S; the identification of S with S induces a map $: S_- \to S_+$ with

$$\mathrm{Int}\, M^0 = CC(M^0_-; g_-) \cup CC(M^0_+; g_+).$$

The hyperbolic surface S is bent along L with dihedral angle $\frac{1}{2}$. The following lemma concludes the proof of Proposition 4.

Lemma 5 *The map $: S_- \to S_+$ is isotopic to an isometry.*

Proof The cover $(N ; h)$ of $(\mathrm{Int}\, M^0; g)$ corresponding to the surface S is geometrically finite. Since S is incompressible we obtain that N is homeomorphic to the interior of $S \times [-1; 1]$ and the parabolic locus of $(N ; h)$ is the collection $P \times \{1\}$. The convex surface $S \subset \mathrm{Int}\, M^0$ lifts to one of the components of the boundary of the convex-core of $(N ; h)$; the other components are spheres with three punctures, and hence totally geodesic. The map can be extended to the map $\tilde{} : N_- = S_- \times (-1; 1) \to N_+ = S_+ \times (-1; 1)$ given by $(x; t) \mapsto ((x); -t)$. The map $\tilde{}$ maps, up to isotopy, P_- to P_+ and L to L. Hence, the uniqueness part of Bonahon and Otal's Realization theorem implies that $\tilde{}$ is isotopic to an isometry and this gives the desired result. □

Concluding remarks

Recall that in Theorem 1 we do not claim that the singular locus of d is independent of . If M is the double of a compact manifold with incompressible boundary whose interior admits a convex-cocompact hyperbolic metric, then, using Otal's trick, it is possible to construct a link L such that M admits a hyperbolic cone-manifold structure with singular locus L and cone-angle $2 -$ for all . Proposition 4 suggests that this may be a more general phenomenon but the author does not think that it is always possible to choose the singular locus independently of .

Question 1 *Let L be a link in $\mathbb{S}^2 \times \mathbb{S}^1$ which intersects an essential sphere n times. Is there a hyperbolic cone-manifold structure on $\mathbb{S}^2 \times \mathbb{S}^1$ with singular locus L and with cone-angle greater than $\frac{n-2}{n} 2$?*

Question 2 *Is there a link $L \subset \mathbb{S}^3$ such that for every > 0 there is a hyperbolic cone-manifold structure on $\mathbb{S}^3$ with singular locus L and with cone-angle $2 -$?*

We suspect that both questions have a negative answer.

We define, as suggested by Michel Boileau, the *hyperbolic volume* $\mathrm{Hypvol}(M)$ of a closed 3–manifold M as the infimum of the volumes of all possible hyperbolic cone–manifold structures on M with cone–angle less or equal to 2π. It follows from [5] and from the Schläfli formula that the hyperbolic volume of a manifold M is achieved if and only if M is hyperbolic. A sequence of hyperbolic cone–manifold structures realizes the hyperbolic volume if the associated volumes converge to $\mathrm{Hypvol}(M)$. From the arguments used in the proof of the Orbifold theorem [1] it is easy to deduce that the hyperbolic volume is realized by a sequence of hyperbolic cone–manifold structures whose cone–angles are all greater or equal to π.

Question 3 *Is there a sequence of metrics realizing the hyperbolic volume and such that the associated cone–angles tend to* 2π*?*

As remarked in the introduction, it follows from our construction that there are sequences of hyperbolic cone–manifold structures whose cone–angles tend to 2π and which have uniformly bounded volume.

Let M now be a closed orientable and irreducible 3–manifold M. We say that M is *geometrizable* if Thurston's Geometrization Conjecture holds for it. If M is geometrizable then let M_{hyp} be the associated complete finite volume hyperbolic manifold. In [12] we proved:

Theorem *Let M be a closed, orientable, geometrizable and prime 3–manifold. Then the minimal volume* $\mathrm{Minvol}(M)$ *of M is equal to* $\mathrm{vol}(M_{\mathrm{hyp}})$ *and moreover, the manifolds (M, g_i) converge in geometrically to M_{hyp} for every sequence (g_i) of metrics realizing* $\mathrm{Minvol}(M)$*. In particular, the minimal volume is achieved if and only if M is hyperbolic.*

Recall that the minimal volume $\mathrm{Minvol}(M)$ of M is the infimum of the volumes $\mathrm{vol}(M, g)$ of all Riemannian metrics g on M with sectional curvature bounded in absolute value by one. A sequence of metrics (g_i) realizes the minimal volume if their sectional curvatures are bounded in absolute value by one and if $\mathrm{vol}(M, g_i)$ converges to $\mathrm{Minvol}(M)$.

Under the assumption that the manifold M is geometrizable and prime, it follows with the same arguments as in [12] that the hyperbolic volume can be bounded from below by the minimal volume.

Question 4 *If M is geometrizable and prime, do the hyperbolic and the minimal volume coincide?*

This question has a positive answer if the manifold M is the double of a manifold which admits a convex{cocompact metric and the answer should be also positive without this restriction. If this is the case, then it should also be possible to show that the Gromov{Hausdor limit of every sequence of hyperbolic cone{ manifold structures which realizes the hyperbolic volume is isometric to M_{hyp}. We do not dare to ask if the assumption on M to be geometrizable can be dropped.

References

[1] M Boileau, J Porti, *Geometrization of 3-orbifolds of cyclic type*, Asterisque No. 272 (2001)

[2] F Bonahon, J{P Otal, *Laminations mesurees de plissage des varietes hyperboliques de dimension 3*, preprint (2001) http://math.usc.edu/~fbonahon/Research/Preprints/Preprints.html

[3] K Bromberg, *Rigidity of geometrically nite hyperbolic cone{manifolds*, preprint (2002) arXiv:math.GT/0009149

[4] D Canary, D Epstein, P Green *Notes on notes ofThurston* in *Analytical and geometric aspects of hyperbolic space*, London Math. Soc. Lecture Note Ser. 111, Cambridge Univ. Press, Cambridge (1987) 3{92

[5] C Hodgson, S Kerckho , *Rigidity of hyperbolic cone{manifolds and hyperbolic Dehn surgery*, J. Di . Geom. 48 (1998) 1{59

[6] K Matsuzaki, M Taniguchi, *Hyperbolic Manifolds and Kleinian Groups*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, (1998)

[7] S Kerckho , *The measure of the limit set of the handlebody group*, Topology 29 (1990) 27{40

[8] S Kojima, *Deformations of hyperbolic3{cone{manifolds*, J. Di . Geom. 49 (1998) 469{516

[9] H. Masur, *Measured foliations and handlebodies*, Ergodic Theory Dyn. Syst. 6, (1986) 99{116

[10] J{P Otal, *Courants geodesiques et produits libres*, These d'Etat, Universite Paris{Sud, Orsay (1988)

[11] J{P Otal, *Thurston's hyperbolization of Haken manifolds*, Surveys in di erential geometry, Vol. III (1998) 77{194

[12] J Souto, *Geometric structures on 3{manifolds and their deformations*, Ph.D.{ thesis Universität Bonn. Bonner Mathematischen Schriften nr. 342 (2001)

[13] W P Thurston, *The Geometry and Topology of 3{manifolds*, Lecture notes (1979)