ISSN 1464-8997 (on line) 1464-8989 (printed) **123**

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part I, section 15, pages 123–135

15. On the structure of the Milnor K-groups of complete discrete valuation fields

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15.0. Introduction

For a discrete valuation field K the unit group K^* of K has a natural decreasing filtration with respect to the valuation, and the graded quotients of this filtration are written in terms of the residue field. The Milnor K-group $K_q(K)$ is a generalization of the unit group and it also has a natural decreasing filtration defined in section 4. However, if K is of mixed characteristic and has absolute ramification index greater than one, the graded quotients of this filtration are known in some special cases only.

Let K be a complete discrete valuation field with residue field $k = k_K$; we keep the notations of section 4. Put $v_p = v_{\mathbb{Q}_p}$.

A description of $\mathrm{gr}_n K_q(K)$ is known in the following cases:

(i) (Bass and Tate [BT]) $\mathrm{gr}_0 K_q(K) \simeq K_q(k) \oplus K_{q-1}(k)$.

(ii) (Graham [G]) If the characteristic of K and k is zero, then $\mathrm{gr}_n K_q(K) \simeq \Omega_k^{q-1}$ for all $n \geqslant 1$.

(iii) (Bloch [B], Kato [Kt1]) If the characteristic of K and of k is $p > 0$ then

$$\mathrm{gr}_n K_q(K) \simeq \mathrm{coker}\left(\Omega_k^{q-2} \longrightarrow \Omega_k^{q-1}/B_s^{q-1} \oplus \Omega_k^{q-2}/B_s^{q-2}\right)$$

where $\omega \longmapsto (\mathrm{C}^{-s}(d\omega), (-1)^q m\, \mathrm{C}^{-s}(\omega)$ and where $n \geqslant 1$, $s = v_p(n)$ and $m = n/p^s$.

(iv) (Bloch–Kato [BK]) If K is of mixed characteristic $(0,p)$, then

$$\mathrm{gr}_n K_q(K) \simeq \mathrm{coker}\left(\Omega_k^{q-2} \longrightarrow \Omega_k^{q-1}/B_s^{q-1} \oplus \Omega_k^{q-2}/B_s^{q-2}\right)$$

where $\omega \longmapsto (\mathrm{C}^{-s}(d\omega), (-1)^q m\, \mathrm{C}^{-s}(\omega))$ and where $1 \leqslant n < ep/(p-1)$ for $e = v_K(p)$, $s = v_p(n)$ and $m = n/p^s$; and

$$\mathrm{gr}_{\frac{ep}{p-1}} K_q(K) \simeq \mathrm{coker}\left(\Omega_k^{q-2} \longrightarrow \Omega_k^{q-1}/(1 + a\,\mathrm{C})B_s^{q-1} \oplus \Omega_k^{q-2}/(1+a\,\mathrm{C})B_s^{q-2}\right)$$

where $\omega \longmapsto ((1 + a\,\mathrm{C})\,\mathrm{C}^{-s}(d\omega), (-1)^q m(1 + a\,\mathrm{C})\,\mathrm{C}^{-s}(\omega))$ and where a is the residue class of p/π^e for fixed prime element of K, $s = v_p(ep/(p-1))$ and $m = ep/(p-1)p^s$.

(v) (Kurihara [Ku1], see also section 13) If K is of mixed characteristic $(0, p)$ and absolutely unramified (i.e., $v_K(p) = 1$), then $\mathrm{gr}_n K_q(K) \simeq \Omega_k^{q-1}/B_{n-1}^{q-1}$ for $n \geqslant 1$.

(vi) (Nakamura [N2]) If K is of mixed characteristic $(0, p)$ with $p > 2$ and $p \nmid e = v_K(p)$, then

$$\mathrm{gr}_n K_q(K) \simeq \begin{cases} \text{as in (iv)} & (1 \leqslant n \leqslant ep/(p-1)) \\ \Omega_k^{q-1}/B_{l_n+s_n}^{q-1} & (n > ep/(p-1)) \end{cases}$$

where l_n is the maximal integer which satisfies $n - l_n e \geqslant e/(p-1)$ and $s_n = v_p(n - l_n e)$.

(vii) (Kurihara [Ku3]) If K_0 is the fraction field of the completion of the localization $\mathbb{Z}_p[T]_{(p)}$ and $K = K_0(\sqrt[p]{pT})$ for a prime $p \neq 2$, then

$$\mathrm{gr}_n K_2(K) \simeq \begin{cases} \text{as in (iv)} & (1 \leqslant n \leqslant p) \\ k/k^p & (n = 2p) \\ k^{p^{l-2}} & (n = lp, l \geqslant 3) \\ 0 & (\text{otherwise}). \end{cases}$$

(viii) (Nakamura [N1]) Let K_0 be an absolutely unramified complete discrete valuation field of mixed characteristic $(0, p)$ with $p > 2$. If $K = K_0(\zeta_p)(\sqrt[p]{\pi})$ where π is a prime element of $K_0(\zeta_p)$ such that $d\pi^{p-1} = 0$ in $\Omega^1_{\mathcal{O}_{K_0(\zeta_p)}}$, then $\mathrm{gr}_n K_q(K)$ are determined for all $n \geqslant 1$. This is complicated, so we omit the details.

(ix) (Kahn [Kh]) Quotients of the Milnor K-groups of a complete discrete valuation field K with perfect residue field are computed using symbols.

Recall that the group of units $U_{1,K}$ can be described as a topological $\mathbb{Z}_p$-module. As a generalization of this classical result, there is an appraoch different from (i)-(ix) for higher local fields K which uses topological convergence and

$$K_q^{\mathrm{top}}(K) = K_q(K)/\cap_{l \geqslant 1}\, lK_q(K)$$

(see section 6). It provides not only the description of $\mathrm{gr}_n K_q(K)$ but of the whole $K_q^{\mathrm{top}}(K)$ in characteristic p (Parshin [P]) and in characteristic 0 (Fesenko [F]). A complete description of the structure of $K_q^{\mathrm{top}}(K)$ of some higher local fields with small ramification is given by Zhukov [Z].

Below we discuss (vi).

15.1. Syntomic complex and Kurihara's exponential homomorphism

15.1.1. Syntomic complex. Let $A = \mathcal{O}_K$ and let A_0 be the subring of A such that A_0 is a complete discrete valuation ring with respect to the restriction of the valuation of K, the residue field of A_0 coincides with $k = k_K$ and A_0 is absolutely unramified. Let π be a fixed prime of K. Let $B = A_0[[X]]$. Define

$$\mathcal{J} = \ker[B \xrightarrow{X\mapsto\pi} A]$$
$$\mathcal{I} = \ker[B \xrightarrow{X\mapsto\pi} A \xrightarrow{\text{mod } p} A/p] = \mathcal{J} + pB.$$

Let D and $J \subset D$ be the PD-envelope and the PD-ideal with respect to $B \to A$, respectively. Let $I \subset D$ be the PD-ideal with respect to $B \to A/p$. Namely,

$$D = B\left[\frac{x^j}{j!}\ ;\ j \geqslant 0, x \in \mathcal{J}\right], \quad J = \ker(D \to A), \quad I = \ker(D \to A/p).$$

Let $J^{[r]}$ (resp. $I^{[r]}$) be the r-th divided power, which is the ideal of D generated by

$$\left\{\frac{x^j}{j!}\ ;\ j \geqslant r, x \in \mathcal{J}\right\}, \ \left(\text{resp. } \left\{\frac{x^i}{i!}\frac{p^j}{j!}\ ;\ i + j \geqslant r, x \in \mathcal{I}\right\}\right).$$

Notice that $I^{[0]} = J^{[0]} = D$. Let $I^{[n]} = J^{[n]} = D$ for a negative n. We define the complexes $\mathbb{J}^{[q]}$ and $\mathbb{I}^{[q]}$ as

$$\mathbb{J}^{[q]} = [J^{[q]} \xrightarrow{d} J^{[q-1]} \otimes_B \widehat{\Omega}^1_B \xrightarrow{d} J^{[q-2]} \otimes_B \widehat{\Omega}^2_B \longrightarrow \cdots]$$
$$\mathbb{I}^{[q]} = [I^{[q]} \xrightarrow{d} I^{[q-1]} \otimes_B \widehat{\Omega}^1_B \xrightarrow{d} I^{[q-2]} \otimes_B \widehat{\Omega}^2_B \longrightarrow \cdots]$$

where $\widehat{\Omega}^q_B$ is the p-adic completion of Ω^q_B. We define $\mathbb{D} = \mathbb{I}^{[0]} = \mathbb{J}^{[0]}$.

Let $\mathbb{T}$ be a fixed set of elements of A_0^* such that the residue classes of all $T \in \mathbb{T}$ in k forms a p-base of k. Let f be the Frobenius endomorphism of A_0 such that $f(T) = T^p$ for any $T \in \mathbb{T}$ and $f(x) \equiv x^p \mod p$ for any $x \in A_0$. We extend f to B by $f(X) = X^p$, and to D naturally. For $0 \leqslant r < p$ and $0 \leqslant s$, we get

$$f(J^{[r]}) \subset p^r D, \quad f(\widehat{\Omega}^s_B) \subset p^s \widehat{\Omega}^s_B,$$

since

$$f(x^{[r]}) = (x^p + py)^{[r]} = (p!x^{[p]} + py)^{[r]} = p^{[r]}((p-1)!x^{[p]} + y)^r,$$
$$f\big(z\frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_s}{T_s}\big) = z\frac{dT_1^p}{T_1^p} \wedge \cdots \wedge \frac{dT_s^p}{T_s^p} = zp^s\frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_s}{T_s},$$

where $x \in \mathcal{J}$, y is an element which satisfies $f(x) = x^p + py$, and $T_1, \ldots, T_s \in \mathbb{T} \cup \{X\}$. Thus we can define

$$f_q = \frac{f}{p^q} \colon J^{[r]} \otimes \widehat{\Omega}^{q-r}_B \longrightarrow D \otimes \widehat{\Omega}^{q-r}_B$$

for $0 \leqslant r < p$. Let $\mathscr{S}(q)$ and $\mathscr{S}'(q)$ be the mapping fiber complexes (cf. Appendix) of

$$\mathbb{J}^{[q]} \xrightarrow{1-f_q} \mathbb{D} \quad \text{and} \quad \mathbb{I}^{[q]} \xrightarrow{1-f_q} \mathbb{D}$$

respectively, for $q < p$. For simplicity, from now to the end, we assume p is large enough to treat $\mathscr{S}(q)$ and $\mathscr{S}'(q)$. $\mathscr{S}(q)$ is called the *syntomic complex* of A with respect to B, and $\mathscr{S}'(q)$ is also called the *syntomic complex* of A/p with respect to B (cf. [Kt2]).

Theorem 1 (Kurihara [Ku2]). *There exists a subgroup* S^q *of* $H^q(\mathscr{S}(q))$ *such that* $U_X H^q(\mathscr{S}(q)) \simeq U_1\widehat{K_q}(A)$ *where* $\widehat{K_q}(A) = \varprojlim K_q(A)/p^n$ *is the* p*-adic completion of* $K_q(A)$ (*see subsection 9.1*).

Outline of the proof. Let $U_X(D \otimes \widehat{\Omega}_B^{q-1})$ be the subgroup of $D \otimes \widehat{\Omega}_B^{q-1}$ generated by $XD \otimes \widehat{\Omega}_B^{q-1}$, $D \otimes \widehat{\Omega}_B^{q-2} \wedge dX$ and $I \otimes \widehat{\Omega}_B^{q-1}$, and let

$$S^q = U_X(D \otimes \widehat{\Omega}_B^{q-1})/((dD \otimes \widehat{\Omega}_B^{q-2} + (1-f_q)J \otimes \widehat{\Omega}_B^{q-1}) \cap U_X(D \otimes \widehat{\Omega}_B^{q-1})).$$

The infinite sum $\sum_{n\geqslant 0} f_q^n(dx)$ converges in $D \otimes \widehat{\Omega}_B^q$ for $x \in U_X(D \otimes \widehat{\Omega}_B^{q-1})$. Thus we get a map

$$\begin{aligned} U_X(D \otimes \widehat{\Omega}_B^{q-1}) &\longrightarrow H^q(\mathscr{S}(q)) \\ x &\longmapsto \bigl(x, \sum_{n=0}^{\infty} f_q^n(dx)\bigr) \end{aligned}$$

and we may assume S^q is a subgroup of $H^q(\mathscr{S}(q))$. Let E_q be the map

$$\begin{aligned} E_q \colon U_X(D \otimes \widehat{\Omega}_B^{q-1}) &\longrightarrow \widehat{K}_q(A) \\ x\frac{dT_1}{T_1} \wedge \dots \wedge \frac{dT_{q-1}}{T_{q-1}} &\longmapsto \{E_1(x), T_1, \dots, T_{q-1}\}, \end{aligned}$$

where $E_1(x) = \exp\circ(\sum_{n\geqslant 0} f_1^n)(x)$ is *Artin–Hasse's exponential homomorphism*. In [Ku2] it was shown that E_q vanishes on

$$(dD \otimes \widehat{\Omega}_B^{q-2} + (1-f_q)J \otimes \widehat{\Omega}_B^{q-1}) \cap U_X(D \otimes \widehat{\Omega}_B^{q-1}),$$

hence we get the map

$$E_q \colon S^q \longrightarrow \widehat{K}_q(A).$$

The image of E_q coincides with $U_1\widehat{K}_q(A)$ by definition.

On the other hand, define $s_q \colon \widehat{K}_q(A) \longrightarrow S^q$ by

$$\begin{aligned} &s_q(\{a_1, \dots, a_q\}) \\ &= \sum_{i=1}^{q} (-1)^{i-1} \frac{1}{p} \log\Bigl(\frac{f(\widetilde{a}_i)}{\widetilde{a}_i^{\,p}}\Bigr) \frac{d\widetilde{a}_1}{\widetilde{a}_1} \wedge \dots \wedge \frac{d\widetilde{a_{i-1}}}{\widetilde{a_{i-1}}} \wedge f_1\Bigl(\frac{d\widetilde{a_{i+1}}}{\widetilde{a_{i+1}}}\Bigr) \wedge \dots \wedge f_1\Bigl(\frac{d\widetilde{a}_q}{\widetilde{a}_q}\Bigr) \end{aligned}$$

(cf. [Kt2], compare with the series Φ in subsection 8.3), where $\widetilde{a}$ is a lifting of a to D. One can check that $s_q \circ E_q = -\,\mathrm{id}$. Hence $S^q \simeq U_1\widehat{K}_q(A)$. Note that if $\zeta_p \in K$, then one can show $U_1\widehat{K}_q(A) \simeq U_1\widehat{K}_q(K)$ (see [Ku4] or [N2]), thus we have $S^q \simeq U_1\widehat{K}_q(K)$. □

Example. We shall prove the equality $s_q \circ E_q = -\,\mathrm{id}$ in the following simple case. Let $q = 2$. Take an element $adT/T \in U_X(D \otimes \widehat{\Omega}_B^{q-1})$ for $T \in \mathbb{T} \cup \{X\}$. Then

$$\begin{aligned}
&s_q \circ E_q\big(a\frac{dT}{T}\big)\\
&= s_q(\{E_1(\widetilde{a}), T\})\\
&= \frac{1}{p}\log\left(\frac{f(E_1(a))}{E_1(a)^p}\right) f_1\left(\frac{dT}{T}\right)\\
&= \frac{1}{p}\left(\log\circ f\circ\exp\circ\sum_{n\geqslant 0} f_1^n(a) - p\log\circ\exp\circ\sum_{n\geqslant 0} f_1^n(a)\right)\frac{dT}{T}\\
&= \left(f_1\sum_{n\geqslant 0} f_1^n(a) - \sum_{n\geqslant 0} f_1^n(a)\right)\frac{dT}{T}\\
&= -a\frac{dT}{T}.
\end{aligned}$$

15.1.2. Exponential Homomorphism. The usual exponential homomorphism

$$\begin{aligned}
\exp_\eta\colon A &\longrightarrow A^*\\
x &\longmapsto \exp(\eta x) = \sum_{n\geqslant 0}\frac{x^n}{n!}
\end{aligned}$$

is defined for $\eta \in A$ such that $v_A(\eta) > e/(p-1)$. This map is injective. Section 9 contains a definition of the map

$$\begin{aligned}
\exp_\eta\colon \widehat{\Omega}_A^{q-1} &\longrightarrow \widehat{K}_q(A)\\
x\frac{dy_1}{y_1}\wedge\dots\wedge\frac{dy_{q-1}}{y_{q-1}} &\longmapsto \{\exp(\eta x), y_1, \dots, y_{q-1}\}
\end{aligned}$$

for $\eta \in A$ such that $v_A(\eta) \geqslant 2e/(p-1)$. This map is not injective in general. Here is a description of the kernel of $\exp_\eta$.

Theorem 2. *The following sequence is exact:*

$$(*)\qquad H^{q-1}(\mathscr{S}'(q)) \xrightarrow{\psi} \Omega_A^{q-1}/pd\widehat{\Omega}_A^{q-2} \xrightarrow{\exp_p} \widehat{K}_q(A).$$

Sketch of the proof. There is an exact sequence of complexes

$$0 \to \mathrm{MF}\begin{pmatrix} \mathbb{J}^{[q]} \\ {\scriptstyle 1-f_q}\downarrow \\ \mathbb{D} \end{pmatrix} \to \mathrm{MF}\begin{pmatrix} \mathbb{I}^{[q]} \\ {\scriptstyle 1-f_q}\downarrow \\ \mathbb{D} \end{pmatrix} \to \mathbb{I}^{[q]}/\mathbb{J}^{[q]} \to 0,$$

$$\begin{matrix} \| & & \| \\ \mathscr{S}(q) & & \mathscr{S}'(q) \end{matrix}$$

where MF means the mapping fiber complex. Thus, taking cohomologies we have the following diagram with the exact top row

$$\begin{array}{ccccc} H^{q-1}(\mathscr{S}'(q)) & \xrightarrow{\psi} & H^{q-1}(\mathbb{I}^{[q]}/\mathbb{J}^{[q]}) & \xrightarrow{\delta} & H^q(\mathscr{S}(q)) \\ & & {\scriptstyle (1)}\uparrow & & {\scriptstyle \mathrm{Thm.1}}\uparrow \\ & & \widehat{\Omega}_A^{q-1}/pd\widehat{\Omega}_A^{q-2} & \xrightarrow{\exp_p} & U_1\widehat{K}_q(A), \end{array}$$

where the map (1) is induced by

$$\widehat{\Omega}_A^{q-1} \ni \omega \longmapsto p\widetilde{\omega} \in I \otimes \widehat{\Omega}_B^{q-1}/J \otimes \widehat{\Omega}_B^{q-1} = (\mathbb{I}^{[q]}/\mathbb{J}^{[q]})^{q-1}.$$

We denoted the left horizontal arrow of the top row by ψ and the right horizontal arrow of the top row by δ. The right vertical arrow is injective, thus the claims are

(1) is an isomorphism,

(2) this diagram is commutative.

First we shall show (1). Recall that

$$H^{q-1}(\mathbb{I}^{[q]}/\mathbb{J}^{[q]}) = \mathrm{coker}\left(\frac{I^{[2]} \otimes \widehat{\Omega}_B^{q-2}}{J^{[2]} \otimes \widehat{\Omega}_B^{q-2}} \longrightarrow \frac{I \otimes \widehat{\Omega}_B^{q-2}}{J \otimes \widehat{\Omega}_B^{q-2}}\right).$$

From the exact sequence

$$0 \longrightarrow J \longrightarrow D \longrightarrow A \longrightarrow 0,$$

we get $D \otimes \widehat{\Omega}_B^{q-1}/J \otimes \widehat{\Omega}_B^{q-1} = A \otimes \widehat{\Omega}_B^{q-1}$ and its subgroup $I \otimes \widehat{\Omega}_B^{q-2}/J \otimes \widehat{\Omega}_B^{q-2}$ is $pA \otimes \widehat{\Omega}_B^{q-1}$ in $A \otimes \widehat{\Omega}_B^{q-1}$. The image of $I^{[2]} \otimes \widehat{\Omega}_B^{q-2}$ in $pA \otimes \widehat{\Omega}_B^{q-1}$ is equal to the image of

$$\mathfrak{I}^2 \otimes \widehat{\Omega}_B^{q-2} = \mathfrak{J}^2 \otimes \widehat{\Omega}_B^{q-2} + p\mathfrak{J}\widehat{\Omega}_B^{q-2} + p^2\widehat{\Omega}_B^{q-2}.$$

On the other hand, from the exact sequence

$$0 \longrightarrow \mathfrak{J} \longrightarrow B \longrightarrow A \longrightarrow 0,$$

we get an exact sequence

$$(\mathfrak{J}/\mathfrak{J}^2) \otimes \widehat{\Omega}_B^{q-2} \xrightarrow{d} A \otimes \widehat{\Omega}_B^{q-1} \longrightarrow \widehat{\Omega}_A^{q-1} \longrightarrow 0.$$

Thus $d\mathcal{J}^2 \otimes \widehat{\Omega}_B^{q-2}$ vanishes on $pA \otimes \widehat{\Omega}_B^{q-1}$, hence

$$H^{q-1}(\mathbb{I}^{[q]}/\mathbb{J}^{[q]}) = \frac{pA \otimes \widehat{\Omega}_B^{q-1}}{pd\mathcal{J}\widehat{\Omega}_B^{q-2} + p^2 d\widehat{\Omega}_B^{q-2}} \overset{p^{-1}}{\simeq} \frac{A \otimes \widehat{\Omega}_B^{q-1}}{d\mathcal{J}\widehat{\Omega}_B^{q-2} + pd\widehat{\Omega}_B^{q-2}} \simeq \widehat{\Omega}_A^{q-1}/pd\widehat{\Omega}_A^{q-2},$$

which completes the proof of (1).

Next, we shall demonstrate the commutativity of the diagram on a simple example. Consider the case where $q = 2$ and take $adT/T \in \widehat{\Omega}_A^1$ for $T \in \mathbb{T} \cup \{\pi\}$. We want to show that the composite of

$$\widehat{\Omega}_A^1/pdA \xrightarrow{(1)} H^1(\mathbb{I}^{[2]}/\mathbb{J}^{[2]}) \xrightarrow{\delta} S^q \xrightarrow{E_q} U_1\widehat{K}_2(A)$$

coincides with $\exp_p$. By (1), the lifting of adT/T in $(\mathbb{I}^{[2]}/\mathbb{J}^{[2]})^1 = I \otimes \widehat{\Omega}_B^1/J \otimes \widehat{\Omega}_B^1$ is $p\widetilde{a} \otimes dT/T$, where $\widetilde{a}$ is a lifting of a to D. Chasing the connecting homomorphism δ,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (J\otimes\widehat{\Omega}_B^1)\oplus D & \longrightarrow & (I\otimes\widehat{\Omega}_B^1)\oplus D & \longrightarrow & (I\otimes\widehat{\Omega}_B^1)/(J\otimes\widehat{\Omega}_B^1) & \longrightarrow & 0 \\
 & & \Big\downarrow d & & \Big\downarrow d & & \Big\downarrow d & & \\
0 & \longrightarrow & (D\otimes\widehat{\Omega}_B^2)\oplus(D\otimes\widehat{\Omega}_B^1) & \longrightarrow & (D\otimes\widehat{\Omega}_B^2)\oplus(D\otimes\widehat{\Omega}_B^1) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \Big\downarrow d & & \Big\downarrow d & & \Big\downarrow d & &
\end{array}$$

(the left column is $\mathscr{S}(2)$, the middle is $\mathscr{S}'(2)$ and the right is $\mathbb{I}^{[2]}/\mathbb{J}^{[2]}$); $p\widetilde{a}dT/T$ in the upper right goes to $(pd\widetilde{a} \wedge dT/T, (1-f_2)(p\widetilde{a} \otimes dT/T))$ in the lower left. By E_2, this element goes

$$\begin{aligned}
&E_2\big((1-f_2)\big(p\widetilde{a} \otimes \frac{d}{T}\big)\big) = E_2\big((1-f_1)(p\widetilde{a}) \otimes \frac{dT}{T}\big) \\
&= \{E_1((1-f_1)(p\widetilde{a})), T\} = \big\{\exp \circ \big(\sum_{n\geqslant 0} f_1^n\big) \circ (1-f_1)(p\widetilde{a}), T\big\} \\
&= \{\exp(pa), T\}.
\end{aligned}$$

in $U_1\widehat{K}_2(A)$. This is none other than the map $\exp_p$. □

By Theorem 2 we can calculate the kernel of $\exp_p$. On the other hand, even though $\exp_p$ is not surjective, the image of $\exp_p$ includes $U_{e+1}\widehat{K}_q(A)$ and we already know $\mathrm{gr}_i\widehat{K}_q(K)$ for $0 \leqslant i \leqslant ep/(p-1)$. Thus it is enough to calculate the kernel of $\exp_p$ in order to know all $\mathrm{gr}_i\widehat{K}_q(K)$. Note that to know $\mathrm{gr}_i\widehat{K}_q(K)$, we may assume that $\zeta_p \in K$, and hence $\widehat{K}_q(A) = U_0\widehat{K}_q(K)$.

15.2. Computation of the kernel of the exponential homomorphism

15.2.1. Modified syntomic complex. We introduce a modification of $\mathscr{S}'(q)$ and calculate it instead of $\mathscr{S}'(q)$. Let $\mathbb{S}_q$ be the mapping fiber complex of

$$1 - f_q\colon (\mathbb{J}^{[q]})^{\geqslant q-2} \longrightarrow \mathbb{D}^{\geqslant q-2}.$$

Here, for a complex $C^\cdot$, we put

$$C^{\geqslant n} = (0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow C^n \longrightarrow C^{n+1} \longrightarrow \cdots).$$

By definition, we have a natural surjection $H^{q-1}(\mathbb{S}_q) \to H^{q-1}(\mathscr{S}'(q))$, hence $\psi(H^{q-1}(\mathbb{S}_q)) = \psi(H^{q-1}(\mathscr{S}'(q)))$, which is the kernel of $\exp_p$.

To calculate $H^{q-1}(\mathbb{S}_q)$, we introduce an X-filtration. Let $0 \leqslant r \leqslant 2$ and $s = q-r$. Recall that $B = A_0[[X]]$. For $i \geqslant 0$, let $\mathrm{fil}_i(I^{[r]}\otimes_B\widehat{\boldsymbol{\Omega}}^s_B)$ be the subgroup of $I^{[r]}\otimes_B\widehat{\boldsymbol{\Omega}}^s_B$ generated by the elements

$$\left\{ X^n \frac{(X^e)^j}{j!}\frac{p^l}{l!} a\omega : n+ej \geqslant i, n \geqslant 0, j+l \geqslant r, a \in D, \omega \in \widehat{\boldsymbol{\Omega}}^s_B \right\}$$
$$\cup \left\{ X^n \frac{(X^e)^j}{j!}\frac{p^l}{l!} av \wedge \frac{dX}{X} : n+ej \geqslant i, n \geqslant 1, j+l \geqslant r, a \in D, v \in \widehat{\boldsymbol{\Omega}}^{s-1}_B \right\}.$$

The map $1 - f_q\colon I^{[r]} \otimes \widehat{\boldsymbol{\Omega}}^s_B \to D \otimes \widehat{\boldsymbol{\Omega}}^s_B$ preserves the filtrations. By using the latter we get the following

Proposition 3. *$H^{q-1}(\mathrm{fil}_i\mathbb{S}_q)_i$ form a finite decreasing filtration of $H^{q-1}(\mathbb{S}_q)$. Denote*

$$\mathrm{fil}_i H^{q-1}(\mathbb{S}_q) = H^{q-1}(\mathrm{fil}_i\mathbb{S}_q),$$
$$\mathrm{gr}_i H^{q-1}(\mathbb{S}_q) = \mathrm{fil}_i H^{q-1}(\mathbb{S}_q)/\mathrm{fil}_{i+1} H^{q-1}(\mathbb{S}_q).$$

Then $\mathrm{gr}_i H^{q-1}(\mathbb{S}_q)$

$$= \begin{cases} 0 & (\textit{ if } i > 2e) \\ X^{2e-1}dX \wedge \left(\widehat{\Omega}^{q-3}_{A_0}/p\right) & (\textit{ if } i = 2e) \\ X^i\left(\widehat{\Omega}^{q-2}_{A_0}/p\right) \oplus X^{i-1}dX \wedge (\widehat{\Omega}^{q-3}_{A_0}/p) & (\textit{ if } e < i < 2e) \\ X^e\left(\widehat{\Omega}^{q-2}_{A_0}/p\right) \oplus X^{e-1}dX \wedge \left(\mathfrak{Z}_1\widehat{\Omega}^{q-3}_{A_0}/p^2\widehat{\Omega}^{q-3}_{A_0}\right) & (\textit{ if } i = e, p \mid e) \\ X^{e-1}dX \wedge \left(\mathfrak{Z}_1\widehat{\Omega}^{q-3}_{A_0}/p^2\widehat{\Omega}^{q-3}_{A_0}\right) & (\textit{ if } i = e, p \nmid e) \\ \left(X^i \frac{\left(p^{\max(\eta'_i - v_p(i),0)}\widehat{\Omega}^{q-2}_{A_0} \cap \mathfrak{Z}_{\eta_i}\widehat{\Omega}^{q-2}_{A_0}\right) + p^2\widehat{\Omega}^{q-2}_{A_0}}{p^2\widehat{\Omega}^{q-2}_{A_0}}\right) & \\ \qquad \oplus \left(X^{i-1}dX \wedge \frac{\mathfrak{Z}_{\eta_i}\widehat{\Omega}^{q-3}_{A_0} + p^2\widehat{\Omega}^{q-3}_{A_0}}{p^2\widehat{\Omega}^{q-3}_{A_0}}\right) & (\textit{ if } 1 \leqslant i < e) \\ 0 & (\textit{ if } i = 0). \end{cases}$$

Here η_i *and* η'_i *are the integers which satisfy* $p^{\eta_i - 1}i < e \leqslant p^{\eta_i}i$ *and* $p^{\eta'_i - 1}i - 1 < e \leqslant p^{\eta'_i}i - 1$ *for each* i*,*

$$\mathfrak{Z}_n\widehat{\Omega}^q_{A_0} = \ker\left(\widehat{\Omega}^q_{A_0} \xrightarrow{d} \widehat{\Omega}^{q+1}_{A_0}/p^n\right)$$

for positive n*, and* $\mathfrak{Z}_n\widehat{\Omega}^q_{A_0} = \widehat{\Omega}^q_{A_0}$ *for* $n \leqslant 0$.

Outline of the proof. From the definition of the filtration we have the exact sequence of complexes:

$$0 \longrightarrow \mathrm{fil}_{i+1}\mathbb{S}_q \longrightarrow \mathrm{fil}_i\mathbb{S}_q \longrightarrow \mathrm{gr}_i\mathbb{S}_q \longrightarrow 0$$

and this sequence induce a long exact sequence

$$\cdots \to H^{q-2}(\mathrm{gr}_i\mathbb{S}_q) \to H^{q-1}(\mathrm{fil}_{i+1}\mathbb{S}_q) \to H^{q-1}(\mathrm{fil}_i\mathbb{S}_q) \to H^{q-1}(\mathrm{gr}_i\mathbb{S}_q) \to \cdots .$$

The group $H^{q-2}(\mathrm{gr}_i\mathbb{S}_q)$ is

$$H^{q-2}(\mathrm{gr}_i\mathbb{S}_q) = \ker\begin{pmatrix} \mathrm{gr}_i I^{[2]} \otimes \widehat{\Omega}^{q-2}_B \longrightarrow (\mathrm{gr}_i I \otimes \widehat{\Omega}^{q-1}_B) \oplus (\mathrm{gr}_i D \otimes \widehat{\Omega}^{q-2}_B) \\ x \longmapsto (dx, (1-f_q)x) \end{pmatrix}.$$

The map $1 - f_q$ is equal to 1 if $i \geqslant 1$ and $1 - f_q\colon p^2\widehat{\Omega}^{q-2}_{A_0} \to \widehat{\Omega}^{q-2}_{A_0}$ if $i = 0$, thus they are all injective. Hence $H^{q-2}(\mathrm{gr}_i\mathbb{S}_q) = 0$ for all i and we deduce that $H^{q-1}(\mathrm{fil}_i\mathbb{S}_q)_i$ form a decreasing filtration on $H^{q-1}(\mathbb{S}_q)$.

Next, we have to calculate $H^{q-2}(\mathrm{gr}_i\mathbb{S}_q)$. The calculation is easy but there are many cases which depend on i, so we omit them. For more detail, see [N2].

Finally, we have to compute the image of the last arrow of the exact sequence

$$0 \longrightarrow H^{q-1}(\mathrm{fil}_{i+1}\mathbb{S}_q) \longrightarrow H^{q-1}(\mathrm{fil}_i\mathbb{S}_q) \longrightarrow H^{q-1}(\mathrm{gr}_i\mathbb{S}_q)$$

because it is not surjective in general. Write down the complex $\mathrm{gr}_i\mathbb{S}_q$:

$$\cdots \to (\mathrm{gr}_i I \otimes \widehat{\Omega}_B^{q-1}) \oplus (\mathrm{gr}_i D \otimes \widehat{\Omega}_B^{q-2}) \xrightarrow{d} (\mathrm{gr}_i D \otimes \widehat{\Omega}_B^{q}) \oplus (\mathrm{gr}_i D \otimes \widehat{\Omega}_B^{q-1}) \to \cdots,$$

where the first term is the degree $q-1$ part and the second term is the degree q part. An element (x,y) in the first term which is mapped to zero by d comes from $H^{q-1}(\mathrm{fil}_i\mathbb{S}_q)$ if and only if there exists $z \in \mathrm{fil}_i D \otimes \widehat{\Omega}_B^{q-2}$ such that $z \equiv y$ modulo $\mathrm{fil}_{i+1} D \otimes \widehat{\Omega}_B^{q-2}$ and

$$\sum_{n \geqslant 0} f_q^n(dz) \in \mathrm{fil}_i I \otimes \widehat{\Omega}_B^{q-1}.$$

From here one deduces Proposition 3. □

15.2.2. Differential modules. Take a prime element π of K such that $\pi^{e-1}d\pi = 0$. We assume that $p \nmid e$ in this subsection. Then we have

$$\widehat{\Omega}_A^q \simeq \Bigg(\bigoplus_{i_1<i_2<\cdots<i_q} A\frac{dT_{i_1}}{T_{i_1}} \wedge \cdots \wedge \frac{dT_{i_q}}{T_{i_q}} \Bigg) \oplus \Bigg(\bigoplus_{i_1<i_2<\cdots<i_{q-1}} A/(\pi^{e-1})\frac{dT_{i_1}}{T_{i_1}} \wedge \cdots \wedge \frac{dT_{i_{q-1}}}{T_{i_{q-1}}} \wedge d\pi \Bigg),$$

where $\{T_i\} = \mathbb{T}$. We introduce a filtration on $\widehat{\Omega}_A^q$ as

$$\mathrm{fil}_i\widehat{\Omega}_A^q = \begin{cases} \widehat{\Omega}_A^q & (\text{ if } i = 0) \\ \pi^i\widehat{\Omega}_A^q + \pi^{i-1}d\pi \wedge \widehat{\Omega}_A^{q-1} & (\text{ if } i \geqslant 1). \end{cases}$$

The subquotients are

$$\begin{aligned} \mathrm{gr}_i\widehat{\Omega}_A^q &= \mathrm{fil}_i\widehat{\Omega}_A^q/\mathrm{fil}_{i+1}\widehat{\Omega}_A^q \\ &= \begin{cases} \Omega_F^q & (\text{ if } i = 0 \text{ or } i \geqslant e) \\ \Omega_F^q \oplus \Omega_F^{q-1} & (\text{ if } 1 \leqslant i < e), \end{cases} \end{aligned}$$

where the map is

$$\begin{aligned} &\Omega_F^q \ni \omega \longmapsto \pi^i\widetilde{\omega} \in \pi^i\widehat{\Omega}_A^q \\ &\Omega_F^{q-1} \ni \omega \longmapsto \pi^{i-1}d\pi \wedge \widetilde{\omega} \in \pi^{i-1}d\pi \wedge \widehat{\Omega}_A^{q-1}. \end{aligned}$$

Here $\widetilde{\omega}$ is the lifting of ω. Let $\mathrm{fil}_i(\widehat{\Omega}_A^q/pd\widehat{\Omega}_A^{q-1})$ be the image of $\mathrm{fil}_i\widehat{\Omega}_A^q$ in $\widehat{\Omega}_A^q/pd\widehat{\Omega}_A^{q-1}$. Then we have the following:

Proposition 4. *For $j \geqslant 0$,*

$$\mathrm{gr}_j\left(\widehat{\Omega}^q_A/pd\widehat{\Omega}^{q-1}_A\right) = \begin{cases} \Omega^q_F & (j=0) \\ \Omega^q_F \oplus \Omega^{q-1}_F & (1 \leqslant j < e) \\ \Omega^q_F/B^q_l & (e \leqslant j), \end{cases}$$

where l be the maximal integer which satisfies $j - le \geqslant 0$.

Proof. If $1 \leqslant j < e$, $\mathrm{gr}_j\widehat{\Omega}^q_A = \mathrm{gr}_j(\widehat{\Omega}^q_A/pd\widehat{\Omega}^{q-1}_A)$ because $pd\widehat{\Omega}^{q-1}_A \subset \mathrm{fil}_e\widehat{\Omega}^q_A$. Assume that $j \geqslant e$ and let l be as above. Since $\pi^{e-1}d\pi = 0$, $\widehat{\Omega}^{q-1}_A$ is generated by elements $p\pi^i d\omega$ for $0 \leqslant i < e$ and $\omega \in \widehat{\Omega}^{q-1}_{A_0}$. By [I] (Cor. 2.3.14), $p\pi^i d\omega \in \mathrm{fil}_{e(1+n)+i}\widehat{\Omega}^q_A$ if and only if the residue class of $p^{-n}d\omega$ belongs to B_{n+1}. Thus $\mathrm{gr}_j(\widehat{\Omega}^q_A/pd\widehat{\Omega}^{q-1}_A) \simeq \Omega^q_F/B^q_l$. □

By definition of the filtrations, $\exp_p$ preserves the filtrations on $\widehat{\Omega}^{q-1}_A/pd\widehat{\Omega}^{q-2}_A$ and $\widehat{K}_q(K)$. Furthermore, $\exp_p\colon \mathrm{gr}_i(\widehat{\Omega}^{q-1}_A/pd\widehat{\Omega}^{q-2}_A) \to \mathrm{gr}_{i+e}K_q(K)$ is surjective and its kernel is the image of $\psi(H^{q-1}(\mathbb{S}_q)) \cap \mathrm{fil}_i(\widehat{\Omega}^{q-1}_A/pd\widehat{\Omega}^{q-2}_A)$ in $\mathrm{gr}_i(\widehat{\Omega}^{q-1}_A/pd\widehat{\Omega}^{q-2}_A)$. Now we know both $\widehat{\Omega}^{q-1}_A/pd\widehat{\Omega}^{q-2}_A$ and $H^{q-1}(\mathbb{S}_q)$ explicitly, thus we shall get the structure of $K_q(K)$ by calculating ψ. But ψ does not preserve the filtration of $H^{q-1}(\mathbb{S}_q)$, so it is not easy to compute it. For more details, see [N2], especially sections 4-8 of that paper. After completing these calculations, we get the result in (vi) in the introduction.

Remark. Note that if $p \mid e$, the structure of $\widehat{\Omega}^{q-1}_A/pd\widehat{\Omega}^{q-2}_A$ is much more complicated. For example, if $e = p(p-1)$, and if $\pi^e = p$, then $p\pi^{e-1}d\pi = 0$. This means the torsion part of $\widehat{\Omega}^{q-1}_A$ is larger than in the the case where $p \nmid e$. Furthermore, if $\pi^{p(p-1)} = pT$ for some $T \in \mathbb{T}$, then $p\pi^{e-1}d\pi = pdT$, this means that $d\pi$ is not a torsion element. This complexity makes it difficult to describe the structure of $K_q(K)$ in the case where $p \mid e$.

Appendix. The mapping fiber complex.

This subsection is only a note on homological algebra to introduce the mapping fiber complex. The mapping fiber complex is the degree -1 shift of the mapping cone complex.

Let $C^{\cdot} \xrightarrow{f} D^{\cdot}$ be a morphism of non-negative cochain complexes. We denote the degree i term of $C^{\cdot}$ by C^i.

Then the mapping fiber complex $\mathrm{MF}(f)^{\cdot}$ is defined as follows.

$$\mathrm{MF}(f)^i = C^i \oplus D^{i-1},$$

$$\text{differential} \quad d\colon C^i \oplus D^{i-1} \longrightarrow C^{i+1} \oplus D^i$$

$$(x,y) \longmapsto (dx, f(x) - dy).$$

By definition, we get an exact sequence of complexes:

$$0 \longrightarrow D[-1]^{\cdot} \longrightarrow \mathrm{MF}(f)^{\cdot} \longrightarrow C^{\cdot} \longrightarrow 0,$$

where $D[-1]^{\cdot} = (0 \to D^0 \to D^1 \to \cdots)$ (degree -1 shift of $D^{\cdot}$.)

Taking cohomology, we get a long exact sequence

$$\cdots \to H^i(\mathrm{MF}(f)^{\cdot}) \to H^i(C^{\cdot}) \to H^{i+1}(D^{\cdot}[-1]) \to H^{i+1}(\mathrm{MF}(f)^{\cdot}) \to \cdots ,$$

which is the same as the following exact sequence

$$\cdots \to H^i(\mathrm{MF}(f)^{\cdot}) \to H^i(C^{\cdot}) \xrightarrow{f} H^i(D^{\cdot}) \to H^{i+1}(\mathrm{MF}(f)^{\cdot}) \to \cdots .$$

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