ISSN 1464-8997 (on line) 1464-8989 (printed)

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part I, section 8, pages 81–89

8. Explicit formulas for the Hilbert symbol

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Recall that the Hilbert symbol for a local field K with finite residue field which contains a primitive p^n th root of unity ζ_{p^n} is a pairing

$$(\ , \ )_{p^n}\colon K^*/K^{*\,p^n} \times K^*/K^{*\,p^n} \to \langle \zeta_{p^n} \rangle, \quad (\alpha,\beta)_{p^n} = \gamma^{\Psi_K(\alpha)-1}, \quad \gamma^{p^n} = \beta,$$

where $\Psi_K\colon K^* \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is the reciprocity map.

8.1. History of explicit formulas for the Hilbert symbol

There are two different branches of explicit reciprocity formulas (for the Hilbert symbol).

8.1.1. The first branch (Kummer's type formulas).

Theorem (E. Kummer 1858)**.** *Let* $K = \mathbb{Q}_p(\zeta_p)$, $p \neq 2$. *Then for principal units* ε,η

$$\boxed{(\varepsilon,\eta)_p = \zeta_p^{\,\mathrm{res}(\log\,\eta(X)\,d\log\,\varepsilon(X)X^{-p})}}$$

where $\varepsilon(X)|_{X=\zeta_p-1} = \varepsilon$, $\eta(X)|_{X=\zeta_p-1} = \eta$, $\varepsilon(X),\eta(X) \in \mathbb{Z}_p[[X]]^*$.

The important point is that one associates to the elements ε,η the series $\varepsilon(X),\eta(X)$ in order to calculate the value of the Hilbert symbol.

Theorem (I. Shafarevich 1950)**.** *Complete explicit formula for the Hilbert norm residue symbol* $(\alpha,\beta)_{p^n}$, $\alpha,\beta \in K^*$, $K \supset \mathbb{Q}_p(\zeta_{p^n})$, $p \neq 2$, *using a special basis of the group of principal units.*

This formula is not very easy to use because of the special basis of the group of units and certain difficulties with its verification for $n > 1$. One of applications of this formula was in the work of Yakovlev on the description of the absolute Galois group of a local field in terms of generators and relations.

Complete formulas, which are simpler that Shafarevich's formula, were discovered in the seventies:

Published 10 December 2000:

Theorem (S. Vostokov 1978), (H. Brückner 1979)**.** *Let a local field K with finite residue field contain $\mathbb{Q}_p(\zeta_{p^n})$ and let $p \neq 2$. Denote $\mathcal{O}_0 = W(k_K)$, $\mathrm{Tr} = \mathrm{Tr}_{\mathcal{O}_0/\mathbb{Z}_p}$. Then for $\alpha, \beta \in K^*$*

$$\boxed{(\alpha,\beta)_{p^n} = \zeta_{p^n}^{\mathrm{Tr}\ \mathrm{res}\ \Phi(\alpha,\beta)/\underline{s}}, \quad \Phi(\alpha,\beta) = l(\underline{\beta})\underline{\alpha}^{-1}d\underline{\alpha} - l(\underline{\alpha})\frac{1}{p}\underline{\beta}^{-\triangle}d\underline{\beta}^{\triangle}}$$

where $\underline{\alpha} = \theta X^m(1+\psi(X))$, $\theta \in \mathcal{R}$, $\psi \in X\mathcal{O}_0[[X]]$, is such that $\underline{\alpha}(\pi) = \alpha$, $\underline{s} = \underline{\zeta_{p^n}}^{p^n} - 1$,

$$l(\underline{\alpha}) = \frac{1}{p}\log\bigl(\underline{\alpha}^p/\underline{\alpha}^{\triangle}\bigr),$$

$$\Bigl(\sum a_i X^i\Bigr)^{\triangle} = \sum \mathrm{Frob}_K(a_i)X^{pi}, \quad a_i \in \mathcal{O}_0.$$

Note that for the term X^{-p} in Kummer's theorem can be written as $X^{-p} = 1/(\underline{\zeta_p}^p - 1) \mod p$, since $\zeta_p = 1 + \pi$ and so $\underline{s} = \underline{\zeta_p}^p - 1 = (1+X)^p - 1 = X^p \mod p$.

The works [V1] and [V2] contain two different proofs of this formula. One of them is to construct the explicit pairing

$$(\alpha,\beta) \to \zeta_{p^n}^{\mathrm{Tr}\ \mathrm{res}\ \Phi(\alpha,\beta)/\underline{s}}$$

and check the correctness of the definition and all the properties of this pairing completely independently of class field theory (somewhat similarly to how one works with the tame symbol), and only at the last step to show that the pairing coincides with the Hilbert symbol. The second method, also followed by Brükner, is different: it uses Kneser's (1951) calculation of symbols and reduces the problem to a simpler one: to find a formula for $(\varepsilon,\pi)_{p^n}$ where π is a prime element of K and ε is a principal unit of K. Whereas the first method is very universal and can be extended to formal groups and higher local fields, the second method works well in the classical situation only.

For $p = 2$ explicit formulas were obtained by G. Henniart (1981) who followed to a certain extent Brückner's method, and S. Vostokov and I. Fesenko (1982, 1985).

8.1.2. The second branch (Artin–Hasse's type formulas).

Theorem (E. Artin and H. Hasse 1928)**.** *Let $K = \mathbb{Q}_p(\zeta_{p^n})$, $p \neq 2$. Then for a principal unit ε and prime element $\pi = \zeta_{p^n} - 1$ of K*

$$\boxed{(\varepsilon,\zeta_{p^n})_{p^n} = \zeta_{p^n}^{\mathrm{Tr}\,(-\log\varepsilon)/p^n}, \quad (\varepsilon,\pi)_{p^n} = \zeta_{p^n}^{\mathrm{Tr}\,(\pi^{-1}\zeta_{p^n}\log\varepsilon)/p^n}}$$

where $\mathrm{Tr} = \mathrm{Tr}_{K/\mathbb{Q}_p}$.

Theorem (K. Iwasawa 1968)**.** *Formula for $(\varepsilon,\eta)_{p^n}$ where $K = \mathbb{Q}_p(\zeta_{p^n})$, $p \neq 2$, ε,η are principal units of K and $v_K(\eta - 1) > 2v_K(p)/(p-1)$.*

To some extent the following formula can be viewed as a formula of Artin–Hasse's type. Sen deduced it using his theory of continuous Galois representations which itself is a generalization of a part of Tate's theory of p-divisible groups. The Hilbert symbol is interpreted as the cup product of H^1.

Theorem (Sh. Sen 1980)**.** *Let* $|K:\mathbb{Q}_p|<\infty$, $\zeta_{p^n}\in K$, *and let* π *be a prime element of* $\mathcal{O}_K$. *Let* $g(T),h(T)\in W(k_K)[T]$ *be such that* $g(\pi)=\beta\not\equiv 0$, $h(\pi)=\zeta_{p^m}$. *Let* $\alpha\in\mathcal{O}_K$, $v_K(\alpha)\geqslant 2v_K(p)/(p-1)$. *Then*

$$\boxed{(\alpha,\beta)_{p^m}=\zeta_{p^m}^c,\quad c=\frac{1}{p^m}\operatorname{Tr}_{K/\mathbb{Q}_p}\left(\frac{\zeta_{p^m}}{h'(\pi)}\frac{g'(\pi)}{\beta}\log\alpha\right).}$$

R. Coleman (1981) gave a new form of explicit formulas which he proved for $K=\mathbb{Q}_p(\zeta_{p^n})$. He uses formal power series associated to norm compatible sequences of elements in the tower of finite subextensions of the p-cyclotomic extension of the ground field and his formula can be viewed as a generalization of Iwasawa's formula.

8.2. History: Further developments

8.2.1. Explicit formulas for the (generalized) Hilbert symbol in the case where it is defined by an appropriate class field theory.

Definition. Let K be an n-dimensional local field of characteristic 0 which contains a primitive p^m th root of unity. The p^m th Hilbert symbol is defined as

$$K_n^{\mathrm{top}}(K)/p^m\times K^*/K^{*p^m}\to\langle\zeta_{p^m}\rangle,\quad(\alpha,\beta)_{p^m}=\gamma^{\Psi_K(\alpha)-1},\quad\gamma^{p^m}=\beta,$$

where $\Psi_K\colon K_n^{\mathrm{top}}(K)\to\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is the reciprocity map.

For higher local fields and $p>2$ complete formulas of Kummer's type were constructed by S. Vostokov (1985). They are discussed in subsections 8.3 and their applications to K-theory of higher local fields and p-part of the existence theorem in characteristic 0 are discussed in subsections 6.6, 6.7 and 10.5. For higher local fields, $p>2$ and Lubin–Tate formal group complete formulas of Kummer's type were deduced by I. Fesenko (1987).

Relations of the formulas with syntomic cohomologies were studied by K. Kato (1991) in a very important work where it is suggested to use Fontaine–Messing's syntomic cohomologies and an interpretation of the Hilbert symbol as the cup product explicitly computable in terms of the cup product of syntomic cohomologies; this approach implies Vostokov's formula. On the other hand, Vostokov's formula appropriately generalized defines a homomorphism from the Milnor K-groups to cohomology groups of a syntomic complex (see subsection 15.1.1). M. Kurihara (1990) applied syntomic cohomologies to deduce Iwasawa's and Coleman's formulas in the multiplicative case.

For higher local fields complete formulas of Artin–Hasse's type were constructed by M. Kurihara (1998), see section 9.

8.2.2. Explicit formulas for p-divisible groups.

Definition. Let F be a formal p-divisible group over the ring $\mathcal{O}_{K_0}$ where K_0 is a subfield of a local field K. Let K contain p^n-division points of F. Define the Hilbert symbol by

$$K^* \times F(\mathcal{M}_K) \to \ker[p^n], \quad (\alpha, \beta)_{p^n} = \Psi_K(\alpha)(\gamma) -_F \gamma, \quad [p^n](\gamma) = \beta,$$

where $\Psi_K\colon K^* \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is the reciprocity map.

For formal Lubin–Tate groups, complete formulas of Kummer's type were obtained by S. Vostokov (1979) for odd p and S. Vostokov and I. Fesenko (1983) for even p. For relative formal Lubin–Tate groups complete formulas of Kummer's type were obtained by S. Vostokov and A. Demchenko (1995).

For local fields with finite residue field and formal Lubin–Tate groups formulas of Artin–Hasse's type were deduced by A. Wiles (1978) for K equal to the $[\pi^n]$-division field of the isogeny $[\pi]$ of a formal Lubin–Tate group; by V. Kolyvagin (1979) for K containing the $[\pi^n]$-division field of the isogeny $[\pi]$; by R. Coleman (1981) in the multiplicative case and some partial cases of Lubin–Tate groups; his conjectural formula in the general case of Lubin–Tate groups was proved by E. de Shalit (1986) for K containing the $[\pi^n]$-division field of the isogeny $[\pi]$. This formula was generalized by Y. Sueyoshi (1990) for relative formal Lubin–Tate groups. F. Destrempes (1995) extended Sen's formulas to Lubin–Tate formal groups.

J.–M. Fontaine (1991) used his crystalline ring and his and J.–P. Wintenberger's theory of field of norms for the p-cyclotomic extension to relate Kummer theory with Artin–Schreier–Witt theory and deduce in particular some formulas of Iwasawa's type using Coleman's power series. D. Benois (1998) further extended this approach by using Fontaine–Herr's complex and deduced Coleman's formula. V. Abrashkin (1997) used another arithmetically profinite extension ($L = \cup F_i$ of F, $F_i = F_{i-1}(\pi_i)$, $\pi_i^p = \pi_{i-1}$, π_0 being a prime element of F) to deduce the formula of Brückner–Vostokov.

For formal groups which are defined over an absolutely unramified local field K_0 ($e(K_0|\mathbb{Q}_p) = 1$) and therefore are parametrized by Honda's systems, formulas of Kummer's type were deduced by D. Benois and S. Vostokov (1990), for $n = 1$ and one-dimensional formal groups, and by V. Abrashkin (1997) for arbitrary n and arbitrary formal group with restriction that K contains a primitive p^n th root of unity. For one dimensional formal groups and arbitrary n without restriction that K contains a primitive p^n th root of unity in the ramified case formulas were obtained by S. Vostokov and A. Demchenko (2000). For arbitrary n and arbitrary formal group without restrictions on K Abrashkin's formula was established by Benois (2000), see subsection 6.6 of Part II.

Sen's formulas were generalized to all p-divisible groups by D. Benois (1997) using an interpretation of the Hilbert pairing in terms of an explicit construction of p-adic periods. T. Fukaya (1998) generalized the latter for higher local fields.

8.2.3. Explicit formulas for p-adic representations. The previously discussed explicit formulas can be viewed as a description of the exponential map from the tangent space of a formal group to the first cohomology group with coefficients in the Tate module. Bloch and Kato (1990) defined a generalization of the exponential map to de Rham representations. An explicit description of this map is closely related to the computation of Tamagawa numbers of motives which play an important role in the Bloch–Kato conjecture. The description of this map for the $\mathbb{Q}_p(n)$ over cyclotomic fields was given by Bloch–Kato (1990) and Kato (1993); it can be viewed as a vast generalization of Iwasawa's formula (the case $n = 1$). B. Perrin-Riou constructed an Iwasawa theory for crystalline representations over an absolutely unramified local field and conjectured an explicit description of the cup product of the cohomology groups. There are three different approaches which culminate in the proof of this conjecture by P. Colmez (1998), K. Kato–M. Kurihara–T. Tsuji (unpublished) and for crystalline representations of finite height by D. Benois (1998).

K. Kato (1999) gave generalizations of explicit formulas of Artin–Hasse, Iwasawa and Wiles type to p-adically complete discrete valuation fields and p-divisible groups which relates norm compatible sequences in the Milnor K-groups and trace compatible sequences in differential forms; these formulas are applied in his other work to give an explicit description in the case of p-adic completions of function fields of modular curves.

8.3. Explicit formulas in higher dimensional fields of characteristic 0

Let K be an n-dimensional field of characteristic 0, $\text{char}(K_{n-1}) = p$, $p > 2$. Let $\zeta_{p^m} \in K$.

Let $t_1, \dots, t_n$ be a system of local parameters of K.

For an element

$$\alpha = t_n^{i_n} \dots t_1^{i_1} \theta (1 + \sum a_J t_n^{j_n} \dots t_1^{j_1}), \quad \theta \in \mathcal{R}^*, a_J \in W(K_0),$$

$(j_1, \dots, j_n) > (0, \dots, 0)$ denote by $\underline{\alpha}$ the following element

$$X_n^{i_n} \dots X_1^{i_1} \theta (1 + \sum a_J X_n^{j_n} \dots X_1^{j_1})$$

in $F\{\{X_1\}\} \dots \{\{X_n\}\}$ where F is the fraction field of $W(K_0)$. Clearly $\underline{\alpha}$ is not uniquely determined even if the choice of a system of local parameters is fixed.

Independently of class field theory define the following explicit map

$$V(\ ,\ )_m\colon (K^*)^{n+1} \to \langle \zeta_{p^m} \rangle$$

by the formula

$$V(\alpha_1, \dots, \alpha_{n+1})_m = \zeta_{p^m}^{\operatorname{Tr} \operatorname{res} \ \Phi(\alpha_1, \dots, \alpha_{n+1})/\underline{s}}, \quad \Phi(\alpha_1, \dots, \alpha_{n+1})$$
$$= \sum_{i=1}^{n+1} \frac{(-1)^{n-i+1}}{p^{n-i+1}} l\left(\underline{\alpha_i}\right) \frac{d\underline{\alpha_1}}{\underline{\alpha_1}} \wedge \dots \wedge \frac{d\underline{\alpha_{i-1}}}{\underline{\alpha_{i-1}}} \wedge \frac{d\underline{\alpha_{i+1}}^{\triangle}}{\underline{\alpha_{i+1}}^{\triangle}} \wedge \dots \wedge \frac{d\underline{\alpha_{n+1}}^{\triangle}}{\underline{\alpha_{n+1}}^{\triangle}}$$

where $\underline{s} = \underline{\zeta_{p^m}}^{p^m} - 1$, $\operatorname{Tr} = \operatorname{Tr}_{W(K_0)/\mathbb{Z}_p}$, $\operatorname{res} = \operatorname{res}_{X_1,\dots,X_n}$,

$$l(\underline{\alpha}) = \frac{1}{p} \log\left(\underline{\alpha}^p / \underline{\alpha}^{\triangle}\right), \quad \left(\sum a_J X_n^{j_n} \cdots X_1^{j_1}\right)^{\triangle} = \sum \operatorname{Frob}(a_J) X_n^{pj_n} \cdots X_1^{pj_1}.$$

Theorem 1. *The map $V(\ ,\ )_m$ is well defined, multilinear and symbolic. It induces a homomorphism*

$$K_n(K)/p^m \times K^*/K^{*\,p^m} \to \mu_{p^m}$$

and since V is sequentially continuous, a homomorphism

$$V(\ ,\ )_m\colon K_n^{\text{top}}(K)/p^m \times K^*/K^{*\,p^m} \to \mu_{p^m}$$

which is non-degenerate.

Comment on Proof. A set of elements $t_1, \dots, t_n$, $\varepsilon_{\mathbf{j}}, \omega$ (where $\mathbf{j}$ runs over a subset of $\mathbb{Z}^n$) is called a *Shafarevich basis* of K^*/K^{*p^m} if

(1) every $\alpha \in K^*$ can be written as a convergent product $\alpha = t_1^{i_1} \dots t_n^{i_n} \prod_{\mathbf{j}} \varepsilon_{\mathbf{j}}^{b_{\mathbf{j}}} \omega^c$ mod K^{*p^m}, $b_{\mathbf{j}}, c \in \mathbb{Z}_p$.

(2) $V\left(\{t_1, \dots, t_n\}, \varepsilon_{\mathbf{j}}\right)_m = 1$, $\quad V\left(\{t_1, \dots, t_n\}, \omega\right)_m = \zeta_{p^m}$.

An important element of a Shafarevich basis is $\omega(a) = E(as(X))|_{X_n = t_n, \dots, X_1 = t_1}$ where

$$E(f(X)) = \exp\left(\left(1 + \frac{\triangle}{p} + \frac{\triangle^2}{p^2} + \cdots\right)(f(X))\right),$$

$a \in W(K_0)$.

Now take the following elements as a Shafarevich basis of K^*/K^{*p^m}:

— elements $t_1, \dots, t_n$,

— elements $\varepsilon_J = 1 + \theta t_n^{j_n} \dots t_1^{j_1}$ where $p \nmid \gcd(j_1, \dots, j_n)$, $0 < (j_1, \dots, j_n) < p(e_1, \dots, e_n)/(p-1)$, where $(e_1, \dots, e_n) = \mathbf{v}(p)$, $\mathbf{v}$ is the discrete valuation of rank n associated to $t_1, \dots, t_n$,

— $\omega = \omega(a)$ where a is an appropriate generator of $W(K_0)/(\mathbf{F} - 1)W(K_0)$.

Using this basis it is relatively easy to show that $V(\ ,\ )_m$ is non-degenerate.

In particular, for every $\theta \in \mathcal{R}^*$ there is $\theta' \in \mathcal{R}^*$ such that

$$V\big(\{1+\theta t_n^{i_n}\dots t_1^{i_1}, t_1,\dots,\widehat{t_l},\dots,t_n\}, 1+\theta' t_n^{pe_n/(p-1)-i_n}\dots t_1^{pe_1/(p-1)-i_1}\big)_m = \zeta_{p^m}$$

where i_l is prime to p, $0 < (i_1,\dots,i_n) < p(e_1,\dots,e_n)/(p-1)$ and $(e_1,\dots,e_n) = \mathbf{v}(p)$.

Theorem 2. *Every open subgroup N of finite index in $K_n^{\text{top}}(K)$ such that $N \supset p^m K_n^{\text{top}}(K)$ is the orthogonal complement with respect to $V(\ ,\ )_m$ of a subgroup in K^*/K^{*p^m}.*

Remark. Given higher local class field theory one defines the Hilbert symbol for l such that l is not divisible by $\operatorname{char}(K)$, $\mu_l \leqslant K^*$ as

$$(\ ,\ )_l\colon K_n(K)/l \times K^*/K^{*l} \to \langle\zeta_l\rangle, \quad (x,\beta)_l = \gamma^{\Psi_K(x)-1}$$

where $\gamma^l = \beta$, $\Psi_K\colon K_n(K) \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is the reciprocity map.

If l is prime to p, then the Hilbert symbol $(\ ,\ )_l$ coincides (up to a sign) with the $(q-1)/l$ th power of the tame symbol of 6.4.2. If $l = p^m$, then the p^m th Hilbert symbol coincides (up to a sign) with the symbol $V(\ ,\ )_m$.

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