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Asymptotics and $6j$-symbols

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Abstract Recent interest in the Kashaev-Murakami-Murakami hyperbolic volume conjecture has made it seem important to be able to understand the asymptotic behaviour of certain special functions arising from representation theory — for example, of the quantum $6j$-symbols for $SU(2)$. In 1998 I worked out the asymptotic behaviour of the *classical* $6j$-symbols, proving a formula involving the geometry of a Euclidean tetrahedron which was conjectured by Ponzano and Regge in 1968. In this note I will try to explain the methods and philosophy behind this calculation, and speculate on how similar techniques might be useful in studying the quantum case.

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1 Introduction

The Kashaev-Murakami-Murakami hyperbolic volume conjecture [19, 12, 17] is a conjecture about the asymptotic behaviour of a certain sequence of "coloured Jones polynomial" knot invariants $J_N(K)$, indexed by natural numbers N. In its simplest form, it states that if the knot K is hyperbolic, then the invariants grow exponentially, with growth rate equal to the hyperbolic volume divided by 2π. We do not yet have any conceptual explanation of why this conjecture might be true, and this seems a serious impediment to attempts to prove it, despite the progress of Thurston [24], Yokota [30], etc.

Attempts to prove and generalise this conjecture have led to renewed interest in the asymptotics of the *quantum* $6j$*-symbols* for $SU(2)$ and of the closely-related *Witten-Reshetikhin-Turaev* and *Turaev-Viro* invariants of 3-manifolds. The hope is that each of these will display asymptotic behaviour governed by geometry in an interesting and useful way.

What I want to describe in this note is a *philosophy*, a method by which results of this form might be proved. In 1998 I proved an essentially similar statement relating the asymptotic behaviour of the *classical* $6j$*-symbols* to the geometry

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of Euclidean tetrahedra [23]. This theorem had been conjectured in 1968 by the physicists Ponzano and Regge [20]; while well-known to and much used by physicists, it had remained unproven and largely unexplained.

The method is *geometric quantization*: the idea is that if we want to understand the asymptotic behaviour of some kind of representation-theoretic quantity, then we should first write is as an integral over some *geometrically meaningful* space, and use the method of stationary phase to evaluate it in terms of local contributions from (geometrically meaningful) critical points.

The plan of the paper is as follows. I start by defining $6j$-symbols algebraically and describing some formulae for them. I then explain their heuristic physical interpretation, and how this enabled Wigner to give a rough asymptotic formula for them. I describe the general method of geometric quantization in representation theory, with special reference to the classical $6j$-symbol example. Finally I explain how, at least in principle, one should be able to adapt these techniques to deal with the quantum $6j$-symbol. I have tried to complement rather than overlap the paper [23] as much as possible.

2 The algebra of $6j$-symbols

Suppose we have a category, such as the category of representations of a compact group, possessing reasonable notions of tensor product, duality, and decomposition into irreducibles. Let I be a set indexing the irreps, and let V_a denote the irrep corresponding to $a \in I$. Then there is an isomorphism

$$V_a \otimes V_b \cong \bigoplus_{c \in I} V_c \otimes \mathrm{Hom}(V_c, V_a \otimes V_b)$$

describing the decomposition of a tensor product of two irreps. The space $\mathrm{Hom}(V_c, V_a \otimes V_b)$ might also be written as $\mathrm{Inv}(V_c^* \otimes V_a \otimes V_b)$, a space of trilinear invariants or *multiplicity space*. With this rule we can express arbitrary spaces of invariants in terms of trilinear ones. Two obvious ways of decomposing a space of 4-linear invariants are

$$\mathrm{Inv}(V_a \otimes V_b \otimes V_c \otimes V_d) \cong \bigoplus_{e \in I} \mathrm{Inv}(V_a \otimes V_b \otimes V_e) \otimes \mathrm{Inv}(V_e^* \otimes V_c \otimes V_d)$$

$$\mathrm{Inv}(V_a \otimes V_b \otimes V_c \otimes V_d) \cong \bigoplus_{f \in I} \mathrm{Inv}(V_a \otimes V_c \otimes V_f) \otimes \mathrm{Inv}(V_f^* \otimes V_b \otimes V_d)$$

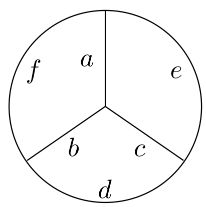
and the $6j$-symbol

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$$

is defined to be the part of the resulting "change-of-basis" operator mapping

$$\mathrm{Inv}(V_a \otimes V_b \otimes V_e) \otimes \mathrm{Inv}(V_e^* \otimes V_c \otimes V_d) \to \mathrm{Inv}(V_a \otimes V_c \otimes V_f) \otimes \mathrm{Inv}(V_f^* \otimes V_b \otimes V_d).$$

The two most important properties of $6j$-symbols are their *tetrahedral symmetry* and the *Elliott-Biedenharn* or *pentagon* identity. The tetrahedral symmetry is a kind of equivariance property under permutation of the six labels, summarised by associating it with a labelled Mercedes badge:



The Elliott-Biedenharn identity expresses the fact that the composition of five successive change-of-basis operators inside a space of 5-linear invariants is the identity. For further details on all of this see Carter, Flath and Saito [3].

For the group $SU(2)$, things can be made much more concrete. Let V denote the fundamental representation on $\mathbb{C}^2$, so that the irreducible representations of G are the symmetric powers $V_a = S^a V$ $(a = 0, 1, 2, \dots)$ with dimensions $a+1$. They are all *self-dual*: $V_a \cong V_a^*$.

The spaces of trilinear invariants $\mathrm{Inv}(V_a \otimes V_b \otimes V_c)$ are either one-dimensional or zero-dimensional, according to whether a, b, c, satisfy the following condition "(Δ)" or not:

$$a \le b + c \qquad b \le c + a \qquad c \le a + b \qquad a + b + c \text{ is even.}$$

The triangle inequality here is the simplest example of the "geometry governs algebra" phenomenon with which we are concerned, and it will be fully explained later.

Because these non-zero multiplicity spaces are one-dimensional, the $6j$-symbols for $SU(2)$ are maps between one-dimensional vector spaces, so by means of a suitable normalisation convention we can think of them as numbers (in fact, they turn out to be *real* numbers) rather than operators. By defining its value to be zero if any of the triples don't satisfy (Δ), we can think of the $6j$-symbol for $SU(2)$ as simply a real-valued function of six natural numbers, which is invariant under the group S_4 of symmetries of a tetrahedron.

There are various formulae for the $6j$-symbols. The *spin network* method gives a straightforward but impractical combinatorial formula. There is a one-variable summation of ratios of factorials, which is the most efficient. There is also a generating function approach. See Westbury [26], or [3].

The representation theory of the corresponding quantum group (Hopf algebra) $U_q(\mathfrak{sl}(2))$ has all the properties needed for definition of $6j$-symbols, and has the same indexing of irreducibles, resulting in the $\mathbb{Q}(q)$-valued *quantum* $6j$*-symbols* defined by Kirillov and Reshetikhin [13], which specialise at $q = 1$ to the classical ones. At a root of unity $q = e^{2\pi i/r}$ the representation category may be quotiented to obtain one with finitely many irreducibles, indexed by $0, 1, \dots, r-2$. Turaev and Viro [25] used the (real-valued) quantum $6j$-symbols associated to this category to make an invariant of 3-manifolds, and this is the main reason for topologists to be interested in $6j$-symbols.

Let T be a triangulation T of a closed 3-manifold M. Define a *state* s to be an assignment of numbers in the range $0, 1, \dots, r-2$ to the edges of T. Given a state s, we can associate to an edge e labelled $s(e)$ the *quantum dimension* $d(s(e))$ of the associated irrep, and to each tetrahedron t the *quantum* $6j$*-symbol* $\tau(s(t))$ corresponding to the labels on its edges. The real-valued state-sum

$$Z(T) = \sum_s \prod_e d(s(e)) \prod_t \tau(s(t))$$

is invariant under the $2-3$ Pachner move, because of the Elliott-Biedenharn identity. A minor renormalisation brings invariance under the $1-4$ move too and so we obtain an invariant of M, the *Turaev-Viro invariant* at $q = e^{2\pi i/r}$.

The TV invariant turns out to be the square of the modulus of the surgery-based *Witten-Reshetikhin-Turaev* invariant of M, which therefore contains more information. But because it is computable in terms of intrinsic structure (a triangulation), it should be easier to relate to the geometry of M.

3 The physics of classical $6j$-symbols

To a physicist, the representation V_a of $SU(2)$ is the space of states of a quantum particle with spin $\frac{1}{2}a$. A *composite* system of (for example) four particles with spins $\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c, \frac{1}{2}d$ is described by the *tensor product* of state spaces. On this space there is a *total spin* operator (the Casimir for the diagonal $SU(2)$, in fact) whose eigenspaces are the irreducible summands; thus, for example, the invariant space $\mathrm{Inv}(V_a \otimes V_b \otimes V_c \otimes V_d)$ is the subspace of states of the system in which the total spin is zero.

The action of $SU(2)$ on the *first two* factors commutes with the total spin operator, is and *its* Casimir gives the decomposition

$$\mathrm{Inv}(V_a \otimes V_b \otimes V_c \otimes V_d) \cong \bigoplus_e \mathrm{Inv}(V_a \otimes V_b \otimes V_e) \otimes \mathrm{Inv}(V_e \otimes V_c \otimes V_d),$$

into states in which the total spin of the first two (and therefore also last two) particles is $\frac{1}{2}e$.

The similar Casimir for the first and third particles does *not* commute with this one and so gives a different eigenspace decomposition. Standard quantum mechanics principles imply that the square of the relevant matrix element

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}^2,$$

is the *probability*, starting with the system in the state where the first two particles have total spin $\frac{1}{2}e$, that measuring the total spin of the first and third combined gives $\frac{1}{2}f$.

The possible states of a classical particle with angular momentum of magnitude j are the vectors in $\mathbb{R}^3$ of length j. A *random* such particle therefore has a state represented by a rotationally-symmetric probability distribution on $\mathbb{R}^3$ supported on a sphere of radius j. For a *quantum* particle of a given spin j, one can imagine the space of states as a space of certain complex-valued wave-functions on $\mathbb{R}^3$, whose pointwise norms give (in general, rather spread-out) probability distributions for the value of a hypothetical angular momentum vector. The *semi-classical limit* requires that quantum particles with very large spin should have distributions very close to those of the corresponding classical particles, becoming more and more localised near the appropriate sphere in $\mathbb{R}^3$.

Wigner [27] gave an asymptotic formula for the $6j$-symbols by adopting this point of view. The *classical* version of the experiment described above, whose output is the square of the $6j$-symbol, is as follows. Suppose one has four random vectors of lengths $\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c, \frac{1}{2}d$ which form a closed quadrilateral; given that one diagonal is $\frac{1}{2}e$, what is the probability (density) that the other is $\frac{1}{2}f$? This analysis yielded the formula

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}^2 \approx \frac{1}{3\pi V},$$

with V the volume of the Euclidean tetrahedron with edge-lengths $a, b, \ldots, f$, supposing it exists. It should be taken as a local root-mean-square average over the rapidly oscillatory behaviour of the $6j$-symbol.

There is a classical version of the Turaev-Viro state-sum, using edges labelled by arbitrary irreps of $SU(2)$, which was written down by the physicists Ponzano and Regge [20] in 1968. Their version is an *infinite* state-sum which turns out to diverge for closed 3-manifolds; the Turaev-Viro invariant can be viewed as a successful "regularisation" of their sum.

Their state-sum is a lattice model of *Euclidean quantum gravity*, which involves a path integral over the space of all Riemannian metrics on a 3-manifold. The states are interpreted as *piecewise-Euclidean metrics* on T, made by gluing Euclidean tetrahedra along faces, and from the asymptotic formula for $6j$-symbols (below) one sees that the "integrand" measures the curvature of the metric at the edges of T. Stationary points of the "integral" (classical solutions) should be metrics in which the dihedral angles of the Euclidean simplexes glued around every edge sum to 2π, or at least to multiples of 2π. (The resulting ramification does seem to cause some problems in this model.)

Remarkably, the Turaev-Viro invariant with $q = e^{2\pi i/r}$ can be interpreted in this context as a lattice model of quantum gravity *with a positive cosmological constant*. Its stationary points should correspond to metrics with *constant positive curvature*, and so we should expect that the asymptotic behaviour of the TV invariant (and likewise of the quantum $6j$-symbols themselves) as $r \to \infty$ will reflect this. For further details see the survey by Regge and Williams [22]. Additional insight into the state-sum can be obtained from Witten's paper [28] or the work of Dijkgraaf and Witten [6].

We can associate to the six labels $a, b, \ldots f$ a metric tetrahedron τ with these as side lengths. The conditions (Δ) guarantee that the individual faces may be realised in Euclidean 2-space, but as a whole the tetrahedron has an isometric embedding into *Euclidean* or *Minkowskian* 3-space according to the sign of the *Cayley determinant*, a cubic polynomial in the squares of the edge-lengths. If τ is Euclidean, let $\theta_a, \theta_b, \ldots, \theta_f$ be its corresponding exterior dihedral angles and V its volume.

Theorem [23] *As $k \to \infty$ (for $k \in \mathbb{Z}$) there is an asymptotic formula*

$$\begin{Bmatrix} ka & kb & kc \\ kd & ke & kf \end{Bmatrix} \sim \begin{cases} \sqrt{\dfrac{2}{3\pi V k^3}} \cos\left\{ \sum (ka+1)\dfrac{\theta_a}{2} + \dfrac{\pi}{4} \right\} & \text{if } \tau \text{ is Euclidean,} \\ \text{exponentially decaying} & \text{if } \tau \text{ is Minkowskian.} \end{cases}$$

(*The sum is over the six edges of the tetrahedron.*)

To have a hope of proving this one needs to start from the *right formula* for the $6j$-symbol. An approach very much in the spirit of Wigner's is explained next.

4 The geometry of classical $6j$-symbols

Geometric quantization is a collection of procedures for turning symplectic manifolds (classical phase spaces) into Hilbert spaces (quantum state spaces). We will here consider *Kähler quantization* only.

If M is a symplectic manifold with an *integral* symplectic form (one that evaluates to an integer on all classes in $H_2(M;\mathbb{Z})$) then it is possible to find a smooth line bundle $\mathcal{L}$ on M with a connection whose curvature form is $(-2\pi i)^{-1}\omega$. The quantization $Q(M)$ is then a subspace of the space of sections of $\mathcal{L}$, specified by a choice of *polarisation* of M.

If M is *Kähler* (complex in a way compatible with the symplectic form) then there is a standard way to polarise it: the bundle $\mathcal{L}$ can be taken to be *holomorphic*, and the relevant subspace $Q(M)$ is its space of *holomorphic sections*.

Such a bundle can also be given a smooth *hermitian metric* $\langle -,-\rangle$ compatible with its connection. When M is compact, the space $Q(M)$ will be finite-dimensional, and we can define an obvious Hilbert space inner product of two sections by the integral formula

$$(s_1, s_2) = \int_M \langle s_1, s_2\rangle \frac{\omega^n}{n!}.$$

The *dimension* of $Q(M)$ can be computed cohomologically via the *Riemann-Roch formula*: at least, the *Euler characteristic* $\chi(\mathcal{L})$ of the set of sheaf cohomology groups $H^*(M;\mathcal{L})$ is given by

$$\int_M e^{c_1(M)}\,\mathrm{td}(TM),$$

and in many cases one can prove a *vanishing theorem* showing that the space of holomorphic sections $H^0(M;\mathcal{L})$ is the only non-trivial space, and thereby obtain a direct formula for its dimension.

Note that we can rescale the symplectic form by a factor of $k \in \mathbb{N}$, replacing $\mathcal{L}$ by $\mathcal{L}^{\otimes k}$, and repeat the construction. Examining the behaviour as $k \to \infty$ corresponds to examining the behaviour of the quantum system as $\hbar = 1/k$ tends to zero; this is the *semi-classical limit*. If M has dimension $2n$ then the formula for $\chi(\mathcal{L}^{\otimes k})$ is a polynomial in k with leading term $k^n\,\mathrm{vol}(M)$, where the volume is measured with respect to the symplectic measure $\omega^n/n!$. This phenomenon is the simplest possible manifestation of the kind of geometric asymptotic behaviour we are studying.

Note also that if there is an equivariant action of a compact group G on $\mathcal{L} \to M$ which preserves the Kähler structure and hermitian form then it acts on the sections of $\mathcal{L}$, giving a *unitary* representation of G. In this case there is an *equivariant index formula* giving the character of $H^*(M;\mathcal{L})$ in cohomological terms, and also a *fixed-point formula* for the character which may be regarded as a kind of *exact* semi-classical approximation.

If G is a Lie group with Lie algebra $\mathfrak{g}$ then its *coadjoint representation* $\mathfrak{g}^*$ decomposes as a union of symplectic *coadjoint orbits* under the action of G. Kirillov's *orbit principle* [14] is that quantization induces a correspondence between the irreducible unitary representations of G and certain of the coadjoint orbits, though the association does depend on the method of quantization used. For a *compact* group G, the coadjoint orbits are G-invariant Kähler manifolds, and the *integral* ones are parametrised by (in fact, are the orbits through) the weights in the positive Weyl chamber. Kähler quantization turns the orbit through the weight λ into the irrep with highest weight λ. This is (part of) the *Borel-Weil-Bott theorem*, which is described more algebro-geometrically in Segal [4] or Fulton and Harris [8].

The correspondence between Kähler manifolds and representations is very helpful in understanding invariant theory for Lie groups. There are three essential ideas: first, the above association between irreps and integral coadjoint orbits; second, that tensor products of representations correspond to products of Kähler manifolds; third, that taking the space of G-invariants of a representation corresponds to taking the *Kähler quotient* of a manifold.

The G-actions we are dealing with are *Hamiltonian*, meaning that the vector fields defining the infinitesimal action of G are symplectic gradients and that we can define an equivariant *moment map* $\mu : M \to \mathfrak{g}^*$ collecting them all up according to the formula

$$d\mu(\xi) = \iota_{X_\xi}\omega \quad (= \omega(X_\xi, -)),$$

where $\xi \in \mathfrak{g}$ and X_ξ is the corresponding vector field. The Kähler quotient is then defined as $M_G = \mu^{-1}(0)/G$. The theorem of Guillemin and Sternberg [9] is that $Q(M_G) = \mathrm{Inv}(Q(M))$. Note that for a coadjoint orbit the moment map turns out to be simply the inclusion map $M \subseteq \mathfrak{g}^*$, and for a product of manifolds, the moment map is the sum of the individual ones.

For $SU(2)$ the coadjoint space is Euclidean $\mathbb{R}^3$, with the group acting by $SO(3)$ rotations. All coadjoint orbits other than the origin are spheres, and the integral ones are those S^2_a with integral radius a. Kähler quantization entails thinking of S^2_a as the Riemann sphere, equipped with the ath tensor power of the hyperplane bundle; the space of holomorphic sections is the irrep V_a, and Riemann-Roch gives its dimension (correctly!) as $a+1$.

To compute the space $\mathrm{Inv}(V_a \otimes V_b \otimes V_c)$, we first form the product M of the three spheres of radii a, b, c. Its moment map is just the sum of the three inclusion maps into $\mathbb{R}^3$, so that $\mu^{-1}(0)$ is the space of closed triangles of vectors of lengths a, b, c. Now M_G is the space of such things up to overall rotation: it is either a point or empty, and its quantization $Q(M_G) = \mathrm{Inv}(V_a \otimes V_b \otimes V_c)$ is either $\mathbb{C}$ or zero, according to the triangle inequalities, whose role in $SU(2)$ representation theory is now apparent. (The additional parity condition can only be seen by considering the lift of the $SU(2)$ action to the line bundle $\mathcal{L}$.) Higher "polygon spaces" arise similarly: for example, $\mathrm{Inv}(V_a \otimes V_b \otimes V_c \otimes V_d)$ is the quantization of the moduli space of shapes of quadrilaterals of sides a, b, c, d in $\mathbb{R}^3$.

A fundamental ingredient of Guillemin and Sternberg's proof that quantization commutes with reduction is the fact that a G-invariant section s of the equivariant bundle $\mathcal{L} \to M$ has maximal pointwise norm on the set $\mu^{-1}(0)$. In fact, the norm of s decays in a Gaussian exponential fashion in the transverse directions (and will in fact reach the value zero on the *unstable points* of M).

The kth power s^k is an invariant section of $\mathcal{L}^{\otimes k}$ whose norm decays faster; we can imagine in the limiting case $k \to \infty$ that such a section becomes *localised* to a delta-function-like distribution supported on $\mu^{-1}(0)$. *Pairings* of such sections will become localised to the *intersections* of these support manifolds, and this is the main idea of the proof of the asymptotic formula for the $6j$-symbol.

In [23] it is written as a pairing between two 12-linear invariants, and thus as an integral over the symplectic quotient of the product of twelve spheres, whose radii depend on the six labels. The intersection locus amounts either to two points corresponding to mirror-image Euclidean tetrahedra, if they exist, or is empty. In the first case one gets a sum of two local contributions, each a Gaussian integral, and after rather messy calculations the formula emerges; exponential decay is automatic in the second case.

5 The geometry of quantum $6j$-symbols

Quantum $6j$-symbols, evaluated at a root of unity, come from a category which might be considered as the category of representations of a quantum group at this root of unity, or of a loop group at a corresponding level. The loop group picture leads to a beautiful and conceptually very valuable analogue of the geometric framework described above. In principle it also allows an analogous calculation of the asymptotic behaviour, though in practice this seems quite difficult.

For a compact group G we saw above the association between irreps and integral coadjoint orbits. Let us now consider the analogous correspondence for the category of positive energy representations of its loop group LG at level k. (For the actual construction of the representations see Pressley and Segal [21].)

Instead of coadjoint orbits we use *conjugacy classes* in G itself. Notice that the foliation of $\mathfrak{g}$ by adjoint orbits is a *linearisation* of the foliation of G by conjugacy classes at the identity, so that the quantum orbit structure is a sort of *curved counterpart* of the classical case. The conjugacy classes correspond under the exponential map to points of a *Weyl alcove*, a truncation of a Weyl chamber inside a Cartan subalgebra. The "integral" conjugacy classes giving the irreps at level k are those obtained by exponentiating k^{-1} times the elements of the weight lattice lying in a k-fold dilation of this alcove.

The definition of the *fusion* tensor product of such irreps is subtle. However, given integral conjugacy classes $C_1, C_2, \ldots, C_n$ corresponding to level-k irreps of LG, it is not hard to describe a symplectic manifold $\mathcal{M}(C_1, C_2, \ldots, C_n)$ which will correspond to the invariant part of their tensor product.

Let $\mathcal{M}$ be the moduli space of flat G-connections on an n-punctured sphere. These are just representations, up to conjugacy, of its fundamental group, which we take to have one generator for each boundary circle and the relation that their product is 1. This space $\mathcal{M}$ is a Poisson manifold, and traces of the puncture holonomies give Casimir functions on it. Their common level sets, the symplectic leaves, are the spaces $\mathcal{M}(C_1, C_2, \ldots, C_n)$ comprising representations with the generators mapping to given conjugacy classes.

In the case of $SU(2)$, the exponential map gives a bijective correspondence between the unit interval in the Cartan subalgebra $\mathbb{R}$ and the conjugacy classes. At level k, the allowable highest weights are therefore $0, 1, \ldots, k$, corresponding to the conjugacy classes C_a with trace equal to $2\cos(\pi a/k)$ for some $a = 0, 1, \ldots, k$, and to the positive energy irreps V_a of $LSU(2)$ at level k.

The space of trilinear invariants $\mathrm{Inv}(V_a \otimes V_b \otimes V_c)$ (where $0 \leq a, b, c \leq k$) corresponds to the space $\mathcal{M}(C_a, C_b, C_c)$ of triples of matrices inside $C_a \times C_b \times C_c$ whose product is 1, considered up to conjugacy. This amounts to the space of shapes of triangles of sides $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}$ in *spherical* 3-space, and is either a single point or empty according to the *quantum triangle inequalities*, meaning the condition (Δ) together with the extra rule $a+b+c \leq 2k$. This corresponds to the well-known fusion rule for quantum $SU(2)$ at root of unity $q = e^{2\pi i/(k+2)}$. Similarly, spaces of quadrilinear invariants correspond to spaces of spherical quadrilaterals with prescribed lengths, and so on. This is the real justification for the use of the word "curved" above!

The explicit construction of a vector space (of tensor invariants) from a symplectic manifold such as $\mathcal{M}(C_1, C_2, \ldots, C_n)$ is achieved as before by using the Kähler quantization technique. The technical difference here is that such spaces have *many* natural complex structures, and so the procedure is more subtle. A choice of complex structure on the underlying punctured sphere induces a complex structure on $\mathcal{M}(C_1, C_2, \ldots, C_n)$ which can be used to construct a holomorphic line bundle and a finite-dimensional space of holomorphic sections, the *space of conformal blocks*. These spaces depend smoothly on the chosen complex structure and in fact form a bundle over the Teichmüller space of such structures with a natural *projectively flat connection*, described by Hitchin [10]. The connection enables canonical and coherent identications of all the different fibre spaces, at least up to scalars.

As before, there is an index formula, the *Verlinde formula*, for the dimensions of such spaces. We can consider the semi-classical limit by sending the level k to infinity but keeping the conjugacy classes fixed, because the highest weight of the irrep corresponding to a fixed conjugacy class scales with the level. The formula is then a polynomial in k with leading term given by the volume of $\mathcal{M}(C_1, C_2, \ldots, C_n)$. See Witten [28] or Jeffrey and Weitsman [11] for more details here, though perhaps the theory of *quasi-Hamiltonian spaces* developed by Alexeev, Malkin and Meinrenken [1] will ultimately give the best framework.

The asymptotic problem for the $SU(2)$ quantum $6j$-symbol is as follows. Pick six rational numbers $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ between 0 and 1. For a level k such that the the six products $a = \alpha k$ etc. are integers, we want to evaluate the quantum $6j$-symbol

$$\begin{Bmatrix} k\alpha & k\beta & k\gamma \\ k\delta & k\epsilon & k\zeta \end{Bmatrix}$$

at $q = e^{2\pi i/(k+2)}$ and then look at the asymptotic expansion as $k \to \infty$. The guess is that this should have something to do with the geometry of a *spherical* tetrahedron, since we have everywhere replaced geometry of the original coadjoint $\mathbb{R}^3$ with the group $SU(2) = S^3$.

We can express this quantum $6j$-symbol as a hermitian pairing between a certain pair of vectors in the space $\mathrm{Inv}(V_a \otimes V_b \otimes V_c \otimes V_d)$. This means working over a 2-dimensional symplectic manifold, the space of spherical quadrilaterals of given edge-lengths.

Now the classical version of this manifold, the space of Euclidean quadrilaterals, has well-known Hamiltonian circle actions corresponding to the lengths of the diagonals of the quadrilateral. This extends to the quantum, spherical case: the lengths are in fact the traces of the holonomies around curves separating the punctures into pairs, and generate Goldman's flows [11].

If such a circle action were to preserve the Kähler structure then it would act on the quantization, thereby decomposing the space of quadrilinear invariants into one-dimensional weight spaces. It would be natural to assume that these would generate the different bases mentioned in section 2 and the vectors we need to pair to compute the $6j$-symbol.

In the classical case these flows do *not* preserve the natural Kähler structure on the product of four spheres. To proceed one would need additional machinery to show that the quantization is independent of the Kähler polarisation; then one would recover the action of the circles on the quantization and perhaps be able to carry the idea through, obtaining an alternative to the proof in [23].

In the quantum case we have *moduli* of Kähler structures coming from the choices of complex structure on the sphere with 4 distinguished points. The moduli space is a Riemann sphere minus three points; these correspond to "stable curve" degenerations and we may add them in to compactify it.

At each singular point there is a *Verlinde decomposition* of the space of conformal blocks into a sum of tensor products of one-dimensional trilinear invariant spaces, and these spaces *are* the eigenspaces of the Hamiltonian flow which preserves the degenerate complex structure. So we ought to be able to specify geometrically the two sections we need to pair. Unfortunately they live in the fibres over different points in the moduli space, so we then need to parallel transport one using the the holonomy of the projectively flat connection before we can pair them easily. Dealing with this might be difficult; it seems for example that even the unitarity of the holonomy is still not established.

If we view the moduli space as $\mathbb{C} - \{0, 1\}$ then we seek the holonomy along the unit interval from 0 to 1. Now *asymptotically* the connection we are examining becomes the *Knizhnik-Zamolodchikov connection*, and this holonomy is nothing more than the *Drinfeld associator*. (See Bakalov and Kirillov [2], for example.) This is the *geometric* explanation for the equivalence of the $6j$-symbol and associator pointed out recently by Bar-Natan and Thurston. Of course, one could try to compute a nice tetrahedrally symmetric formula for the associator hoping that the asymptotic formula for the $6j$-symbol would follow: this would be a completely alternative approach to the asymptotic problem.

6 Related problems

Problem 1 *Compute the asymptotics of the quantum $6j$-symbol.*

Remarks One programme for the computation was outlined above. Chris Woodward has recently conjectured [29] a precise formula and checked it empirically. Suppose there is a spherical tetrahedron with sides l equal to π times $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, and associated dihedral angles θ_l. Let V be its volume, and let G be the determinant of the spherical Gram matrix, the symmetric 4×4 matrix with ones on the diagonal and the quantities $\cos(l)$ off the diagonal. Then he conjectures that

$$\begin{Bmatrix} k\alpha & k\beta & k\gamma \\ k\delta & k\epsilon & k\zeta \end{Bmatrix}_{q=e^{2\pi i/(k+2)}} \sim \sqrt{\frac{4\pi^2}{k^3\sqrt{G}}} \cos\left\{ \sum (kl+1)\frac{\theta_l}{2} - \frac{k}{\pi}V + \frac{\pi}{4} \right\}.$$

Problem 2 *Compute the asymptotics of the Turaev-Viro invariant of a closed 3-manifold.*

Remarks Such a formula might result from an asymptotic formula for the quantum $6j$-symbol, though this would not be straightforward. The hope is that the asymptotics might relate to the existence of spherical geometries on a 3-manifold, although technical issues related to ramified gluings of spherical tetrahedra make this seem likely to be a fairly weak connection.

Problem 3 *Prove the Minkowskian part of Ponzano and Regge's formula.*

Remarks The methods of [23] don't give a precise formula for the exponentially decaying asymptotic regime which occurs when the stationary points have become "imaginary". It is possible *formally* to write down a "Wick rotated" integral over a product of hyperboloids, instead of spheres, as a formula for the same classical $6j$-symbol. This integral has well-defined stationary points corresponding to Minkowskian tetrahedra, whose local contributions seem correct, but the problem is that it does not converge! Some kind of argument involving deformation of the contour of integration is probably required.

Problem 4 *Try to compute asymptotic expansions of similar quantities.*

Remarks Classical $6j$-symbols can be generalised to so-called $3nj$-symbols, associated to arbitrary trivalent labelled graphs drawn on a sphere. The asymptotic behaviour here will be governed by many stationary points corresponding to the different isometric embeddings of such a graph into $\mathbb{R}^3$, but it's not completely clear what the expected contribution from each should be — the volume, or something more complicated.

Stefan Davids [5] studied $6j$-symbols for the non-compact group $SU(1,1)$. There are various different cases corresponding to unitary irreps from the discrete and continuous series, and some surprising relations between the discrete series symbols and geometry of Minkowskian tetrahedra.

One could study the Frenkel-Turaev elliptic and trigonometric $6j$-symbols [7]. I have no idea what they might correspond to geometrically.

The $6j$-symbols for higher rank groups are not simply scalar-valued quantities, because the trilinear invariant spaces typically have dimension bigger than one. This makes them trickier to handle and the question of asymptotics less interesting. One could at least study their norms as operators and expect a geometrical result. There are possibly some nice special cases: Knutson and Tao [15] showed that for $GL(N)$ the property of three irreps having multiplicity one is stable under rescaling their highest weights by k, so one can expect some scalar-valued $6j$-symbols with interesting asymptotics.

Problem 5 *The hyperbolic volume conjecture.*

Remarks A basic approach to the conjecture is to try to give a formula for Kashaev's coloured Jones polyomial invariant in terms of some quantum dilogarithms associated to an ideal triangulation of the knot complement, and then relate their asymptotics to geometry. In fact the quantum dilogarithm, the basic ingredient in Kashaev's invariant, does seem to behave as a kind of $6j$-symbol, satisfying a pentagon-type identity and having an asymptotic relationship with volumes of ideal hyperbolic tetrahedra. It would seem helpful to be able to interpret it as arising from geometric quantization of some suitable space of hyperbolic tetrahedra, with a view to gaining conceptual understanding of the conjecture.

Jun Murakami and Yano [18] have applied Kashaev's non-rigorous stationary phase methods to the Kirillov-Reshetikhin sum formula for the quantum $6j$-symbol. The (false) result is exponential growth, with growth rate given by the volume of the hyperbolic tetrahedron with the appropriate dihedral angles.

Similarly, Hitoshi Murakami [16] has obtained "fake" exponential asymptotics for the Turaev-Viro invariants of some hyperbolic 3-manifolds.

These strange results are very interesting. In each case we start with something which can be expressed as an $SU(2)$ path integral and in an alternative way as a sum. Applying perturbation theory methods to the path integral suggests the correct polynomial asymptotics. But "approximating" the sum by a contour integral in the most obvious way and applying stationary phase gives very different asymptotic behaviour, seemingly reflecting a *complexification* of the original path integral. There is a certain similarity to the appearance of the "imaginary" Minkowskian critical points in problem 3. As in that case, the problem appears to be making sense of the complexified integral in the first place. It is presumably this quantity which we should be interested in as a genuine exponentially growing invariant, and which we should try to learn to compute using some kind of TQFT techniques.

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