Part II : Microbundles

1 Semisimplicial sets

The construction of simplicial homology and singular homology of a simplicial complex or a topological space is based on a simple combinatorial idea, that of incidence or equivalently of face operator.

In the context of singular homology, a new operator was soon considered, namely the degeneracy operator, which locates all of those simplices which factorise through the projection onto one face. Those were, rightly, called degenerate simplices and the guess that such simplices should not contribute to homology turned out to be by no means trivial to check.

Semisimplicial complexes, later called semisimplicial sets, arose round about 1950 as an abstraction of the combinatorial scheme which we have just referred to (Eilenberg and Zilber 1950, Kan 1953). Kan in particular showed that there exists a homotopy theory in the semisimplicial category, which encapsulates the combinatorial aspects of the homotopy of topological spaces [Kan 1955].

Furthermore, the semisimplicial sets, despite being purely algebraically defined objects, contain in their DNA an intrinsic topology which proves to be extremely useful and transparent in the study of some particular function spaces upon which there is not given, it is not desired to give or it is not possible to give in a straightforward way, a topology corresponding to the posed problem. Thus, for example, while the space of loops on an ordered simplicial complex is not a simplicial complex, it can nevertheless be defined in a canonical way as a semisimplicial set.

The most complete bibliographical reference to the study of semisimplicial objects is [May 1967]; we also recommend [Moore 1958] for its conciseness and clarity.

1.1 The semisimplicial category

Recall that the standard simplex $\Delta^m \subset \mathbb{R}^m$ is

$$\Delta^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0 \text{ and } \sum x_i \leq 1\}.$$

The vertices of Δ^m are ordered $0, e_1, e_2, \ldots, e_m$, where e_i is the unit vector in the i^{th} coordinate. Let Δ be the category whose objects are the standard simplices $\Delta^k \subset \mathbb{R}^k$ $(k = 0, 1, 2, \ldots)$ and whose morphisms are the simplicial monotone maps $\theta\colon \Delta^j \to \Delta^k$. A *semisimplicial object* in a category $\mathcal{C}$ is a contravariant functor

$$X\colon \Delta \to \mathcal{C}.$$

If $\mathcal{C}$ is the category of sets, X is called a *semisimplicial set*. If $\mathcal{C}$ is the category of monoids (or groups), X is called a *semisimplicial monoid* (or *group*, respectively).

We will focus, for the moment, on semisimplicial sets, abbreviated ss–sets.

We write $X^{(k)}$ instead of $X(\Delta^k)$ and call $X^{(k)}$ the *set of k–simplices of X*. The morphism induced by θ will be denoted by $\theta^\#\colon X^{(k)} \to X^{(j)}$. A simplex of X is called *degenerate* if it is of the form $\theta^\# \sigma$, with θ *non* injective; if, on the contrary, θ is injective, $\theta^\# \sigma$ is said to be a *face* of σ.

A simplicial complex K is said to be *ordered* if a partial order is given on its vertices, which induces a total order on the vertices of each simplex in K. In this case K determines an ss–set K defined as follows:

$$\mathrm{K}^{(n)} = \{f\colon \Delta^n \to K : f \text{ is a simplicial monotone map}\}.$$

If $\theta \in \Delta$, then $\theta^\# f$ is defined as $f \circ \theta$. In particular, if Δ^k is a standard simplex, it determines an ss–set Δ^k.

The most important example of an ss–set is the *singular complex*, $\mathrm{Sing}(A)$, of a topological space A. A k–simplex of $\mathrm{Sing}(A)$ is a map $f\colon \Delta^k \to A$ and, if $\theta\colon \Delta^j \to \Delta^k$ is in Δ, then $\theta^\#(f) = f \circ \theta$.

We notice that, if A is a one–point set $\ast$, each simplex of dimension > 0 in $\mathrm{Sing}(\ast)$ is degenerate.

If X, Y are ss–sets, a *semisimplicial map* $f\colon X \to Y$, (abbreviated to ss–*map*), is a natural transformation of functors from X to Y. Therefore, for each k, we have maps $f^{(k)}\colon X^{(k)} \to Y^{(k)}$ which make the following diagrams commute

$$\begin{array}{ccc} X^{(k)} & \xrightarrow{f^{(k)}} & Y^{(k)} \\ \Big\downarrow{\scriptstyle \theta^\#} & & \Big\downarrow{\scriptstyle \theta^\#} \\ X^{(j)} & \xrightarrow[f^{(j)}]{} & Y^{(j)} \end{array}$$

for each $\theta\colon \Delta^j \to \Delta^k$ in Δ.

Examples

(a) A map $g\colon A \to B$ induces an ss–map $\mathrm{Sing}(A) \to \mathrm{Sing}(B)$ by composition.

(b) If X is an ss–set, a k–simplex σ of X determines a *characteristic map* $\sigma\colon \Delta^k \to X$ defined by setting

$$\sigma(\lambda) := \lambda^{\#}(\sigma).$$

The composition of two ss–maps is again an ss–map. Therefore we can define the *semisimplicial category* (denoted by SS) of semisimplicial sets and maps. Finally, there are obvious notions of *sub* ss*–set* $A \subset X$ and *pair* $(X; A)$ of ss–sets.

1.2 Semisimplicial operators

In order to have a concrete understanding of the category SS we will examine in more detail the category Δ.

Each morphism of Δ is a composition of morphisms of two distinct types:

(a) $\sigma_i\colon \Delta^m \to \Delta^{m-1}$, $0 \le i \le m-1$,

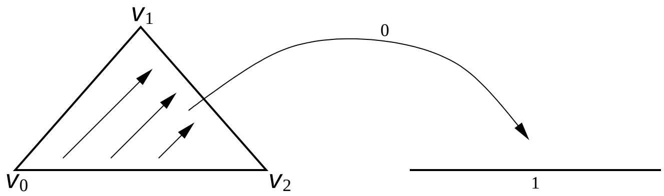
$$\sigma_0(t_1, \ldots, t_m) = (t_2, \ldots, t_m)$$
$$\sigma_i(t_1, \ldots, t_m) = (t_1, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_m) \text{ for } i > 0$$

(b) $\delta_i\colon \Delta^m \to \Delta^{m+1}$, $0 \le i \le m+1$,

$$\delta_0(t_1, \ldots, t_m) = (1 - \textstyle\sum_1^n t_i, t_1, \ldots, t_m).$$
$$\delta_i(t_1, \ldots, t_m) = (t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_m) \text{ for } i > 0.$$

The morphism σ_i flattens the simplex on the face opposite the vertex v_i, preserving the order.

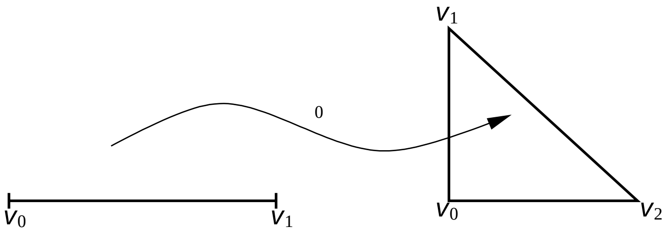
Example



$\sigma_0\colon \Delta^2 \to \Delta^1$

The morphism δ_i embeds the simplex into the face opposite to the vertex v_i.

Example



The following relations hold:

$$
\begin{array}{lll}
{}_j\ {}_i = & {}_i\ {}_{j-1} & i < j \\
{}_j\ {}_i = & {}_i\ {}_{j+1} & i \quad j \\
{}_j\ {}_i = & {}_i\ {}_{j-1} & i < j \\
{}_j\ {}_j = & {}_j\ {}_{j+1} = 1 & \\
{}_j\ {}_i = & {}_{i-1}\ {}_j & i > j+1
\end{array}
$$

If 2 is injective, then is a composition of morphisms of type ${}_i$, otherwise is a composition of morphisms ${}_i$ and morphisms ${}_j$. Therefore, if X is an ss{set and if we denote ${}^{\#}_i$ by s_i and ${}^{\#}_j$ by $@_j$, we get a description of X as a sequence of sets

$$X^0 \leftleftarrows\!\!\to X^1 \; X^2 \; X^3$$

where the arrows pointing left are the face operators $@_j$ and the remaining arrows are the degeneracy operators s_i. Obviously, we require the following relations to hold:

$$
\begin{array}{lll}
@_i @_j = @_{j-1} @_i & & i < j \\
s_i s_j = s_{j+1} s_i & & i \quad j \\
@_j s_j = @_{j+1} s_j = 1 & & \\
@_i s_j = s_{j-1} @_i & & i < j \\
@_i s_j = s_j @_{i-1} & & i > j+1
\end{array}
$$

In the case of the singular complex Sing(A), the map $@_i$ is the usual face operator, ie, if $f\colon\ {}^k\ !\ A$ is a k{singular simplex in A, then $@_i f$ is the $(k-1)${singular simplex in A obtained by restricting f to the i{th face of ^{k}:

$$@_i f\colon\ {}^{k-1} \overset{i}{-!}\ {}^k \overset{f}{-!} A:$$

On the other hand, $s_j f$ is the $(k+1)${singular simplex in A obtained by projecting ${}^{k+1}$ on the j{th face and then applying f:

$$s_j f\colon\ {}^{k+1} \overset{j}{-!}\ {}^k \overset{f}{-!} A:$$

The following lemma is easy to check and the theorem is a corollary.

Lemma (Unique decomposition of the morphisms of Δ) *If φ is a morphism of Δ, then φ can be written, in a unique way, as*

$$\varphi = \underbrace{(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_p})}_{\text{injective}}\underbrace{(s_{j_1}\cdots s_{j_t})}_{\text{surjective}} = \varphi_1\varphi_2. \qquad \square$$

Theorem (Eilenberg–Zilber) *If X is an ss–set and σ is an n–simplex in X, then there exist a unique non-degenerate simplex τ and a unique surjective morphism $\varphi \in \Delta$, such that*

$$\varphi^\#(\tau) = \sigma. \qquad \square$$

1.3 Homotopy

If X, Y are ss–sets, their *product*, $X \times Y$, is defined as follows:

$$(X \times Y)^{(k)} := X^{(k)} \times Y^{(k)}$$
$$\varphi^\#(x, y) := (\varphi^\# x, \varphi^\# y)$$

Example $\mathrm{Sing}(A \times B) \cong \mathrm{Sing}(A) \times \mathrm{Sing}(B)$.

Let us write $I = \Delta^1$, I $= \Delta^1$. Then I has three non-degenerate simplices, ie $0, 1, I$, or, more precisely, $\Delta^0 \to 0$, $\Delta^0 \to 1$, $\Delta^1 \to I$. Write $\mathbf{0}$ for the ss–set obtained by adding to the simplex 0 all of its degeneracies, corresponding to the simplicial maps

$$\Delta^k \to 0, \tag{1.3.1}$$

$k = 1, 2, \ldots$. Hence, $\mathbf{0}$ has a k–simplex in each dimension. For $k > 0$, the k–simplex is degenerate and it consists of the singular simplex (1.3.1).

Proceed in a similar manner for $\mathbf{1}$. One could also say, more concisely,

$$\mathbf{0} = \mathrm{Sing}(0) \qquad \mathbf{1} = \mathrm{Sing}(1).$$

Now, let $f_0, f_1 \colon X \to Y$ be two semisimplicial maps.

A *homotopy* between f_0 and f_1 is a semisimplicial map

$$F \colon \mathrm{I} \times X \to Y$$

such that $F|\mathbf{0} \times X = f_0$ and $F|\mathbf{1} \times X = f_1$ through the canonical isomorphisms $\mathbf{0} \times X \equiv X \equiv \mathbf{1} \times X$.

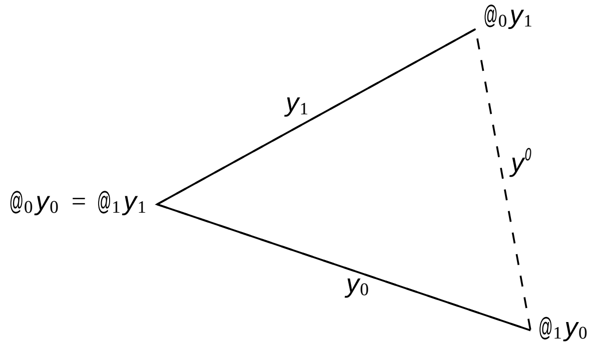
In this case, we say that f_0 is *homotopic* to f_1, and write $f_0 \simeq f_1$. Unfortunately homotopy is *not* an equivalence relation. Let us look at the simplest situation: $X = \Delta^0$. Suppose we have two homotopies $F, G\colon I \to Y$, with $F(1) = G(0)$. If we set $F(I) = y_0 \in Y^{(1)}$ and $G(I) = y_1 \in Y^{(1)}$, we have

$$\partial_0 y_0 = \partial_1 y_1.$$

What transitivity requires, is the existence of an element $y' \in Y^{(1)}$ such that

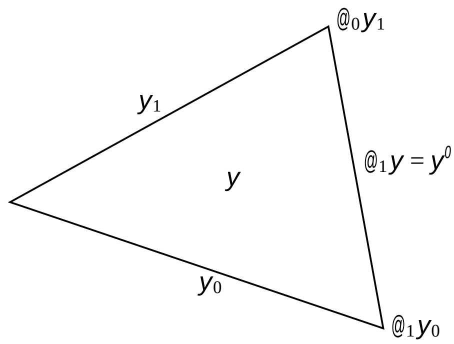
$$\partial_1 y' = \partial_1 y_0 \qquad \partial_0 y' = \partial_0 y_1.$$

In general such an element does not exist.



It was first observed by Kan (1957) that this difficulty can be avoided by assuming in Y the existence of an element $y \in Y^{(2)}$ such that

$$y_0 = \partial_2 y \quad \text{and} \quad y_1 = \partial_0 y$$



If such a simplex y exists, then $y' = \partial_1 y$ is the simplex we were looking for. In fact

$$\partial_1 y' = \partial_1 \partial_1 y = \partial_1 \partial_2 y = \partial_1 y_0$$
$$\partial_0 y' = \partial_0 \partial_1 y = \partial_0 \partial_0 y = \partial_0 y_1.$$

We are now ready for the general definition:

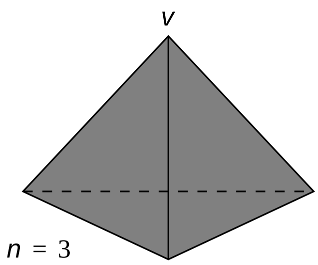
Definition An ss–set Y *satisfies the Kan condition* if, given simplices

$$y_0, \ldots, y_{k-1}, y_{k+1}, \ldots, y_{n+1} \in Y^{(n)}$$

such that $\partial_i y_j = \partial_{j-1} y_i$ for $i < j$ and $i, j \neq k$, there exists $y \in Y^{(n+1)}$ such that $\partial_i y = y_i$ for $i \neq k$.

Such an ss{set is said to be *Kan*. We shall prove later that for semisimplicial maps with values in a Kan ss{set, homotopy is an equivalence relation. $[f]_{SS}$, or $[f]$ for short, denotes the homotopy class of f. We abbreviate Kan ss{set to kss{set.

Example Sing(A) is a kss{set. This follows from the fact that the star $S(v; \_)$ is a deformation retract of for each vertex v 2 = ^{n}.



The union of three faces of the pyramid is a retract of the whole pyramid.

Exercise If is a standard simplex, a *horn* of is, by de nition, the star $S(v; \_)$, where v is a vertex of . Check that an ss{set X is Kan if and only if each ss{map ! X extends to an ss{map ! X.

This exercise gives us an alternative de nition of a kss{set.

Note The extension property allowed D M Kan to develop the homotopy theory in the whole category of ss{sets. The original work of Kan in this direction was based on *semicubical complexes*, but it was soon clear that it could be translated to the semisimplicial environment. For technical reasons, the category of ss{sets replaced the analogous semicubical category, which, recently, regained a certain attention in several contexts, not the least in computing sciences.

In brief the greatest inconvenience in the semicubical category is the fact that the cone on a cube is not a combinatorial cube, while the cone on a simplex is still a simplex.

1.4 The topological realisation of an ss{set (Milnor 1958)

Let X be an ss{set and

$$\overline{X} = \coprod_n \ ^n \quad X^{(n)};$$

where $X^{(n)}$ has the discrete topology and denotes the disjoint union.

We de ne the *topological realisation of* X, written jXj, to be the quotient space of $\overline{X}$ with respect to the equivalence relation generated by the following identi cations

$$(t; \ ^{\#}) \quad (\ (t); \ );$$

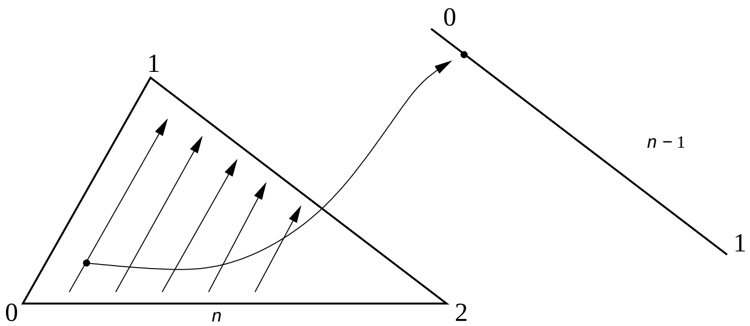
where $t\ 2 \quad ^n$, $\ 2$ and $\ 2\ X$.

Thus, the starting point is an in nite union of standard simplices each labelled by an element of X: We denote those simplices by $\ ^n$ instead of $\ ^n$ $(\ 2\ X^{(n)})$.

The relation is de ned on labelled simplices by using the composition of the two elementary operations (a) and (b) described below. Let us consider $\ ^{n-1}$ and $\ ^n$:

(a) if $= @_i$ for some $i = 0; : : : ; n$, then identi es $\ ^{n-1}$ to $@_i(\ ^n)$, ie, glues to each simplex its faces

(b) if $= s_j$ for some $j = 0; : : : ; n-1$, then squeezes the simplex $\ ^n$ on its j{th face, which in turn is identi ed with $\ ^{n-1}$.



$j = 0$

As a result jXj acquires a cw{structure, with a k{cell for each *non degenerate* k{simplex of X with a canonical characteristic map $\ ^k\ !\ X$.

Examples

(a) If K is a simplicial complex and K is its associated ss{set, then jK$j = K$. In particular

$$j \ ^n j = \ ^n; \quad j\mathrm{I}j = I = [0; 1]; \quad j0j = 0 \ ; \quad j1j = 1:$$

(b) jSing$(\ )j = \ $.

(c) In general it can be proved that, for each cw{complex X, the realisation jSing$(X)j$ is homotopicy equivalent to X by the map

$$[t; \ ]\ 7!\ \ (t)$$

where $: {}^n \to X$ and $t \in {}^n$ and [ ] indicates equivalence class in $|\mathrm{Sing}(X)|$.

(d) If X, Y are ss–sets then $|X \times Y|$ can be identified with $|X| \times |Y|$.

1.5 Approximation

Now we want to describe the realisation of an ss–map. If $f\colon X \to Y$ is such a map, we define its *realisation* $|f|\colon |X| \to |Y|$ by setting

$$[t, \ ] \mapsto [t, f(\ )].$$

Clearly $|f|$ is well defined, since if $[t, \ ] = [s, \ ]$ and there is $\ \in \ $, with ${}^{\#}(\ ) = \ $ and $\ (t) = s$, then

$$|f|[t, \ ] = [t, f(\ )] = [t, f(\ {}^{\#}(\ ))] = [t, \ {}^{\#} f(\ )] =$$
$$= [\ (t), f(\ )] = |f|[\ (t), \ ] = |f|[s, \ ].$$

We say that a (continuous) map $h\colon |X| \to |Y|$ is *realized* if $h = |f|$ for some $f\colon X \to Y$.

The following result is very useful.

Semisimplicial Approximation Theorem *Let $Z \subset X$ and Y be ss–sets, with Y a kss–set, and let $g\colon |X| \to |Y|$ be such that its restriction to $|Z|$ is the realisation of an ss–map. Then there is a homotopy*

$$g \simeq g' \ \mathrm{rel}\ |Z|$$

such that g' is the realisation of an ss–map. □

A very short and elegant proof of the approximation theorem is due to [Sanderson 1975].

1.5.1 Corollary *Let Y be a kss–set. Two ss–maps with values in Y are homotopic if and only if their realisations are homotopic.* □

1.5.2 Corollary *Homotopy between ss–maps is an equivalence relation, if the codomain is a kss–set.* □

This is the result announced after Definition 1.3.

Exercise Convince yourself that an ordered simplicial complex seldom satisfies the Kan condition.

It is not a surprise that the semisimplicial approximation theorem provides a quick proof of Zeeman's relative simplicial approximation theorem (1964), given here in an intrinsic form:

Theorem (Zeeman 1959) *Let X, Y be polyhedra, Z a closed subpolyhedron in X and let $f\colon X \to Y$ be a map such that $f|Z$ is PL. Then, given $\varepsilon > 0$, there exists a PL map $g\colon X \to Y$ such that*

$$(1)\quad f|Z = g|Z \qquad (2)\quad \mathrm{dist}\,(f,g) < \varepsilon \qquad (3)\quad f \simeq g \ \mathrm{rel}\, Z. \qquad \square$$

The above theorem is important because, as observed by Zeeman himself, if $L \subset K$ and T are simplicial complexes, a standard result of Alexander (1915) tells us that each map $f\colon |K| \to |T|$, with $f|L$ simplicial, may be approximated by a simplicial map $g\colon K' \to T$, where $K' \lhd K$ such that $f|L$ in turn is *approximated* by $g|L'$. However, while this is sufficient in algebraic topology, in geometric topology we frequently need the strong version

$$f|L' = g|L'.$$

The interested reader might wish to consult [Glaser 1970, pp. 97–103], [Zeeman 1964].

1.6 Homotopy groups

If X is an ss–set, we call the *base point* of X a 0–simplex $*_X \in X^{(0)}$ or, equivalently, the sub ss–set $* \subset X$, generated by $*_X$. An ss–map $f\colon X \to Y$ is a *pointed map* if $f(*_X) = *_Y$.

As a consequence of the semisimplicial approximation theorem, the homotopy theory of ss–sets coincides with the usual homotopy theory of their realisations.

More precisely, let X, Y be pointed ss–sets, with $* \subset Y \subset X$. We define *homotopy groups* by setting

$$\pi_n(X,*) := \pi_n(|X|,*)$$
$$\pi_n(X,Y,*) := \pi_n(|X|,|Y|,*).$$

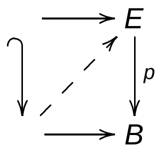
We recall that from the approximation theorem that, if K is a simplicial complex and X a kss–set, then each map $f\colon K \to |X|$ is homotopic to a map $f'\colon K \to |X|$ which is the realisation of an ss–map. Moreover, if f is already the realisation of a map on the subcomplex $L \subset K$, the homotopy can be taken to be constant on L. This property allows us to choose, according to our needs, suitable representatives for the elements of $\pi_n(X,*)$. As an example, we have:

$$\pi_n(X,*) := [I^n, \dot I^n; X, *]_{\mathrm{SS}} = [\Delta^n, \dot\Delta^n; X, *]_{\mathrm{SS}} = [S^n, 1; X, *]_{\mathrm{SS}},$$

where I^n, or S^n, is given the structure of an ss–set by *any* ordered triangulation, which is, for convenience, very often omitted in the notation.

1.7 Fibrations

An ss–map $p\colon E \to B$ is a *Kan fibration* if, for each commutative square of ss–maps



there exists an ss–map $\Delta \to E$, which preserves commutativity. Here Δ and Λ represent a standard simplex and one of its horns respectively.

An equivalent definition of Kan fibration is the following: if $x \in B_{q+1}$ and $y_0, \ldots, y_{k-1}, y_{k+1}, \ldots, y_{q+1} \in E^{(q)}$ are such that $p(y_i) = \partial_i x$ and $\partial_i y_j = \partial_{j-1} y_i$ per $i < j$ and $j \neq k$, then there is $y \in E^{(q+1)}$, such that $\partial_i y = y_i$, for $i \neq k$ and $p(y) = x$.

If F is the preimage in E of the base point, then F is an ss–set, known as the *fibre* over $*$.

Lemma *Let $p\colon E \to B$ be a Kan fibration:*

(a) *if F is the fibre over a point in B, then F is a* kss*–set,*

(b) *if p is surjective, E is Kan if and only if B is Kan.* □

The proof is left to the reader, who may appeal to [May 1967, pp. 25–27].

Theorem [Quillen 1968] *The geometric realisation of a Kan fibration is a Serre fibration.* □

Remark Quillen's proof is very short, but it relies on the theory of minimal fibrations, which we will not introduce in our brief outline of the ss–category as it it is not explicitly used in the rest of the book. The same remark applies to Sanderson's proof of the simplicial approximation lemma. We refer the reader to [May 1967, pages 35–43]

As a consequence of this theorem and the definition of homotopy groups we deduce that, provided $p\colon E \to B$ is a Kan fibration with B a kss–set, the there is a *homotopy long exact sequence*:

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \xrightarrow{p} \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots$$

Suppose now that we have two ss–fibrations $p_i\colon E_i \to B_i$ $(i = 1, 2)$ and let $f\colon E_1 \to E_2$ be an ss–map which covers an ss–map $f_0\colon B_1 \to B_2$. Assume all the ss–sets are Kan and fix a base point in each path component so that p_i, f, f_0 are pointed maps.

Proposition *Let p_i, f, f_0 be as above. Any two of the following properties imply the remaining one:*

(a) *f is a homotopy equivalence,*

(b) *f_0 is a homotopy equivalence,*

(c) *the restriction of f to the bre of E_1 over the base point of each path component B_1 is a homotopy equivalence with the corresponding bre of E_2.*

Proof This result is an immediate consequence of the long exact sequence in homotopy, Whitehead's Theorem and the Five Lemma. □

1.8 The homotopy category of ss{sets

Although it will be used very little, the content of this section is quite important, as it clari es the role of the category of ss{sets in homotopy theory.

We denote by SS (resp KSS) the category of ss{sets (resp kss{sets) and ss{maps, and by CW the category of cw-complexes and continuous maps.

The geometric realisation gives rise to a functor $|\ |\colon \mathrm{SS} \to \mathrm{CW}$. We also consider the singular functor $S\colon \mathrm{CW} \to \mathrm{SS}$.

Theorem (Milnor) *The functors $|\ |$ and S induce inverse isomorphisms between the homotopy category of* kss*{sets and the homotopy category of* cw*{complexes:*

$$h\,\mathrm{KSS} \underset{S}{\overset{|\ |}{\rightleftarrows}} h\,\mathrm{CW}$$ □

For a full proof, see, for instance, [May 1967, pp. 61{62].

Hence, there is a natural bijection between the homotopy classes of ss{maps $[\mathrm{Sing}(X), Y]$ and the homotopy classes of maps $[X, |Y|]$, provided that X has the homotopy type of a cw{complex and Y is a kss{set. Sometimes, we write just $[X, Y]$ for either set.

In conclusion, as indicated earlier, we observe that the semisimplicial structure provides us with a simple, safe and e ective way to introduce a good topology, even a cw structure, on the PL function spaces that we will consider. This topology will allow the application of tools from classical homotopy theory.

Terminology For convenience, whenever there is no possibility of misunderstandings we will confuse X and its realisation $|X|$. Moreover, unless otherwise stated, all the maps from $|X|$ to $|Y|$ are always intended to be realised and, therefore, abusing language, we will refer to such maps as semisimplicial maps.

2 Topological and PL microbundles

Each smooth manifold has a well determined tangent vector bundle. The same does not hold for topological manifolds. However there is an appropriate generalisation of the notion of a tangent bundle, introduced by Milnor (1958) using microbundles.

2.1 Topological microbundles

A *microbundle* , with *base* a topological space B, is a diagram of maps

$$B \xrightarrow{i} E \xrightarrow{p} B$$

with $p \circ i = 1_B$, where i is the *zero–section* and p is the *projection* of .

A microbundle is required to satisfy a *local triviality* condition which we will state after some examples and notation.

Notation We write $E = E(\ )$, $B = B(\ )$, $p = p$, $i = i$ etc. We also write $/B$ and E/B to refer to . Further B is often identified with $i(B)$.

Examples

(a) The *product microbundle*, with *fibre* $\mathbb{R}^m$ and *base* B, is given by

$$\varepsilon^m_B : B \xrightarrow{i} B \times \mathbb{R}^m \xrightarrow{\pi_1} B$$

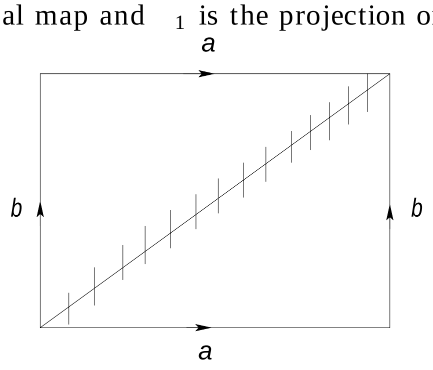
with $i(b) = (b, 0)$ and $\pi_1(b, v) = b$.

(b) More generally, any vector bundle with fibre $\mathbb{R}^m$ is, in a natural way, a microbundle.

(c) If M is a topological manifold without boundary, the *tangent microbundle* of M, written TM, is the diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$$

where Δ is the diagonal map and π_1 is the projection on the first factor.



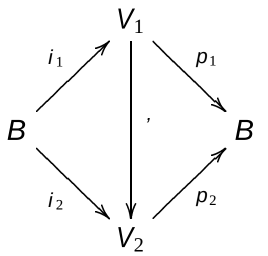
$T(S^1 \times S^1)$

Microbundles maps

2.2 An *isomorphism*, between microbundles on the same base B,

$$: B \xrightarrow{i} E \xrightarrow{p} B \qquad (\ = 1; 2);$$

is a commutative diagram



where V is an open neighbourhood of i (B) in E and ' is a homeomorphism.

2.2.1 *In particular, if* $E{=}B$ *is a microbundle and* U *is an open neighbourhood of* $i(B)$ *in* E*, then* $U{=}B$ *is a microbundle isomorphic to* $E{=}B$*.*

Exercise

Prove that, if M is a smooth manifold, its tangent vector bundle and its tangent microbundle are isomorphic as microbundles.

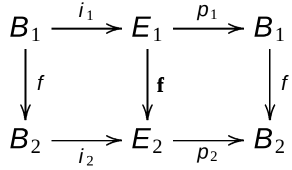
Hint Put a metric on M. If the points $x; y \in M$ are close enough, consider the unique short geodesic from x to y and associate to $(x; y)$ the pair having x as rst component and the velocity vector at x as second component.

Observation Any $(\mathbb{R}^m; 0)${bundle on B is a microbundle, and isomorphic bundles are isomorphic as microbundles.

2.3 More generally, a microbundle *map*

$$: B \xrightarrow{i} E \xrightarrow{p} B \qquad = 1; 2$$

is a commutative diagram



where V_1 is an open neighbourhood of $i_1(B_1)$ in E_1 and $\mathfrak{f}$, f are continuous maps. We write $\mathfrak{f}\colon \xi_1 \to \xi_2$ meaning that $\mathfrak{f}$ covers $f\colon B_1 \to B_2$. Occasionally, in order to be more precise, we will write $(\mathfrak{f}, f)\colon \xi_1 \to \xi_2$. For isomorphisms we shall use the imprecise notation since, by definition, each isomorphism $\varphi\colon \xi_1/B \cong \xi_2/B$ covers 1_B.

A map $f\colon M \to N$ of topological manifolds induces a map between tangent microbundles

$$df\colon TM \to TN,$$

known as the *differential* of f and defined as follows

$$\begin{array}{ccccccc} M & \longrightarrow & M \times M & \longrightarrow & M \\ \big\downarrow f & & \big\downarrow f \times f & & \big\downarrow f \\ N & \longrightarrow & N \times N & \longrightarrow & N \end{array}$$

Note As we have already observed, each microbundle is isomorphic to any open neighbourhood of its zero–section; in other words, what really matters in a microbundle is its behaviour near its zero–section.

In particular, the tangent microbundle TM can, in principle, be constructed by choosing, in a continuous way, a chart U_x around x as a fibre over $x \in M$. Yet, as we do not have canonical charts for M, such a choice is not a topological invariant of M: this is where the notion of microbundle comes in to solve the problem, telling us that we are not forced to select a specific chart U_x, since a *germ* of a chart (defined below) is sufficient. The name *microbundle* is due to Arnold Shapiro.

2.4 Induced microbundle

If ξ is a microbundle on B and $A \subset B$, the *restriction* $\xi|A$ is the microbundle obtained by restricting the total space, ie,

$$\xi|A\colon A \to p^{-1}(A) \xrightarrow{p} A$$

More generally, if ξ/B is a microbundle and $f\colon A \to B$ is a map of topological spaces, the *induced* microbundle $f^*(\xi)$ is defined via the usual categorical construction of pull–back of the map p over the map f.

Example If $f\colon M \to N$ is a map of topological manifolds, then $f^*(TN)$ is the microbundle

$$M \xrightarrow{i} M \times N \xrightarrow{p_1} M$$

with $i(x) = (x, f(x))$.

2.5 Germs

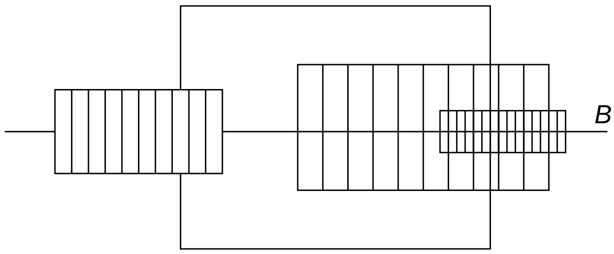
Two microbundle maps $(f, f): \xi_1 \to \xi_2$ and $(g, g): \xi_1 \to \xi_2$ are *germ equivalent* if f and g agree on some neighbourhood of B_1 in E_1. The germ equivalence class of (f, f) is called the *germ of* (f, f) or less precisely the *germ of* f. The notion of the *germ* of a map (or isomorphism) is far more useful and flexible then that of map or isomorphism of microbundles because unlike maps and isomorphisms, *germs can be composed*. Therefore we have the *category of microbundles and germs of maps of microbundles*.

From now on, unless there is any possibility of confusion, we will use interchangeably, both in the notation and in the exposition, the germs and their representatives.

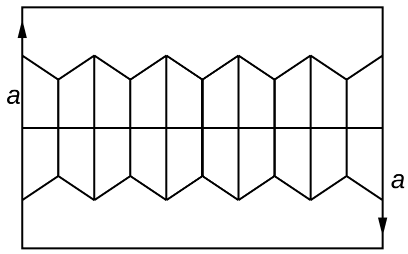
2.6 Local triviality

A microbundle ξ/B is *locally trivial*, *of dimension* or *rank* m, or, more simply, an m–*microbundle*, if it is locally isomorphic to the product microbundle ε^m_B. This means that each point of B has a neighbourhood U in B such that $\varepsilon^m_U \cong \xi|U$.

An m–microbundle ξ/B is *trivial* if it is isomorphic to ε^m_B.



locally trivial



A non trivial microbundle on S^1

Examples

(a) The tangent microbundle TM^m is locally trivial of rank m.

In fact, let $x \in M$ and (U, φ) be a chart of M on a neighbourhood of x such that $\varphi(U) = \mathbb{R}^m$. Define $h_x\colon U \times \mathbb{R}^m \to U \times U$ near $U \times 0$ by

$$h_x(u, v) = (u, \varphi^{-1}(\varphi(u) + v)).$$

(b) If ξ/B is an m–microbundle and $f\colon A \to B$ is continuous, then the induced microbundle $f^*(\xi)$ is locally trivial. This follows from two simple facts:

(1) If ξ is trivial, then $f^*(\xi)$ is trivial.

(2) If $U \subset B$ and $V = f^{-1}(U) \subset A$, then

$$f^*(\xi)|V = (f|V)^*(\xi|U).$$

Terminology From now on the term *microbundle* will always mean *locally trivial microbundle*.

2.7 Bundle maps

With the notation used in 2.3, the germ of a map $(\hat{f}, f)$ of m–microbundles is said to be *locally trivial* if, for each point x, of B_1, $\hat{f}$ restricts to a germ of an isomorphism of $\xi_1|x$ and $\xi_2|f(x)$. Once the local trivialisations have been chosen this germ is nothing but a germ of isomorphism of $(\mathbb{R}^m, 0)$ (as a microbundle over 0) to itself.

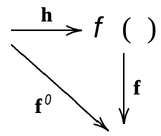
A locally trivial map is called a *bundle map*. Thus a map is a bundle map if, restricted to a convenient neighbourhood of the zero-section, it respects the fibres and it is an open topological embedding on each fibre. Note that an isomorphism between m–microbundles is automatically a bundle map.

Terminology We often refer to an isomorphism between m–microbundles as a *micro–isomorphism*.

Examples

(a) If $f\colon M \to N$ is a homeomorphism of topological manifolds, its differential $df\colon TM \to TN$ is a bundle map. It will be enough to observe that, since it is a local property, it is sufficient to consider the case of a homeomorphism $f\colon \mathbb{R}^m \to \mathbb{R}^m$. This is a simple exercise.

(b) Going back to the induced bundle, there is a natural bundle map $\hat{f}\colon f^*(\xi) \to \xi$. The universal property of the fibre product implies that $\hat{f}$ is, essentially, the only example of a bundle map. In fact, if $f' \colon \xi \to \eta$ is a bundle map which covers f, then there exists a unique isomorphism $h \colon \xi \to f^*(\eta)$ such that $\bar f \circ h = f'$:



(c) It follows from (b) that if $f \colon A \to B$ is a continuous map then each isomorphism $\varphi \colon \xi_1/B \to \xi_2/B$ induces an isomorphism $f^*(\varphi) \colon f^*(\xi_1) \to f^*(\xi_2)$.

2.8 The Kister–Mazur theorem.

Let $\xi \colon B \xrightarrow{i} E \xrightarrow{p} B$ be an m–microbundle, then we say that ξ *admits* or *contains* a bundle, if there exists an open neighbourhood E_1 of $i(B)$ in E, such that $p \colon E_1 \to B$ is a topological bundle with fibre $(\mathbb{R}^m, 0)$ and zero–section $i(B)$. Such a bundle is called *admissible*.

The reader is reminded that an isomorphism of $(\mathbb{R}^m, 0)$–bundles is a topological isomorphism of $\mathbb{R}^m$–bundles, which is the identity on the 0–section.

Theorem (Kister, Mazur 1964) *If an m–microbundle ξ has base B which is an ENR then ξ admits a bundle, unique up to isomorphism.* □

The reader is reminded that ENR is the acronym for *Euclidean Neighbourhood Retract* and therefore the result is valid, in particular, in those cases when B is a locally finite Euclidean polyhedron or a topological manifold. The proof of this difficult theorem, for which we refer the reader to [Kister 1964], is based upon a lemma which is interesting in itself. Let G_0 be the space of the topological embeddings of $(\mathbb{R}^m, 0)$ in itself with the compact open topology and let H_0 be the subspace of proper homeomorphisms of $(\mathbb{R}^m, 0)$. The lemma states that *H_0 is a deformation retract of G_0, ie, there exists a continuous map $F \colon G_0 \times I \to G_0$ so that $F(g, 0) = g$, $F(g, 1) \in H_0$ for each $g \in G_0$ and $F(h, t) \in H_0$ for each $t \in I$ and $h \in H_0$.*

In the light of this result it makes sense to expect the fact that two admissible bundles are not only isomorphic but even *isotopic*. This fact is proved by Kister.

Note In principle Kister's theorem would allow us to work with genuine $\mathbb{R}^m$–bundles which are more familiar objects than microbundles. In fact, according to definition 2.5, a microbundle ξ is micro-isomorphic to each of its admissible bundles.

It is not surprising if Kister's discovery took, at first, some of the sparkle from the idea of microbundle. Nevertheless, it is in the end convenient to maintain the more sophisticated notion of microbundle, since, for instance, the tangent microbundle of a topological manifold is a canonical object while the admissible tangent bundle is defined only up to isomorphism.

2.9 Microbundle homotopy theorem

The microbundle homotopy theorem states that *each microbundle* $=X \quad I$*, where* X *is a paracompact Hausdorff space, admits an isomorphism* $'$: $\quad I$*, where* *is a copy of* $jX \quad 0$. There is also a *relative version* of this result, where, given C a closed subset of X and an isomorphism $'^{0}: (\ jU) \quad I$, where U is an open neighbourhood of C in X, it is possible to chose $'$ to coincide with $'^{0}$ on an appropriate neighbourhood of C.

Kister's result reduces this theorem to the analogous and more familiar result concerning bundles with fibre $\mathbb{R}^m$ [cf Steenrod 1951, section 11].

The following important property follows immediately from the homotopy theorem.

Proposition *If* $f; g$ *are continuous homotopic maps, of a paracompact Hausdorff space* X *to* Y *and if* $=Y$ *is an* m{*microbundle, then* $f\ (\ ) \quad g\ (\ )$. □

2.10 PL microbundles

The category of PL microbundles and maps is defined in analogy to the corresponding topological case using polyhedra and PL maps, with obvious changes. For example, each PL manifold without boundary M admits a well defined PL *tangent microbundle* given by

$$M -! M \quad M -^{1}! M :$$

A PL map $f: M^m \ ! \ N^m$ induces a *differential* $df: TM \ ! \ TN$, which is a PL map of PL m{microbundles. The PL microbundle $f\ (\ )$, *induced* by a PL map of polyhedra, is defined in the usual way through the categorical construction of the pullback and the natural map $f\ (\ ) \ !$ is locally trivial (ie is a PL *bundle map*) if is locally trivial.

As it the topological case PL microbundle will always mean PL *locally trivial microbundle*.

The PL version of Kister{Mazur theorem is proved in [Kuiper{Lashof 1966].

Finally, the *homotopy theorem* for the PL case asserts that, if X is a polyhedron, then $=X \quad I \qquad I$, with $= jX \quad 0$. Nevertheless the proposition that follows from it is less obvious than its topological counterpart.

Proposition *Let $f, g\colon X \to Y$ be PL maps of polyhedra and assume that f, g are continuously homotopic. Let ξ/Y be a PL m–microbundle. Then*

$$f^*(\xi) \cong_{PL} g^*(\xi).$$

Proof Let $F\colon X \times I \to Y$ be homotopy of f and g. By Zeeman's relative simplicial approximation theorem, there exists a homotopy $F'\colon X \times I \to Y$ of f and g, with F' a PL map. The remaining part of the proof is then clear. □

3 The classifying spaces BPL_m and BTop_m

Now we want to prove the existence of classifying spaces for PL m–microbundles and topological m–microbundles. The question fits in the general context of the construction of the classifying space BG of a simplicial group (monoid) G. On this problem, at the time, a large amount of literature was produced and of this we will just cite, also making a reference for the reader, [Eilenberg and MacLane 1953, 1954], [Maclane 1954], [Heller 1955], [Milnor 1961], [Barratt, Gugenheim and Moore 1959], [May 1967], [Rourke and Sanderson 1971]. The first to construct a semisimplicial model for BPL_m and BTop_m was Milnor prior to 1961.

The semisimplicial groups Top_m and PL_m

3.1 We remind the reader that a semisimplicial group G is a contravariant functor from the category Δ to the category of groups. From now on e_m will denote the identity in $G^{(m)} = G(\Delta^m)$.

We define the ss–set Top_m to have typical k–simplex φ a micro-isomorphism

$$\varphi\colon \Delta^k \times \mathbb{R}^m \to \Delta^k \times \mathbb{R}^m.$$

For each $\theta\colon \Delta^l \to \Delta^k$ in Δ, we define

$$\theta^{\#}\colon \mathrm{Top}_m^{(k)} \to \mathrm{Top}_m^{(l)}$$

by setting $\theta^{\#}(\varphi)$ to be equal to the micro-isomorphism induced by φ according to 2.7 (c):

$$\begin{array}{ccc} \Delta^l \times \mathbb{R}^m & \xrightarrow{\theta^{\#}(\varphi)} & \Delta^l \times \mathbb{R}^m \\ \downarrow{\scriptstyle \theta\times 1} & & \downarrow{\scriptstyle \theta\times 1} \\ \Delta^k \times \mathbb{R}^m & \xrightarrow{\varphi} & \Delta^k \times \mathbb{R}^m \end{array}$$

The operation of composition of micro-isomorphisms makes $\mathrm{Top}_m^{(k)}$ into a group and $\theta^{\#}$ a homomorphism of groups. Therefore Top_m is a semisimplicial group.

3.2 In topological m–microbundle theory Top_m plays the role played by the linear group $GL(m, \mathbb{R})$ in vector bundle theory. Furthermore it can be thought of as the singular complex of the space of germs of the homeomorphisms of $(\mathbb{R}^m, 0)$ to itself.

3.3 Since $|\Delta^k| \cong |\Lambda^k \times I|$, it follows that Top_m satisfies the Kan condition. On the other hand we have the following general result, whose proof is left to the reader.

Proposition *Each semisimplicial group satisfies the Kan condition.*

Proof See [May 1967, p. 67]. □

3.4 The semisimplicial group PL_m is defined in a totally analogous manner and, from now on, the exposition will concentrate on the PL case.

3.5 Steenrod's criterion

The classification of bundles of base X in the classical approach of [Steenrod 1951] is done through the following steps:

(a) there is a one to one canonical correspondence

$$[\mathbb{R}^m\text{–vector bundles}] \leftrightarrow [GL(m;\mathbb{R})\text{–principal bundles}]$$

More generally

$$[\text{bundles with fibre } F \text{ and structure group } G] \leftrightarrow [G\text{–principal bundles}]$$

where $[\ \ ]$ indicates the isomorphism classes;

(b) *recognition criterion*: there exists a classifying principal bundle

$$\gamma_G : G \to EG \to BG$$

which is characterised by the fact that E is path connected and $\pi_q(E) = 0$ if $q \geq 1$. The homotopy type of BG is well defined and it is called the *classifying space* of the group G, or also classifying space for principal G–bundles with base a cw–complex.

The correspondence (a) assigns to a bundle ξ, with group G and fibre F, the *associated principal bundle* $\mathrm{Princ}(\xi)$, which is obtained by assuming that the transitions maps of ξ do not operate on F any longer but operate by translation on G itself. The inverse correspondence assigns to a principal G–bundle, E/X, the bundle obtained by *changing the fibre*, ie the bundle

$$F \to E \times_G F \to X.$$

It follows that by changing the fibre of γ_G, we obtain the classifying bundle for the bundles with group G and fibre F, so that BG is the classifying space also for those bundles. Obviously we are assuming that there is a left action of G on the space F, which is not necessarily effective, so that

$$E \times_G F := E \times F/(xg, y) \sim (x, gy), \quad y \in F.$$

We will follow the outline explained above adapting it to the semisimplicial case.

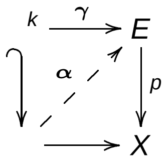
3.6 Semisimplicial principal bundles

Let G be a semisimplicial group. Then a *free action* of G on the ss–set E is an ss–map $E \times G \to E$, such that, for each $\sigma \in E^{(k)}$ and $g', g'' \in G^{(k)}$, we have: (a) $(\sigma g')g'' = \sigma(g'g'')$; (b) $\sigma e_k = \sigma$; (c) $\sigma g' = \sigma g'' \Rightarrow g' = g''$.

The space X of the orbits of E with respect to the action of G is an ss–set and the natural projection $p\colon E \to X$ is called a G–*principal bundle*. The reader can observe that neither E, nor X are assumed to be Kan ss–sets.

Proposition *$p\colon E \to X$ is a Kan fibration.*

Proof Let Λ^k be the k–horn of Δ^k, ie $\Lambda^k = S(v_k; \dot\Delta^k)$. We need to prove the existence of a map α which preserves the commutativity of the diagram below.



To start with consider *any* lifting α' of β, which is not necessarily compatible with γ. Let $\varepsilon\colon \Lambda^k \to G$ be defined by the formula

$$\alpha'(x)\varepsilon(x) = \gamma(x).$$

Since G satisfies the Kan condition, ε extends to $\varepsilon\colon \Delta^k \to G$. If we set

$$\alpha(x) := \alpha'(x)\varepsilon(x);$$

then α is the required lifting. □

The theory of semisimplicial principal G–bundles is analogous to the theory of principal bundles, developed by [Steenrod, 1951] for the topological case. In particular we leave to the reader the task of defining the notion of *isomorphism of G–bundles*, of *trivial* G–bundle, of *G–bundle map*, of *induced G–bundle* and we go straight to the main point.

For each ss–set X let $\mathrm{Princ}(X)$ be the set of isomorphism classes of principal G–bundles on X and, for each ss–map $f\colon X \to Y$, let $f^*\colon \mathrm{Princ}(Y) \to \mathrm{Princ}(X)$ be the induced map: Princ is a contravariant functor with domain the category SS. Our aim is to represent this functor.

3.7 The construction of the universal bundle

Steenrod's recognition criterion 3.5 (b) is carried unchanged to the semisimplicial case with a similar proof. Then it is a matter of constructing a principal G–bundle $\gamma\colon G \to EG \to BG$, such that

(i) EG and BG are Kan ss–sets

(ii) EG is contractible.

▼

We will follow the procedure used by [Heller 1955] and [Rourke–Sanderson 1971]. If X is an ss–set, let

$$X_S := \bigcup_0^\infty X^{(k)}.$$

In other words X_S is the graded set consisting of all the simplexes of X, without the face and degeneracy operators. We will denote with $EG(X)$ the totality of the maps of sets f with domain X_S and range G_S, which have degree zero, ie $f(X^{(k)}) \subset G^{(k)}$.

Since $G^{(k)}$ is a group, then also $EG(X)$ is a group.

Let $G(X)$ be the subgroup consisting of those maps of sets which commute with the semisimplicial operators, ie, those maps of sets which are restrictions of ss–maps. For each $k \geq 0$ we define

$$EG^{(k)} := EG(\Delta^k);$$

and we observe that $G(\Delta^k)$ is a group isomorphic to $G^{(k)}$, the isomorphism being the map which associates to each element of $G^{(k)}$ its characteristic map, $\Delta^k \to G$, thought of as a graded function $\Delta^k_S \to G_S$ (cf II 1.1).

Now it remains to define the semisimplicial operators in

$$EG = \bigcup_0^\infty EG^{(k)}.$$

Let $\mu\colon \Delta^l \to \Delta^k$ be a morphism of Δ and let $\mu_S\colon \Delta^l_S \to \Delta^k_S$ be the corresponding map of sets. For each $\gamma \in EG^{(k)}$ we define

$$\mu^\# \gamma := \gamma \circ \mu_S\colon \Delta^l_S \to G_S$$

where $\mu^\#\colon EG^{(k)} \to EG^{(l)}$ is a homomorphism of groups.

▲

This concludes the definition of an ss–set EG, which even turns out to be a group which has a copy of G as semisimplicial subgroup.

▼

Furthermore, it follows from the definition above, that there is a natural identification:

$$EG(X) \cong \{\text{ss–maps } X \to EG\} \qquad (3.7.1)$$

The reader is reminded that $EG(X)$ is the set of the degree–zero maps of sets from X_S to G_S.

▲

Proposition *EG is Kan and contractible.*

Proof We claim that each ss–map $\partial\Delta^k \to EG$ extends to Δ^k. This follows from (3.7.1) and from the fact that each map of sets of degree zero $\partial\Delta^k_S \to G_S$ obviously admits an extension to Δ^k_S. The result follows straight away from this claim. □

At this point we define

$$BG := EG/G,$$

the ss–set of the right cosets of G in EG, and set $p_\gamma \colon EG \to BG$ to be equal to the natural projection.

In this way we have constructed a principal G–bundle γ/BG with $E(\gamma) = EG$. It follows from Lemma 1.7 that BG is a Kan ss–set.

The following *classification theorem* for semisimplicial principal G-bundles has been established.

Theorem *BG is a classifying space for the group G, ie, the natural transformation*

$$T \colon [X; BG] \to \mathrm{Princ}(X),$$

defined by $T[f] := [f^*(\gamma)]$ is a natural equivalence of functors. □

Corollary *If $H \subset G$ is a semisimplicial subgroup, then there exists, up to homotopy, a fibration*

$$G/H \to BH \to BG.$$

Proof Factorise the universal bundle of G through H and use the fact that, by the Steenrod's recognition principle,

$$EG/H \simeq BH.$$ □

Observation If $H \subset G$ is a subgroup, then the quotient

$$H \to G \to G/H$$

is a principal H–fibration and, by lemma 1.7, G/H is Kan.

Classification of m–microbundles

3.8 So far we have established part (b) of 3.5 for principal G–bundles. Now we assume that $G = \mathrm{PL}_m$ and we will examine part (a). Let K be a locally finite simplicial complex. Order the vertices of K. We consider the associated ss–set K, which consists of all the monotone simplicial maps $f\colon \Delta^q \to K$ $(q = 0, 1, 2, \ldots)$, with $\theta^\#\colon \mathrm{K}^q \to \mathrm{K}^r$ given by $\theta^\#(f) = f\theta$ with $\theta \in \Delta$.

We will denote by Micro(K) the set of the isomorphism classes of m–microbundles on K and by Princ(K) the set of the isomorphism classes of PL principal m–bundles with base K.

Theorem *There is a natural one to one correspondence*

$$\mathrm{Micro}(K) \leftrightarrow \mathrm{Princ}(\mathrm{K}).$$

Proof If ξ/K is an m–microbundle, the *associated principal bundle* Princ(ξ) is defined as follows:

1) a q–simplex of the total space E of Princ(ξ) is a microisomorphism

$$h\colon \Delta^q \times \mathbb{R}^m \to f^*(\xi)$$

with $f \in \mathrm{K}^q$. The semisimplicial operators $\theta^\#\colon E^{(q)} \to E^{(r)}$ are defined by the formula

$$\theta^\#(f, h) := (\theta^\#(f), \theta^*(h))$$

2) the projection $p\colon E^{(q)} \to \mathrm{K}$ is given by $p(h) = f$

3) the action $E^{(q)} \times \mathrm{PL}_m^{(q)} \to E^{(q)}$ is the composition of micro-isomorphisms.

Since $\mathrm{PL}_m^{(q)}$ acts freely on $E^{(q)}$ with orbit space $\mathrm{K}^{(q)}$, then the projection $p\colon E \to \mathrm{K}$ is, by definition, a PL principal m–bundle.

Conversely, given a PL principal m–bundle ξ/K, we can construct an m–microbundle on K as follows: Let $\sigma\colon K \to E(\xi)$ be any map which associates with each ordered q–simplex Δ in K a q–simplex $\sigma(\Delta)$ in $E(\xi)$, such that $p\sigma(\Delta) = \Delta$. Then there exists $\varphi(i, \Delta) \in \mathrm{PL}_m^{(q-1)}$ such that

$$\partial_i \sigma(\Delta) = \sigma(\partial_i \Delta)\varphi(i, \Delta).$$

Furthermore $\varphi(i, \Delta)$ is uniquely determined. Let us now consider the disjoint union of trivial bundles ε^m_Δ with Δ in K. We glue together such bundles by identifying each $\varepsilon^m_{\partial_i \Delta}$ with $\varepsilon^m_\Delta | \partial_i \Delta$ through the micro-isomorphism defined by $\varphi(i, \Delta)$ and by the ordering of the vertices of Δ. The reader can verify that such identifications are compatible when restricted to any face of Δ. Therefore an m–microbundle is defined $\xi[\mathbb{R}^m]/K$. It is not difficult to convince oneself that the two correspondences constructed

$$\xi \longrightarrow \mathrm{Princ}(\xi) \qquad \text{(associated principal bundle)}$$

$$\xi \longrightarrow \xi[\mathbb{R}^n] \qquad \text{(change of fibre)}$$

are inverse of each others. This proves the theorem. □

▲

3.9 A certain amount of technical detail which is necessary for a rigorous treatment of the classi cation of microbundles has been omitted, particularly the part concerning the naturality of various constructions. However the main points have been explained and we move on to state the nal result. To do this we need to de ne a microbundle with base an ss{set X. For what follows it su ces for the reader to think of a microbundle with base X as a microbundle with base jXj. Readers who are concerned about the technical details here may read the following inset material.

▼

It the topological case it is quite satisfactory to regard a microbundle $=X$ as a microbundle $=jXj$, however in the PL case it is not clear how to give jXj the necessary PL structure so that a PL microbundle over jXj makes sense. We avoid this problem by de ning a PL microbundle $=X$ to comprise a collection of PL microbundles with bases the simplexes of X glued together by PL microbundle maps corresponding to the face maps of X.

More precisely, for each $2\ X^{(k)}$ we have a PL microbundle $= {}^k$ and for each pair $2\ X^{(k)};\ 2\ X^{(l)}$ and monotone map $:\ {}^l\ !\ {}^k$ such that ${}^{\#}(\ ) =$ an isomorphism

$$^{\#}\ :$$

which is functorial ie, $(\ )^{\#} = (\ ^{\#}\ )\ ^{\#}$

where $:\ {}^j\ !\ {}^l$ and ${}^{\#}(\ ) =$. Another way of putting this is that we have a lifting of X (as a functor) to the category of PL microbundles and bundle maps. More precisely associate a category $\widetilde{X}$ with X by $\mathrm{Ob}(\widetilde{X}) = \sum_n X^{(n)}$ and $\mathrm{Map}(\widetilde{X})(\ ;\ ) = f(\ ;\ ;\ ) :\ ^{\#}\ =\ g$ for $;\ 2\ \mathrm{Ob}(\widetilde{X})$. Composition of maps in $\widetilde{X}$ is given by $(\ ;\ ;\ )\ (\ ;\ ;\ ) = (\ ;\ ;\ )$. A PL microbundle $=X$ is then a functor from $\widetilde{X}$ to the category of PL microbundles and bundle maps such that for each $2\ X^{(n)}$, $= (\ )$ is a microbundle with base ^{n}.

The de nition implies that the microbundles can be glued to form a (topological) microbundle with base jXj.

▲

Let BPL_m be the classifying space of the group $G = \mathrm{PL}_m$ constructed in 3.7. Theorem 3.7 now implies that we have a PL microbundle $\gamma^m_{\mathrm{PL}}=\mathrm{BPL}_m$ which we call the *classifying bundle* and we have the following classi cation theorem.

Theorem BPL_m *is a classifying space for* PL $m\{$*microbundles which have a polyhedron as base. Precisely, there exists a* PL $m\{$*microbundle* $\gamma^m_{\mathrm{PL}}=\mathrm{BPL}_m$, *such that the set of the isomorphism classes of* PL $m\{$*microbundles on a xed polyhedron* X *is in a natural one to one correspondence with* $[X;\mathrm{BPL}_m]$ *through the induced bundle.* □

3.10 Milnor (1961) also proved that the *homotopy type of* BPL$_m$ *contains a locally finite simplicial complex.*

His argument proceeds through the following steps:

(a) for each finite simplicial complex K the set Micro(K) is countable

(b) by taking K to be a triangulation of the sphere S^q deduce that each homotopy group $\pi_q(\mathrm{BPL}_m)$ is countable

(c) the result then follows from [Whitehead 1949, p. 239].

The theorem of Whitehead, to which we referred, asserts that each countable cw–complex is homotopically equivalent to a locally finite simplicial complex. We still have to prove that each cw–complex whose homotopy groups are countable is homotopically equivalent to a countable cw–complex, for more detail here, see subsection 3.13 below.

Note By virtue of 3.10 and of the Zeeman simplicial approximation theorem it follows that

$$[X, \mathrm{BPL}_m]_{\mathrm{PL}} \cong [X, \mathrm{BPL}_m]_{\mathrm{Top}}.$$

3.11 Let BTop$_m$ be the classifying space of $G = \mathrm{Top}_m$. Then we have, as above:

Theorem BTop$_m$ *classifies topological m–microbundles with base a polyhedron.* □

Addendum BTop$_m$ even classifies the m–microbundles with base X, where X is an ENR. In particular X could be a topological manifold.

Proof of the addendum Let $\gamma^m_{\mathrm{Top}}/\mathrm{BTop}_m$ be a universal m–dimensional microbundle, which certainly exists, and let $N(X)$ be an open neighbourhood of X in a Euclidean space having X as a retract. Let $r\colon N(X) \to X$ be the retraction. Assume that ξ/X is a topological m–bundle and take $r^*(\xi)/N(X)$. By the classification theorem there exists a classifying function

$$(\bar{F}, F)\colon r^*(\xi) \to \gamma^m_{\mathrm{Top}}.$$

Since $r^*(\xi)|X = \xi$, then $(\bar{F}, F)|\xi$ classifies ξ. □

From now on we will write G_m to indicate, without distinction, either Top$_m$ or PL$_m$.

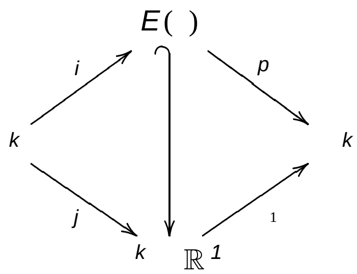
3.12 There are also *relative versions* of the classifying theorems which assert that, if $C \quad X$ is closed and U is an open neighbourhood of C in X and if $f_U: \ |U \to \gamma_G^m$ is a classifying map, then there exists a classifying map $f: \quad \to \gamma_G^m$, such that $f = f_U$ on a neighbourhood of C. In the case where C is a subpolyhedron of X the relative version can be easily obtained using the semisimplicial techniques described above.

3.13 Either for historical reasons or in order to have at our disposal explicit models for BG_m, which should make the exposition and the intuition easier in the rest of the text, we used Milnor's heuristic semisimplicial approach. However the existence of BG_m can be deduced from Brown's theorem [Brown 1962] on representable functors. This was observed for the rst time by Arnold Shapiro. The reader who is interested in this approach is referred to [Kirby–Siebenmann 1977; IV section 8]. Siebenmann observes [ibidem, footnote p. 184] that Brown's proof reduces the unproven statement at the end of 3.10 to an easy exercise. This is true. Let T be a representable homotopy cofunctor de ned on the category of pointed cw–complexes. An easy inspection of Brown's argument ensures that, provided $T(S^n)$ is countable for every $n \quad 0$, T admits a classifying cw–complex which is countable. Now let Y be a path connected cw–complex whose homotopy groups are all countable, and consider $T(X) := [X; Y]$. Then the above observation tells us that $T(X)$ admits a countable classifying Y^0. But Y is homotopically equivalent to Y^0 by the homotopy uniqueness of classifying spaces, which proves what we wanted.

3.14 BG_m as a Grassmannian

We will start by constructing a particular model of EG_m. Let $\mathbb{R}^1$ denote the union $\mathbb{R}^1 \quad \mathbb{R}^2 \quad \mathbb{R}^3 \quad :::$.

An m–microbundle $= \ ^k$ is said to be a *submicrobundle* of $\ ^k \quad \mathbb{R}^1$ if $E(\ ) \quad ^k \quad \mathbb{R}^1$ and the following diagram commutes:



where i is the zero-section of $\ $, p is the projection and $j(x) = (x; 0)$. Having said that, let WG_m be the ss–set whose typical k–simplex is a *monomorphism*

$$f: \ ^k \quad \mathbb{R}^m \to \ ^k \quad \mathbb{R}^1$$

ie, a G_m micro-isomorphism between $\Delta^k \times \mathbb{R}^m$ and a submicrobundle of $\Delta^k \times \mathbb{R}^\infty$. The semisimplicial operators are defined as usual, passing to the induced micro-isomorphism.

Exercise *WG_m is contractible.*

In order to complete the exercise we need to show that each ss–map $\partial\Delta \to WG_m$ extends to $\Delta \to WG_m$, where Δ is any standard simplex. This means that each monomorphism $h\colon \partial\Delta \times \mathbb{R}^m \to \partial\Delta \times \mathbb{R}^\infty$ has to extend to a monomorphism $H\colon \Delta \times \mathbb{R}^m \to \Delta \times \mathbb{R}^\infty$ and this is not difficult to establish. □

In the same way one can verify that WG_m satisfies the Kan condition. WG_m is called the G_m–*Stiefel manifold*.

An action $WG_m \times G_m \to WG_m$ defined by composing the micro–isomorphisms transforms WG_m into the space of a principal fibration

$$\gamma(G_m)\colon G_m \to WG_m \to BG_m. \qquad (3.14.1)$$

By the Steenrod's recognition criterion, BG_m in (3.14.1) is a classifying space for G_m and a typical k–simplex of BG_m is nothing but a G_m–submicrobundle of $\Delta^k \times \mathbb{R}^\infty$. In this way BG_m is presented as a *semisimplicial grassmannian*. Furthermore the *tautological* microbundle γ^m_G/BG_m is obtained by putting on the simplex ξ the microbundle which it represents which we will still denote with ξ. Therefore

$$\gamma^m_G|\xi := \xi.$$

3.15 The ss–set $\mathrm{Top}_m/\mathrm{PL}_m$

In the case of the natural map of grassmannians

$$B\mathrm{PL}_m \xrightarrow{p_m} B\mathrm{Top}_m$$

induced by the inclusion $\mathrm{PL}_m \subset \mathrm{Top}_m$, it is very convenient to have a geometric description of its homotopic fibre. This is very easy to obtain using the semisimplicial language. In fact there is an action also defined by composition,

$$W\mathrm{Top}_m \times \mathrm{PL}_m \to W\mathrm{Top}_m,$$

whose orbit space has the same homotopy type as $B\mathrm{PL}_m$ and gives the required fibration

$$B\colon \mathrm{Top}_m/\mathrm{PL}_m \longrightarrow B\mathrm{PL}_m \xrightarrow{p_m} B\mathrm{Top}_m.$$

This takes us back to the general construction of Corollary 3.7.

Obviously, $\mathrm{Top}_m/\mathrm{PL}_m$ is the ss–set obtained by factoring with respect to the natural action of PL_m on Top_m, so, by Observation 3.7, $\mathrm{Top}_m/\mathrm{PL}_m$ satisfies the Kan condition and

$$\mathrm{PL}_m \to \mathrm{Top}_m \to \mathrm{Top}_m/\mathrm{PL}_m$$

is a Kan fibration.

4 PL structures on topological microbundles

In this section we will consider the problem of the *reduction* of a topological microbundle to a PL microbundle and we will classify reductions in terms of liftings on their classifying spaces. In this way we will put in place the foundations of the obstruction theory which will allow the use apparatus of homotopy theory for the problem of classifying the PL structures on a topological manifold.

4.1 A *structure of* PL *microbundle* on a topological m–microbundle , with base an ss–set X, is an equivalence class of topological micro–isomorphisms f: $\to$, where $/X$ is a PL microbundle. The equivalence relation is f f$'$ if f$'$ = h f, with h a PL micro–isomorphism.

A structure of PL microbundle will also be called a PL –structure (indicates a microbundle). More generally, an ss–set, PL ( ), is defined so that a typical k–simplex is an equivalence class of micro–isomorphisms

$$f\colon \quad {}^k \quad \to$$

where is a PL m–microbundle on ^{k} X. The semisimplicial operators are defined, as usual, passing to the induced micro–isomorphism.

Equivalently, a structure of PL microbundle on

$$: X \xrightarrow{i} E(\ ) \xrightarrow{p} X$$

is a polyhedral structure , defined on an open neighbourhood U of $i(X)$, such that

$$X \xrightarrow{i} U \xrightarrow{p} X$$

is a (locally trivial) PL m–microbundle. If $'$ is another such polyhedral structure then we say that is *equal* to $'$ if the two structures define the same germ in a neighbourhood of the zero–section, ie, if $=$ $'$ in an open neighbourhood of $i(X)$ in $E(\ )$. Then truely represents an equivalence class. Using this language PL ( ) is the ss–set whose typical k–simplex is the germ around ^{k} X of a PL structure on the product microbundle ^{k} .

Going back to the fibration

$$B\colon \mathrm{Top}_m/\mathrm{PL}_m \to B\mathrm{PL}_m \xrightarrow{p_m} B\mathrm{Top}_m$$

constructed in 3.15 we fix, once and for all, a classifying map f: $\to \gamma^m_{\mathrm{Top}}$, which restricts to a continuous map $f\colon X \to B\mathrm{Top}_m$. Let us also fix a classifying map $p_m\colon \gamma^m_{\mathrm{PL}} \to \gamma^m_{\mathrm{Top}}$, with restriction $p_m\colon B\mathrm{PL}_m \to B\mathrm{Top}_m$. A k–*simplex* of the kss–set Lift(f) is a continuous map

$$:\ {}^k \quad X \to B\mathrm{PL}_m$$

such that $p_m \quad = f \quad {}_2$, where ${}_2$ is the projection on X. Therefore a 0{simplex of Lift(f) is nothing but a *lifting* of f to $B\mathrm{PL}_m$, a 1{simplex is a *vertical homotopy class* of such liftings, etc. As usual the liftings are nothing but sections. In fact, passing to the induced bration f (B) (which we will denote later either with ${}_f$ or [Top$_m$=PL$_m$]) we have, giving the symbols the obvious meanings,

$$\mathrm{Lift}(f) \quad \mathrm{Sect} \ [\mathrm{Top}_m{=}\mathrm{PL}_m] \qquad (4{:}1{:}1)$$

where the right hand side is the ss*{set of sections of the bration* [Top$_m$=PL$_m$] *associated with* .

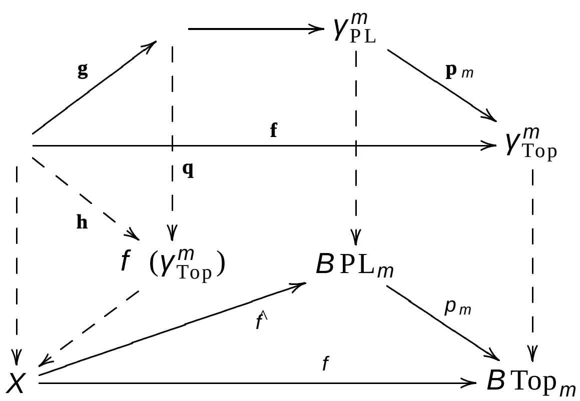
Classi cation theorem for the PL {structures *Using the notation introduced above, there is a homotopy equivalence*

$$: \mathrm{PL} \ (\ ) \ ! \ \mathrm{Lift}(f)$$

which is well de ned up to homotopy.

First we will give an indication of how can be constructed directly, following [Lashof 1971].

First proof Firstly we will observe that f: $! \ \gamma^m_{\mathrm{Top}}$ induces an isomorphism h: $! \ f \ (\gamma^m_{\mathrm{Top}})$.



Let $\hat{f}: X \ ! \ B\mathrm{PL}_m$ be a lifting of f and $= \hat{f} \ (\gamma_{\mathrm{PL}})$. The map of m{ microbundles p$_m$ induces an isomorphism

$$\mathrm{q}: \quad = \hat{f} \ (\gamma_{\mathrm{PL}}) \ ! \ f \ (\gamma_{\mathrm{Top}}){:}$$

In fact, $f \ (\gamma_{\mathrm{Top}}) = (p_m\hat{f}) \ (\gamma_{\mathrm{Top}}) = \hat{f} \ p_m(\gamma_{\mathrm{Top}})$ and there is a canonical isomorphism ' between γ_{PL} and $p_m(\gamma_{\mathrm{Top}})$: Therefore it will su ce to put

$$\mathrm{q} := \hat{f} \ (\ '){:}$$

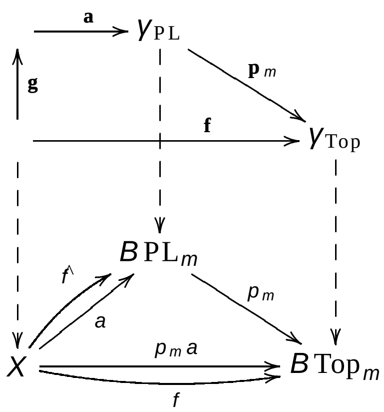
Now we can de ne a PL {structure g on by de ning

$$g := q^{-1}h.$$

In this way we have associated a 0{simplex of PL ( ) with a 0{simplex of Lift(f) .

On the other hand, if $\hat{f}_t$ is a 1-simplex of Lift(f), ie, a vertical homotopy class of liftings of f, then the set of induced bundles $\hat{f}_t(\gamma_{\mathrm{Top}})$ determines, in the way we described above, a 1{simplex g_t of PL {structures on .

Conversely, x a PL {structure g: ! , and let a: ! γ_{PL} be a classifying map which covers $a\colon X \; ! \; B\mathrm{PL}_m$.



The maps $X \; ! \; B\mathrm{Top}_m$ given by $p_m a$ and f are homotopic, since they classify topologically isomorphic microbundles. Therefore, since p_m is a bration and $p_m a$ lifts to a trivially, then f also lifts to a $\hat{f}\colon X \; ! \; B\mathrm{PL}_m$. This way is established a correspondence between a 0{simplex of PL ( ) and a 0{simplex of Lift(f).

4.2 It would be possible to conclude the proof of the theorem in this heuristic way, however we would rather use a less direct argument, which is more elegant and, in some sense, more instructive and illuminating. This argument is due to [Kirby{Siebenmann 1977, pp. 236{239].

▼

Preface If A and B are metrisable topological spaces, then the typical k{ simplex of the SS of the functions B^A is a continuous map

$$^k \quad A \; ! \; B.$$

The semisimplicial operators are de ned by composition of functions. Naturally the path components of B^A are nothing but the homotopy classes $[A; B]$. An ss–map g of a simplicial complex Y in B^A is a continuous map $G\colon Y \times A \to B$, defined by

$$G(y, a) = g(y)(a)$$

for $y \in Y$; furthermore g is homotopic to a constant if and only if G is homotopic to a map of the same type as

$$Y \times A \xrightarrow{\pi_2} A \longrightarrow B.$$

Incidentally we notice that if A has a countable system of neighbourhoods and if we give B^A the compact open topology, then g is continuous if and only if G is continuous.

Second proof of theorem 4.1 Let $\mathrm{M}_{\mathrm{Top}}(X)$ be the ss–set whose typical k–simplex is a topological m–microbundle ξ with base $\Delta^k \times X$. In order to avoid set–theoretical problems we can think of ξ as being represented by a submicrobundle of $\Delta^k \times X \times \mathbb{R}^\infty$. We agree that another such microbundle $\xi' = \Delta^k \times X$ represents the same simplex of $\mathrm{M}_{\mathrm{Top}}(X)$ if ξ coincides with ξ' in a neighbourhood of the zero–section. In practice (cf 3.14) $\mathrm{M}_{\mathrm{Top}}(X)$ can be considered as the *grassmannian* of the m–microbundles on X. Now, if Y is a simplicial complex, then an ss–map $Y \to \mathrm{M}_{\mathrm{Top}}(X)$ is represented by an m–microbundle γ on $Y \times X$ and it is homotopic to a constant if there exists an m–microbundle γ_I on $I \times Y \times X$, such that $\gamma_I|0 \times Y \times X = \gamma$ and $\gamma_I|1 \times Y \times X = Y \times \gamma_1$, where γ_1 is some microbundle on X.

Further, let $\mathrm{M}^+_{\mathrm{Top}}(X)$ be the ss–set whose typical k–simplex is an equivalence class of pairs (ξ, f), where ξ is an m–microbundle on $\Delta^k \times X$ and $\mathrm{f}\colon \xi \to \gamma^m_{\mathrm{Top}}$ is a classifying micro–isomorphism and, also, $(\xi, \mathrm{f}) \sim (\xi', \mathrm{f}')$ if the pairs are identical in a neighbourhood of the two respective zero–sections. In this case an ss–map $g\colon Y \to \mathrm{M}^+_{\mathrm{Top}}(X)$ is represented by an m–microbundle ξ on $Y \times X$, together with a classifying map $\mathrm{f}_\xi\colon \xi \to \gamma^m_{\mathrm{Top}}$. Furthermore g is homotopic to a constant if there exist an m–microbundle ξ_I on $I \times Y \times X$ and a classifying map $\mathrm{F}\colon \xi_I \to \gamma^m_{\mathrm{Top}}$, such that $(\xi_I, \mathrm{F})|0 \times Y \times X = (\xi, \mathrm{f}_\xi)$ and $(\xi_I, \mathrm{F})|1 \times Y \times X$ is of type $(Y \times \xi_1, \mathrm{f}_1\pi_2)$, where π_2 is the projection on $\xi_1 = X$ and f_1 is a classifying map for ξ_1. Consider the two forgetful maps

$$\mathrm{M}_{\mathrm{Top}}(X) \xleftarrow{\alpha_{\mathrm{Top}}} \mathrm{M}^+_{\mathrm{Top}}(X) \xrightarrow{\beta_{\mathrm{Top}}} B\,\mathrm{Top}^X_m,$$

$\alpha_{\mathrm{Top}}(\xi, \mathrm{f}) = \xi$, and $\beta_{\mathrm{Top}}(\xi, \mathrm{f}) = f.$ We leave to the reader the proof that α, β are *homotopy equivalences*, since they induce a bijection between the path components, as well as an isomorphism between the homotopy groups of the corresponding components. For β this is a consequence of the classification theorem for topological m–microbundles, *in its relative version*. In order to find a homotopy inverse for α, we instead use the construction of the induced bundle and of the homotopy theorem for microbundles. In the PL case we have analogous ss–sets and homotopy equivalences, which are defined in the same way as the corresponding topological objects:

$$\mathrm{M}_{\mathrm{PL}}(X) \xleftarrow{\alpha_{\mathrm{PL}}} \mathrm{M}^+_{\mathrm{PL}}(X) \xrightarrow{\beta_{\mathrm{PL}}} B\,\mathrm{PL}^X_m,$$

where k–simplex of $\mathrm{M}_{\mathrm{PL}}(X)$ is now a *topological* m–microbundle on $^k \ X$, *together with a* PL *structure* ; and (;) (^{0}; ^{0}) if such pairs coincide in a neighbourhood of the zero section.

We observe that the proof of the fact that $_{\mathrm{PL}}$ is a homotopy equivalence requires the use of Zeeman's simplicial approximation theorem.

In this way we obtain a commutative diagram of *forgetful* ss–maps

$$\begin{array}{ccccc}
\mathrm{M}_{\mathrm{PL}}(X) & \xleftarrow{\ \mathrm{PL}\ } & \mathrm{M}^+_{\mathrm{PL}}(X) & \xrightarrow{\ \mathrm{PL}\ } & B\mathrm{PL}^X_m \\
\downarrow p' & & \downarrow p & & \downarrow p'' \\
\mathrm{M}_{\mathrm{Top}}(X) & \xleftarrow[\ \mathrm{Top}\ ]{} & \mathrm{M}^+_{\mathrm{Top}}(X) & \xrightarrow[\ \mathrm{Top}\ ]{} & B\mathrm{Top}^X_m
\end{array}$$

where p'' is induced by the projection $p_m\colon B\mathrm{PL}_m \to B\mathrm{Top}_m$ of the fibration B. It is easy to verify that both p' and p'' are Kan fibrations. Furthermore we can assume that p also is a fibration. In fact, if it is not, the Serre's trick makes p a fibration, transforming the diagram above into a new diagram which is *commutative up to homotopy* and where the horizontal morphisms are still homotopy equivalences, while the lateral vertical morphisms p', p'' remain unchanged. At this point the *Proposition* 1.7 ensures that, if (; f) $\in$ $\mathrm{M}^+_{\mathrm{Top}}(X)$, then the fibre p'^{-1}( ) is homotopically equivalent to the fibre $(p'')^{-1}(f)$. However, by definition:

$$(p')^{-1}(\ ) = \mathrm{PL}\ (\ )$$
$$(p'')^{-1}(f) = \mathrm{Lift}(f).$$

The theorem is proved. □