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ON MINIMAX THEOREMS FOR SETS CLOSED IN MEASURE

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The paper is dedicated
to the memory of Yuri Abramovich,
who was dear friend for both of us.

This article is devoted to the Ky Fan minimax theorem for convex sets closed in measure in L^1. In general, these sets do not carry any formal compactness properties for any reasonable topology.

Famous Fan's minimax theorem, stated in its generality, claims that for a function $\Phi(x, y)$ defined on $X \times Y$ (where X and Y have no linear or convex structures) which satisfies a mild *convex-concave-like* conditions, the minimax equality holds provided the set X is compact (see [5] for a simple proof which we will use later for our generalization).

Bukhvalov and Lozanovsky invented in [7] the *Optimization Without Compactness* (OWC) technique for sets in L^1 (see [6] for a survey of the current state of art; an elementary exposition is given in [11]). Roughly speaking norm bounded convex sets in L^1, which are closed with respect to convergence in measure, have many properties usually associated with compact sets only.

Our main goal is to derive a minimax theorem without compactness (Section 1) and further to extend it with milder convexity assumptions (Section 2). Section 3 briefly describes possible applications.

1. Main Theorem

Our assumptions will be the following (all sets below are expected to be non empty). Let (T, Σ, μ) be a σ-finite measure space, and let X be a convex, norm-bounded set in $L^1(\mu)$. Let Y be an arbitrary set and $\Phi : X \times Y \to \mathbb{R}$ be a function such that

(a) for each $y \in Y$ the function $x \mapsto \Phi(x, y)$ is convex;

(b) for each $t \in [0, 1]$ and for every $y_1, y_2 \in Y$ there exists $y_3 \in Y$ with

$$\Phi(x, y_3) \geqslant t\Phi(x, y_1) + (1 - t)\Phi(x, y_2)$$

for every $x \in X$.

Condition (b) is more general than concavity. The latter is too restrictive for many applications. It is said that $\Phi(x, y)$ is *concave-like* in y. This class has a much wider area of applications (see below).

Theorem 1. *Suppose that X, Y and Φ satisfy the conditions above. Suppose that X is closed in measure and $\Phi(\cdot, y)$ is lower semicontinuous in measure on X for each $y \in Y$. Then*

$$\min_{x\in X}\sup_{y\in Y}\Phi(x,y)=\sup_{y\in Y}\min_{x\in X}\Phi(x,y). \tag{1}$$

We will start by justifying 'min' instead of 'inf' in (1). The following well-known consequence of [7] is stated in [13], [11].

Lemma 1. *Let X be a convex norm-bounded set in $L^1(\mu)$ which is closed in measure. Let $f\colon X\to\mathbb{R}$ be a quasi-convex function which is lower semicontinuous in measure. Then f attains its minimum in X.*

Lemma 2. *Let X, Y and Φ be as in Theorem 1. Then the function*

$$f(x) := \sup\{\Phi(x,y)\colon\ y\in Y\}$$

is quasi-convex and lower semicontinuous in measure on X.

Proof of Theorem 1. Let

$$p := \min_{x\in X}\sup_{y\in Y}\Phi(x,y).$$

If $p=-\infty$ there is nothing to prove since the inequality

$$\inf_{x\in X}\sup_{y\in Y}\Phi(x,y)\geqslant\sup_{y\in Y}\inf_{x\in X}\Phi(x,y)$$

always holds.

Let α be any real such that $\alpha<p$. By our assumptions each set

$$C(y) := \{x\in X\colon\ \Phi(x,y)\leqslant\alpha\}$$

is norm bounded, convex and closed in measure for any y. Since the assumption $\alpha<p$ implies that the whole intersection of such sets is empty, from Theorem 1.3 in [6] we deduce that there exist $y_1,\dots,y_n\in Y$ such that $C(y_1)\cap C(y_2)\cap\ldots\cap C(y_n)=\varnothing$. If we assume that

$$\alpha\geqslant\min_{x\in X}\sup_{1\leqslant i\leqslant n}\Phi(x,y_i),$$

then there should exist $x_0\in X$ such that $\alpha\geqslant\Phi(x_0,y_i)$ for all y_i, $i=1,\dots,n$. Hence we would obtain that $x_0\in C(y_i)$, $i=1,\dots,n$. This contradiction shows that

$$\alpha<\min_{x\in X}\sup_{1\leqslant i\leqslant n}\Phi(x,y_i).$$

Consider now the set

$$E := \{(z,r)\in\mathbb{R}^{n+1}:\ (\exists\, x\in X)\ \Phi(x,y_i)\leqslant r+z_i,\ i=1,\dots,n\}.$$

The set E is convex. Indeed, if

$$\Phi(\overline{x},y_i)\leqslant\overline{r}+\overline{z}_i,\ i=1,\dots,n,$$
$$\Phi(\overline{\overline{x}},y_i)\leqslant\overline{\overline{r}}+\overline{\overline{z}}_i,\ i=1,\dots,n;$$

then, for $t \in [0,1]$ and for $x = t\overline{x} + (1-t)\overline{\overline{x}}$ we have

$$\begin{aligned}\Phi(x,y_i) &= \Phi(t\overline{x} + (1-t)\overline{\overline{x}}, y_i) \leqslant t\Phi(\overline{x},y_i) + (1-t)\Phi(\overline{\overline{x}},y_i)\\ &\leqslant t(\overline{r}+\overline{z}_i) + (1-t)(\overline{\overline{r}}+\overline{\overline{z}}_i) = \underbrace{t\overline{r}+(1-t)\overline{\overline{r}}}_{=r} + \underbrace{t\overline{z}_i + (1-t)\overline{\overline{z}}_i}_{=z_i}.\end{aligned}$$

Then, the point

$$\left(0,1 + \max_{1\leqslant i\leqslant n} \Phi(x,y_i)\right) \in \mathbb{R}^{n+1}$$

is interior to E for any $x \in X$. Indeed, if $|r| < \varepsilon$ and $|z_i| < \varepsilon$, $i = 1,\dots,n$, then, putting $\overline{x} = \max_{1\leqslant i\leqslant n} \Phi(x,y_i)$ we get $\Phi(\overline{x},y_i) \leqslant 1 + \Phi(\overline{x},y_i) + z_i + r$ provided $|z_i + r| < 2\varepsilon \leqslant 1$. Also, by construction $(0,\alpha) \notin E$. Indeed, $\Phi(\overline{x},y_i) \leqslant \alpha$, $i = 1,\dots,n$, for some $\overline{x} \in X$ implies that

$$\min_{x\in X} \sup_{i\leqslant i\leqslant n} \Phi(x,y_i) \leqslant \alpha$$

which contradicts the choice of α.

Now we can apply the Separation Theorem to the point $(0,\alpha) \notin E$ and to the convex set E in the finite dimensional space $\mathbb{R}^{n+1}$: note that it is not difficult to prove that the set E is also closed, though we do not need this to apply the Separation Theorem.

So we can find a vector $(\lambda_1,\dots,\lambda_n,\overline{r}) \neq 0$ with

$$\sum_{i=1}^{n} \lambda_i z_i + r\overline{r} \geqslant \overline{r}\alpha \quad ((z,r) \in E). \tag{2}$$

It is clear that $E + \mathbb{R}^{n+1}_{+} \subset E$. If it were $\overline{r} < 0$ then, taking $r > 0$ one would come to a contradiction with (2). This proves that $\overline{r} \geqslant 0$. Analogously, if $\lambda_i < 0$ for some i, then taking the corresponding entry $z_i > 0$ we would come to a contradiction with (2). Again this proves that $\lambda_i \geqslant 0$ for each $i = 1,\dots,n$. Note that actually $\overline{r} > 0$ as the point $(0, 1 + \max_{1\leqslant i\leqslant n} \Phi(x,y_i))$ lies in the interior of E. Indeed, if $\overline{r} = 0$, then $\sum_{i=1}^{n} \lambda_i z_i \geqslant 0$ for all $(z,r) \in E$. Taking (z,r) with z_i negative, and sufficiently close to 0, we find

$$\sum_{i=1}^{n} \lambda_i z_i < 0$$

since $\lambda_i \geqslant 0$, although (z,r) is still a point in E.

The point $(\Phi(x,y_i) + r, -r)$ lies in E for all $x \in X$ and all $r \in \mathbb{R}$. Indeed, $\Phi(x,y_i) \leqslant \Phi(x,y_i) + r - r$. Then, the inequality (2) implies that

$$\sum_{i=1}^{n} \lambda_i \left(\Phi(x,y_i) + r\right) + \overline{r}(-r) \geqslant \overline{r}\alpha. \tag{3}$$

Since $\overline{r} > 0$, dividing (3) by $\overline{r}$ we have

$$\sum_{i=1}^{n} \frac{\lambda_i}{\overline{r}} \Phi(x,y_i) + \left(\sum_{i=1}^{n} \frac{\lambda_i}{\overline{r}} - 1\right) r \geqslant \alpha, \tag{4}$$

for all $x \in X$ and all $r \in \mathbb{R}$.

Since $r \in \mathbb{R}$ is arbitrary, we derive that $\sum_{i=1}^{n} \frac{\lambda_i}{\overline{r}} = 1$. So, (4) can be rewritten as

$$\sum_{i=1}^{n} \frac{\lambda_i}{\overline{r}} \Phi(x, y_i) \geqslant \alpha,$$

where on the left-hand side we have a convex combination of the n points $\Phi(x, y_i)$. Then condition (b) implies, by induction, that there exists $\widehat{y} \in Y$ such that $\Phi(x, \widehat{y}) \geqslant \alpha$ for all $x \in X$, whence

$$\sup_{y \in Y} \min_{x \in X} \Phi(x, y) \geqslant \alpha.$$

Since α is arbitrary underneath p, we achieve

$$\min_{x \in X} \sup_{y \in Y} \Phi(x, y) = \sup_{y \in Y} \min_{x \in X} \Phi(x, y). \ \rhd$$

In Section 3 we will discuss some possible application of Theorem 1; for instance, it is usual to connect minimax equality with results concerning the consistency of a system of inequalities. We explicitly present a result in this direction, which can be derived from (the dual version of) Theorem 1.

Proposition 1. *Let $\mathcal{H}$ be a convex set of concave functionals defined on a set $X \subset L^1(\mu)$ which is closed, norm-bounded and closed in measure. Also assume that each $f \in \mathcal{H}$ is upper semicontinuous in measure on X. Suppose that for every $f \in \mathcal{H}$ there exists $x_f \in X$ such that $f(x_f) \geqslant 0$. Then there exists a point $x_0 \in X$ such that $f(x_0) \geqslant 0$ for all $f \in \mathcal{H}$.*

Remarks. (1) Since OWC technique is true for more general situation then L^1, e. g., for perfect Banach Function Spaces and their vector-valued generalizations when the corresponding Banach space of values is reflexive (see [6]), then the results here and below extend to that more general setting.

(2) Using OWC technique, Levin derived a minimax theorem (see [6; Theorem 4.1]) but he confined himself with convex-concave setting, which is both easier for the proof and not much interesting in applications. Actually, only the approach from [5] gave us the tools for the current general result.

2. Weaker Convexity Conditions

In the literature connected with minimax relationships and related topics, many generalizations of the notion of convexity have been proposed.

Definition 1. A function $\Phi : X \times Y \to \mathbb{R}$ is *finite convex-like with respect to x* if, for each finite set $\{y_1, \ldots, y_n\} \subset Y$, each pair $x_1, x_2 \in X$ and every $t \in [0, 1]$, an element $x_3 \in X$ exists, such that

$$\Phi(x_3, y_i) \leqslant t\Phi(x_1, y_i) + (1 - t)\Phi(x_2, y_i) \tag{5}$$

for every $i = 1, \ldots, n$.

This definition has been introduced by Granas and Liu [9] in the following more general way:

Definition 2. A function $\Phi : X \times Y \to \mathbb{R}$ is *midpoint finite convex-like with respect to x* if, for each finite set $\{y_1, \ldots, y_n\} \subset Y$ and each pair $x_1, x_2 \in X$, an element $x_3 \in X$ exists, such that

$$\Phi(x_3, y_i) \leqslant \frac{1}{2} \left[\Phi(x_1, y_i) + \Phi(x_2, y_i)\right], \tag{6}$$

for every $i = 1, \dots, n$.

It is quite clear that if condition (a) of Section 1 is replaced by

(a_1) Φ is finitely convex-like with respect to x;

(a_2) for every $y \in Y$ the map $x \mapsto \Phi(x, y)$ is quasi-convex,

then Theorem 1 is still true.

It is less trivial to note that the result remains true if (a_1) is replaced by the midpoint finite convex-likeness with respect to x. In fact the following result holds:

Proposition 2. *Let X be a convex, norm-bounded set in $L^1(\mu)$, Y any non-empty set and $\Phi\colon\ X \times Y \to \mathbb{R}$ be such that*

(i) *for each $y \in Y$, the map $x \mapsto \Phi(x, y)$ is lower semicontinuous in measure;*

(ii) *for each $y \in Y$, the map $x \mapsto \Phi(x, y)$ is quasi-convex;*

(iii) *Φ is finitely midpoint convex-like with respect to x.*

Then Φ is finitely convex-like with respect to x.

$\lhd$ Let $\{y_1, \dots, y_n\} \subset Y$ and $x_1, x_2 \in X$ be fixed.

Claim 1: For every $q \in \mathbb{Q}(2)$ (the set of dyadic rational numbers in $[0, 1]$) there exists $x_q \in X$ such that

$$\Phi(x_q, y_i) \leqslant q\Phi(x_1, y_i) + (1 - q)\Phi(x_q, y_i) \tag{7}$$

for each $i = 1, \dots, n$.

This is a standard iterative argument. The proof is that of ([17; Lemma 3.2]).

Claim 2: Let $(C_n)_n$ be a non increasing sequence of convex subsets of $L^1(\mu)$ which are closed in measure. Then

$$\bigcap_n C_n = \{x \in L^1(\mu)\colon\ x = (\mu)\text{-}\lim x_n,\ x_n \in C_n\} =: B.$$

The inclusion $\bigcap_n C_n \subseteq B$ is immediate. Conversely, since the sequence is decreasing, and each set C_n is closed in measure, it is clear that $B \subseteq C_n$ for every n.

Claim 3: Φ is finitely convex-like with respect to x.

Let $t \in [0, 1]$ be fixed, and let $(t_k)_k$ be a sequence in $\mathbb{Q}(2)$ with $t_k \to t$. From Claim 1 we can determine $x_k \in X$ such that

$$\Phi(x_k, y_i) \leqslant t_k\Phi(x_1, y_i) + (1 - t_k)\Phi(x_2, y_i)$$

for every $i = 1, \dots, n$ and every $k \in \mathbb{N}$.

Since $(x_k) \subset X$, and X is by assumption norm bounded, by Theorem 1.4 in [6] there are: a sequence of integers $1 = k_1 < k_2 < \dots$, a sequence of non-negative numbers $(\lambda_p)_p$ with $\displaystyle\sum_{j=k_i}^{k_{i+1}-1} \lambda_j = 1$ and such that the sequence $g_n = \displaystyle\sum_{j=k_i}^{k_{i+1}-1} \lambda_j x_j$ converges μ-a. e. to an element $x \in E$.

For each $i = 1, \dots, n$ and every p, define the number

$$M(i, p) = \max\{t_j\Phi(x_1, y_i) + (1 - t_j)\Phi(x_2, y_i),\ j = k_p, \dots, k_{p+1} - 1\}$$

and note that, since $t_j \to t$, then

$$\lim_j [t_j\Phi(x_1, y_i) + (1 - t_j)\Phi(x_2, y_i)] = t\Phi(x_1, y_i) + (1 - t)\Phi(x_2, y_i) =: \tau_i$$

for every $i = 1, \dots, n$. Hence $\lim_p M(i, p) = \tau_i$, for $i = 1, \dots, n$.

Set now

$$C_{i,p} = C(M(i,p)) = \{x \in X\colon\ \Phi(x,y) \leqslant M(i,p),\ i = 1,\dots,n,\ p \in \mathbb{N}\}$$

and observe that (i) and (ii) ensure that each $C_{i,p}$ is convex and closed in measure. Hence, since for each integer k in $[k_p, k_{p+1} - 1]$ the corresponding $x_k \in C_{i,p}$ for $i = 1,\dots,n$ then $g_p \in C_{i,p}$ for $i = 1,\dots,n$ and $p \in \mathbb{N}$.

By a diagonal argument, we can find a subsequence $\{p_r, r \in \mathbb{N}\}$ such that $(M(i,p))$ is monotone for $i = 1,\dots,n$. Let

$$J_1 = \{i \in \{1,\dots,n\} \text{ such that } (M(i,p_r))_r \text{ is non decreasing}\},$$
$$J_2 = \{i \in \{1,\dots,n\} \text{ such that } (M(i,p_r))_r \text{ is non increasing}\}.$$

Now, for $i \in J_1$ the sequence $(C_{i,p_r})_r$ is non decreasing with respect to r, and $C_{i,p_r} \subseteq C(\tau_i)$. Hence $g_{p_r} \in C(\tau_i)$. Since $C(\tau_i)$ is closed in measure, and $(g_{p_r}$ μ-converges to x, $x \in C(\tau_i)$, namely,

$$\Phi(x,y_i) \leqslant \tau_i = t\phi(x_1,y_i) + (1-t)\Phi(x_2,y_i)$$

for $i \in J_1$.

For $i \in J_2$, the sequence $(C_{i,p_r})_r$ is non increasing; as $g_{p_r} \in C_{i,p_r}$, by Claim 2, $x \in \bigcap_r C_{i,p_r}$. But, as it is easily seen, in this case

$$\bigcap_r C_{i,p_r} \subseteq C(\tau_i).\ \ \triangleright$$

Remark. Note that Proposition 2 remains true in any topological vector space, if X has a sort of the Mazur property, that is, if from every sequence in X we can construct a sequence of convex combinations of terms with far numbers that converges in the topology of E.

3. Applications

Here we present some directions of possible applications of minimax theorems given above and OWC technique in general. In some cases we have developed results, in the other just an understanding of some links.

1. Duality for $\varphi(X,Y)$ spaces. For Banach Function Spaces X and Y on the same measure space and a concave function $\varphi(x,y)$ of two variables one can define the space $\varphi(X,Y)$ (see the details in [14]). The principal duality equality established by Lozanovskii reads as $\varphi(X,Y)' = \hat{\varphi}(X',Y')$, where prime stands for Banach Function Space dual and $\hat{\varphi}$ stands for a suitable convolution operator. This result has been refined both in isomorphic and isometric form by Lozanovskii himself (see [15, 16]), Berezhoi [1, 2] and Reisner [19]. Note that Lozanovskii used OWC technique (separation theorem). Reisner was able to reduce the problem for power function φ to finite-dimensional functions; to do this he had to use Ky Fan's minimax theorem with convex-concave-like assumptions as in our Theorem 1. For Theorem 1 we do not need the machinery of such reduction dealing directly with infinite-dimensional setting. Here we are able to prove all types of duality theorems quoted above.

2. Markov–Kakutani theorem. In [3] a version of the Markov–Kakutani theorem was presented for commuting families of affine contractions on norm bounded convex sets in L^1, which are closed in measure. In [10] a similar result was published for weakly continuous affine mappings (not necessarily contractions). Both proofs are based on OWC technique. Using OWC technique and [20] we are able to prove an analogous result for families of mappings, which are continuous in measure. We do not know the correspondence between the last two results.

3. Stochastic games. Using Theorem 1 and OWC technique it is possible to establish very general infinite-dimensional results on existence of equilibrium in stochastic games (cf. [4; Appendix]).

4. Point estimates in statistics. It is possible to establish some existence theorems for minimax point estimates in mathematical statistics (cf. [12]).

5. Miscellanea. In factorization Theorem 109 [18] we can drop the condition of reflexivity for the Orlich space L_θ because of Proposition 1.

In [8], the proof for existence of an equivalent martingale measure is given. The major step there is to move from an easily constructed element of the second dual to an element from L^1. Though we do not have complete proof by means of OWC we would like to note that this issue is obviously related to our topic (see [8; Appendix 1]).

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