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SOME OPEN QUESTIONS ON POSITIVE OPERATORS IN BANACH LATTICES

E. Yu. Emel'yanov

Dedicated to S. S. Kutateladze
on occasion of his 60th birthday

Recently, some new results on asymptotic behaviour of positive operators in Banach lattices were obtained. Here we discuss some open problems related to these results.

1. Mean ergodicity and weak almost periodicity of positive operators

1.1. Given an operator T in a Banach space X, denote the *Cesàro averages* of T by

$$\mathcal{A}_\tau^T := \frac{1}{\tau}\sum_{k=0}^{\tau-1} T^k.$$

An operator T is called *Cesàro bounded* if $\sup_n \|\mathcal{A}_n^T\| < \infty$, *power bounded* if $\sup\{\|T^n\| : n \in \mathbb{N}\} < \infty$, and *doubly power bounded* if $\sup\{\|T^n\| : n \in \mathbb{Z}\} < \infty$. There exist operators which are Cesàro bounded but not power bounded. An operator T is called *mean ergodic* if the norm limit $\lim\limits_{n\to\infty} \mathcal{A}_n^T x$ exists for all $x \in X$. Each mean ergodic operator T is Cesàro bounded and satisfies

$$\lim_{n\to\infty} \|n^{-1}T^n x\| = 0 \quad (x \in X). \tag{1}$$

An operator T is called *weakly almost periodic* if its *orbit* $\{T^n x : n \geqslant 0\}$ is conditionally weakly compact for all x. It follows from Eberlein's mean ergodic theorem that any weakly almost periodic operator is mean ergodic. For an elementary introduction to the ergodic theorems on operators in Banach spaces, we refer to Krengel [13] and to Lyubich [14].

Below we will use also basic results and terminology from the Banach lattice theory. We refer to [2] and [15] for them.

1.2. The first interesting question arises from the following simple assertion: *given a bounded operator T in a Banach space, if T^m is mean ergodic for some natural m, then T is mean ergodic*. This fact follows immediately from the equality

$$\mathcal{A}_{nm}^T = \frac{1}{m}\sum_{k=0}^{m-1} T^k \mathcal{A}_n^{T^m} \quad (n, m \in \mathbb{N}),$$

and from Eberlein's mean ergodic theorem.

In general, the converse of the assertion above is not true, even for positive contractive operators on a Banach lattice $C(K)$, for appropriate compact Hausdorff space K. Corresponding example was constructed in Sine's paper [19]. However, for a positive power bounded operator S on a Banach lattice with order continuous norm, from the mean ergodicity of S follows the mean ergodicity of S^m for all $m \in \mathbb{N}$, as was shown by Derriennic and Krengel in [4]. The construction of example in [19] is quite technical, and cannot be used directly in any $C(K)$. This motivates the following question:

Open question 1. *Let K be an infinite compact Hausdorff space. Is there a positive power bounded mean ergodic operator T in $C(K)$ such that T^m is not mean ergodic for some $m \in \mathbb{N}$?*

1.3. Another interesting question is related to Sine's mean ergodic theorem [18]:

Let T be a Cesàro bounded operator which satisfies (1). *Then T is mean ergodic if and only if the space* $\mathrm{Fix}(T)$ *of fixed vectors of T separates the space* $\mathrm{Fix}(T^*)$ *of fixed vectors of T^*.*

What can be said in the case of positive operators in Banach lattices? More precisely:

Open question 2. *Let T be a positive Cesàro bounded operator in a Banach lattice such that* (1) *is satisfied and positive fixed vectors of T separates* $\mathrm{Fix}(T^*)$. *Is T mean ergodic*?

1.4. It was shown by Komornik [11] and by Kornfeld and Lin [12] that any mean ergodic Markov operator on L^1-space is weakly almost periodic. In general this result is not true for contractive positive operators in Banach lattices. To show this we can take the operator T on $C(K)$ constructed in [19]. This operator is mean ergodic but not weakly almost periodic, since T^2 is not mean ergodic.

Open question 3. *Let T be a mean ergodic positive power bounded operator on a Banach lattice with order continuous norm. Is T weakly almost periodic*?

Even for the Banach lattice c_0, the answer seems to be unknown. In the case if the answer for c_0 is negative, the following question arises:

Open question 4. *Let T be a mean ergodic positive power bounded operator on a KB-space. Is T weakly almost periodic*? *Does this property of positive power bounded operators characteriz KB-spaces among Banach lattices*?

1.5. Any closed ball in a reflexive Banach space is weakly compact. From this fact and Eberlein's mean ergodic theorem it follows that every power bounded operator in a reflexive Banach space is mean ergodic. Is the converse true? This still open question was suggested by Sucheston [20]. The problem was solved in positive for Banach lattices in [6, 8, 16, 21]; and, for Banach spaces with a basis, in [9]. Moreover, in [21] it was shown that every σ-Dedekind complete Banach lattice in which every positive power bounded operator is mean ergodic is reflexive. The following question is open.

Open question 5. *Let E be a Banach lattice such that every positive power bounded operator on E is mean ergodic. Is E reflexive*?

1.6. We finish this section with the following problem. It was proved by Neubrander [5], that any power bounded positive operator T on an AL-space X such that 0 belongs to the weak-closure of the orbit $\{T^n x\}_{n=1}^{\infty}$ for each $x \in X$ satisfies a formally more strong condition:

$$w\text{-}\lim_{n\to\infty} T^n x = 0 \quad (x \in X).$$

It is well known fact that in general this is not true for power bounded operators in Banach spaces. However it is an interesting open question to extend this property on positive operators in Banach lattices:

Open question 6. *Let T be a positive power bounded operator in a Banach lattice which satisfies $0 \in w\text{-cl}\{T^n x\}_{n=1}^{\infty}$ for each $x \in X$. Does $w\text{-}\lim\limits_{n\to\infty} T^n x = 0$ hold for all $x \in X$?*

2. Positive operator whose spectrum is singleton

2.1. Let T be a positive operator on $E = \mathbb{R}^n$ such that $\sigma(T) = \{1\}$. Using the elementary linear algebra one can show that $T \geqslant I_E$ where I_E is the identity operator on E. This simple result leads to the following natural question:

Open question 7. *Let T be a positive operator on an infinite-dimensional ordered Banach space E such that $\sigma(T) = \{1\}$. Does T satisfy $T \geqslant I_E$?*

This question is probably one of the several most intrigue questions in operator theory like *the invariant subspace problem for operators on separable Hilbert space*, because of there is no infinite-dimensional ordered Banach space for which this question is solved in positive or negative.

2.2. For Banach lattices, Open question 7 was solved in positive in the class of lattice homomorphisms by Arendt, Schaefer, and Wolff [17]. For this and related results, we refer to Meyer-Nieberg's book [15]. It is still unknown if this true for other interesting classes of positive operators on Banach lattices. Sometimes, it is better to consider the following versions of this problem:

Open question 8. *Let T be a contractive* (i. e., $\|T\| \leqslant 1$) *positive operator on a Banach lattice E such that $\sigma(T) = \{1\}$. Does T satisfy $T = I_E$?*

Open question 9. *Let T be a Markov operator on a $C(K)$ such that $\sigma(T) = \{1\}$. Does T satisfy $T = I_{C(K)}$?*

2.3. There are another interesting open question, closely related to ones discussed above. A Banach lattice E is called a *Banach lattice algebra* if, in addition to linear and lattice operations, the operation of multiplication (associative, but not necessary commutative) is defined on E and satisfies:

$$a \geqslant 0, b \geqslant 0 \Rightarrow a \cdot b \geqslant 0, \quad \|a \cdot b\| \leqslant \|a\|\|b\|.$$

The second condition means that E is a Banach algebra. We assume also that E as a Banach algebra possesses a positive unit element. Usually, Banach lattice algebras are considered over $\mathbb{R}$, though this definition can be modified for the complex case.

The famous theorem due to Gelfand and Mazur says that if in a Banach algebra every non-zero element is invertible then this algebra is isometrically isomorphic to $\mathbb{C}$. Despite the Gelfand–Mazur's theorem is known for the long time, its analogue for Banach lattice algebras is a long standing open question.

Open question 10. *Let E be a Banach lattice algebra in which every nonzero positive element has inverse. Is E isomorphic to $\mathbb{R}$?*

This problem was solved in positive only for some special classes of Banach lattice algebras. As far as the author knows, the answer is positive for commutative algebras and for algebras in which every nonzero positive element has positive inverse. The last result is due to Huijsmans [10].

3. Invariant subspace problem for positive operators

3.1. The classical form of the invariant subspace problem is the following: *does every bounded operator in a separable Hilbert space possess a non-trivial invariant subspace*? This problem is still open, however it was solved in positive for many interesting classes of operators in Hilbert spaces. More general such a question can be asked for bounded operators in separable Banach spaces. It was proved by Enflo that the invariant subspace problem for Banach spaces has a negative solution. It is interesting that examples of operators without non-trivial invariant subspaces exist in a such well studied Banach lattice as l^1. But all operators in such examples are non-positive. This leads to the following question:

Open question 11. *Does any positive operator in a separable Banach lattice possess a non-trivial invariant subspace*?

For discussion of this and many related problems we refer to the book [2].

3.1. It was shown by Abramovich, Aliprantis, and Burkinshaw (1993) that every locally quasi-nilpotent positive operator in l^p, for $1 \leqslant p < \infty$, has a non-trivial invariant subspace. This motivates the following two problems:

Open question 12. *Does every* (*locally*) *quasi-nilpotent positive operator in a separable Banach lattice possess a non-trivial invariant subspace*?

Open question 13. *Does every positive operator in* l^p, *for* $1 \leqslant p < \infty$, *possess a non-trivial invariant subspace*?

4. The renorming problem for positive operators

4.1. Let T be a doubly power bounded operator on a Banach space X. Then there exists an equivalent norm on X under which T is a surjective isometry. We can define such a norm $\|\cdot\|_T$ as follows:

$$\|x\|_T := \sup\{\|T^n x\| : n \in \mathbb{Z}\} \qquad (x \in X). \tag{2}$$

However, the situation is different if we consider a positive operator T on a Banach lattice. The norm in (2) may not be a Banach lattice norm, since we do not know if $T^{-1} \geqslant 0$. If this occurs, then it is well known that T is a lattice automorphism and this will make the norm defined in (2) an equivalent lattice norm.

It is known that the positivity of T^{-1} occurs for any doubly power bounded positive operator in a finite-dimensional Banach lattice. On the other hand due to Abramovich's theorem [1] the positivity of T^{-1} occurs also if T is a surjective positive isometry. An example of a positive doubly power bounded operator T in $L^1(\mathbb{R})$ such that $T^{-1} \not\geqslant 0$ was constructed in [7]. Recently [3] this result was extended to all AL-spaces. But the technique which was used in [7] and [3] cannot be applied to other types of Banach lattices. This motivates the following question:

Open question 14. *Let E be an infinite-dimensional Banach lattice. Is there a positive doubly power bounded operator T in E such that T^{-1} is not positive*?

4.2. It follows from the above-mentioned result of [3] that in a dual infinite dimensional $C(K)$-space there is a positive doubly power bounded operator with non-positive inverse. However, not all $C(K)$-spaces are dual.

Open question 15. *Let K be an infinite compact Hausdorff space. Is there a positive doubly power bounded operator T in $C(K)$ such that T^{-1} is not positive*?

References

1. *Abramovich Yu. A.* Isometries of normed lattices // Optimizatsiya.—1988.—V. 43 (60).—P. 74–80.
2. *Abramovich Yu. A., Aliprantis C. D.* An invitation to operator theory // Graduate Studies in Mathematics, V. 50.—2002, Providence, RI: American Mathematical Society.—xiv+530 p.
3. *Alpay S., Binhadjah A., Emel'yanov E. Yu.* A Positive Doubly Power Bounded Operator on an *AL*-Space with a Nonpositive Inverse. [Preprint]
4. *Derriennic Y., Krengel U.* Subadditive mean ergodic theorems // Ergodic Theory Dynamical Systems.—1981.—V. 1, № 1.—P. 33–48.
5. *Eisner T.* Privat communication.
6. *Emel'yanov E. Yu.* Banach lattices on which every power-bounded operator is mean ergodic // Positivity.—1997.—V. 1, № 4.—P. 291–296.
7. *Emel'yanov E. Yu.* A remark to a theorem of Yu. A. Abramovich // Proc. Amer. Math. Soc.—2004.—V. 132, № 3.—P. 781–782.
8. *Emel'yanov E. Yu., Wolff M. P. H.* Mean ergodicity on Banach lattices and Banach spaces // Arch. Math. (Basel).—1999.—V. 72, № 3.—P. 214–218.
9. *Fonf V. P., Lin M., Wojtaszczyk P.* Ergodic Characterization of Reflexivity of Banach spaces // J. Funct. Anal.—2001.—V. 187.—P. 146–162.
10. *Huijsmans C. B.* Lattice-ordered division algebras // Proc. Roy. Irish Acad. Sect.—1992.—V. A 92, № 2.—P. 239–241.
11. *Komornik J.* Asymptotic periodicity of Markov and related operators // Dynamics reported, Dynam. Report. Expositions Dynam. Systems (N.S.).—Berlin: Springer, 1993.—V. 2.—P. 31–68.
12. *Kornfeld I., Lin M.* Weak almost periodicity of L_1-contractions and coboundaries of non-singular transformations // Studia Math.—2000.—V. 138, № 3.—P. 225–240.
13. *Krengel U.* Ergodic Theorems.—Berlin–New York: De Gruyter, 1985.
14. *Lyubich Yu. I.* Introduction to the Theory of Banach Representations of Groups.—Basel, Boston, Berlin: Birkhäuser, 1988.
15. *Meyer-Nieberg P.* Banach Lattices // Universitext.—Berlin: Springer-Verlag, 1991.
16. *Räbiger F.* Ergodic Banach lattices // Indag. Math. (N.S.).—1990.—V. 1, № 4.—P. 483–488.
17. *Schaefer H. H., Wolff M. P. H., Arendt W.* On lattice isomorphisms with positive real spectrum and groups of positive operators // Math. Z.—1978.—V. 164, № 2.—P. 115–123.
18. *Sine R.* A mean ergodic theorem // Proc. Amer. Math. Soc.—1970.—V. 24, № 2.—P. 438–439.
19. *Sine R.* A note on the ergodic properties of homeomorphisms // Proc. Amer. Math. Soc.—1976.—V. 57, № 1.—169–172.
20. *Sucheston L.* Open questions, probability in Banach spaces // Oberwolfach.—Berlin–Heidelberg–New York: Springer, 1975.—P. 285–289. (Lecture Notes in Math.—1976.—№ 526.)
21. *Zaharopol R.* Mean ergodicity of power-bounded operators in countably order complete Banach lattices // Math. Z.—1986.—V. 192, № 1.—P. 81–88.

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Prof. Emel'yanov Eduard Yu.
Turke, Ankaray, Department of Mathematics
Middle East Technical University, 06531
E-mail: `emelanov@math.metu.edu.tr`