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A PRIORI ESTIMATE RESULT FOR AN INVERSE PROBLEM OF TRANSPORT THEORY

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We establish a priori estimate result for an inverse problem of transport theory. We refer to [1], where some existence and uniqueness result are proved.

1. Description of the problem

Consider the following problem :

$$\frac{\partial u}{\partial t} + (v, \nabla_x)u + \Sigma(x, v, t)u(x, v, t) = \int\limits_V J(x, v', t, v)u(x, v', t)dv' + F(x, v, t), \tag{1}$$

$(x, v, t) \in D = G \times V \times (0, T)$, where $u(x, v, t)$ characterize the distribution density of particles in the phase space $G \times V$ at the moment $t \in]0, T[$. The absorption coefficient $\Sigma(x, v, t)$, the dissipation indicator $J(x, v', t, v)$ and the interior source function $F(x, v, t)$ represent the environment where this process moves in. Assume that G is a stricly convex, bounded domain in $\mathbb{R}^n$ and assume also that the boundary ∂G of G is of class $\mathcal{C}^1$. Put

$$V = \{v \in \mathbb{R}^n : 0 < v_0 \leqslant | v | \leqslant v_1\},$$

where v_0, v_1 are two positive reals such that $v_0 < v_1$. Let $\Omega = G \times V$, $F(x, v, t) = f(x, v)g(x, v, t) + h(x, v, t)$. According to [1], if we give all the characteristic of the environment Σ, J, F as well as the out going flow, i. e.

$$u(x, v, t) = \mu(x, v, t),\ (x, v, t) \in \Gamma_+ = \Upsilon_+ \times [0, T], \tag{2}$$

where

$$\Upsilon_+ = \{(x, v) \in \partial G \times V : (v, n_x) > 0\},$$

and n_x is the exterior normal to the boundary ∂G of domain G at the point x; moreover, if the initial state

$$u(x, v, 0) = \varphi(x, v), \quad (x, v) \in \overline{G} \times V \tag{3}$$

and the final state

$$u(x, v, T) = 0,\ (x, v) \in \overline{G} \times V \tag{4}$$

of process are given, then $\exists!(u, f) \in \mathcal{C}^1_{t,(v,\nabla)}(\overline{D}) \times \mathcal{C}(\overline{\Omega})$ such that (1)–(4) hold. In other words, there exists a control $f \in \mathcal{C}(\overline{\Omega})$ such that every initial state $\varphi(x, v)$ is controllable to the

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equilibrium state 0 along a trajectory of the system (1)–(2) on $[0, T]$. We first recall some basic functional spaces which will be used later in order to study the continuous dependence of the solution of the problem (1)–(4). So we use the following notations:

$$\mathcal{C}^1_{t,(v,\nabla)}(\overline{D}) = \left\{ u \in \mathcal{C}(\overline{D}) : \ \frac{\partial u}{\partial t} \in \mathcal{C}(\overline{D}), \ (v, \nabla_x)u \in \mathcal{C}(\overline{D}) \right\}$$

under the norm

$$\|u\|_{\mathcal{C}^1_{t,(v,\nabla)}(\overline{D})} = \|u\|_{\mathcal{C}(\overline{D})} + \|\frac{\partial u}{\partial t}\|_{\mathcal{C}(\overline{D})} + \|(v, \nabla_x)u\|_{\mathcal{C}(\overline{D})};$$

$$\mathcal{C}^1_t(\overline{D}) = \{h \in \mathcal{C}(\overline{D}) : \ \frac{\partial h}{\partial t} \in \mathcal{C}(\overline{D})\}$$

under the norm

$$\|h\|_{\mathcal{C}^1_t(\overline{D})} = \|h\|_{\mathcal{C}(\overline{D})} + \|\frac{\partial h}{\partial t}\|_{\mathcal{C}(\overline{D})};$$

$$\mathcal{C}^1_{(v,\nabla)}(\overline{\Omega}) = \{\varphi \in \mathcal{C}(\overline{\Omega}) : \ (v, \nabla)\varphi \in \mathcal{C}(\overline{\Omega})\}$$

under the norm

$$\|\varphi\|_{\mathcal{C}^1_{(v,\nabla)}(\overline{\Omega})} = \|\varphi\|_{\mathcal{C}(\overline{\Omega})} + \|(v, \nabla)\varphi\|_{\mathcal{C}(\overline{\Omega})};$$

$$\mathcal{C}^1_t(\Gamma_+) = \{\mu \in \mathcal{C}(\Gamma_+) : \ \frac{\partial \mu}{\partial t} \in \mathcal{C}(\Gamma_+)\}$$

under the norm

$$\|\mu\|_{\mathcal{C}^1_t(\Gamma_+)} = \|\mu\|_{\mathcal{C}(\Gamma_+)} + \| \ \frac{\partial \mu}{\partial t}\|_{\mathcal{C}(\Gamma_+)}.$$

Let

$$\alpha(x, v) = \max \{t \in [0, T] : \ x + vt \in \partial G\}, \quad (x, v) \in \overline{G} \times V.$$

Put

$$d = \sup_{(x,v) \in \overline{G} \times V} \alpha(x, v)$$

and assume that $d < T$. Then, according to [1], $\alpha \in \mathcal{C}^1_{(v,\nabla)}(\overline{\Omega})$ and $(v, \nabla)\alpha = -1$.

2. Continuous Dependence

We need first a very useful theorem:

Theorem 1. *Let X be a Banach space, Y be a normed space. Let also $A : X \to X$ and $B : Y \to X$ be two linear continuous operators. Assume that $\|A\| \leqslant q < 1$. Then*

$$(\forall y \in Y)(\exists!\, x \in X) \ x = Ax + By \text{ and } \|x\| \leqslant \frac{1}{1-q}\|B\|\|y\|.$$

The problem is, then, that of controling the system (1)–(4), where

$$J \in \mathcal{C}^1_t(\overline{D} \times V), \ \mu \in \mathcal{C}^1_t(\Gamma_+), \ \varphi \in \mathcal{C}^1_{(v,\nabla)}(\overline{\Omega}), \ \Sigma \in \mathcal{C}^1_t(\overline{D}), \ h \in \mathcal{C}^1_t(\overline{D}), \ g \in \mathcal{C}^1_t(\overline{D}),$$

$$0 < g_0 \leqslant g(x, v, 0) \quad (\forall (x, v) \in \overline{\Omega}).$$

We now prove that there exists a unique solution which depends continuously on the form of the right-hand side of the problem (1)–(3):

Theorem 2. *There is $a > 0$ such that if $d < a$ and if the conditions*

$$\varphi(x, v) = \mu(x, v, 0), \quad (x, v) \in \Upsilon_+,$$

$$\frac{\partial \mu}{\partial t}(x, v, T) = h(x, v, T), \quad (x, v) \in \Upsilon_+,$$

$$\frac{\partial \mu}{\partial t}(x, v, 0) + (v, \nabla)\varphi + \Sigma(x, v, 0)\varphi(x, v) - \int\limits_V J(x, v', 0, v)\varphi(x, v')dv' - h(x, v, 0) = 0,$$

$$(x, v) \in \Upsilon_+, \quad \mu(x, v, T) = 0, \quad (x, v) \in \Upsilon_+$$

holds, then there exists a unique solution $(u, f) \in \mathcal{C}^1_{t,(v,\nabla)}(\overline{D}) \times \mathcal{C}(\overline{\Omega})$ of the problem (1)–(4). Moreover, $f/\Upsilon_+ = 0$ and

$$\|(u, f)\|_{\mathcal{C}^1_t(\overline{D}) \times \mathcal{C}(\overline{\Omega})} \leqslant c\left(\|\mu\|_{\mathcal{C}^1_t(\Gamma_+)} + \|\varphi\|_{\mathcal{C}^1_{(v,\nabla)}(\overline{\Omega})} + \|h\|_{\mathcal{C}^1_t(\overline{D})}\right),$$

where $c > 0$.

$\lhd$ Put $V = \left\{(u, f) \in \mathcal{C}^1_t(\overline{D}) \times \mathcal{C}(\overline{\Omega}) : \ u(x, v, T) = 0, \ (x, v) \in \overline{G} \times V, \ f/\Upsilon_+ = 0\right\}$.

Note that V is a Banach space under the norm $\|(u, f)\|_V = \|u\|_{\mathcal{C}^1_t(\overline{D})} + \|f\|_{\mathcal{C}(\overline{\Omega})}$. According to [1], the problem (1)–(4) is equivalent to fixed point problem $A(u, f) = (u, f)$, where A is an operator from V into V defined by $A(u, f) = (A_1(u, f), A_2(u, f))$, and where A_1 et A_2 are defined as follows:

$$[A_1(u, f)](x, v, t) = \begin{cases} \mu(x + \alpha v, v, t + \alpha) - \int\limits_0^\alpha (Pu + fg + h)(x + v(\alpha - \tau), v, t + \alpha - \tau)d\tau, \\ \qquad\qquad if \ t + \alpha < T, \\ - \int\limits_{\alpha+t-T}^{\alpha} (Pu + fg + h)(x + v(\alpha - \tau), v, t + \alpha - \tau)d\tau, \\ \qquad\qquad if \ t + \alpha \geqslant T; \end{cases}$$

$$(x, v) = \frac{1}{g(x, v, 0)}\Bigg[- \int\limits_0^\alpha f(x + (\alpha - \tau)v, v)\frac{\partial g}{\partial t}(x + (\alpha - \tau)v, v, \alpha - \tau)d\tau - \int\limits_0^\alpha \frac{\partial(Pu + h)}{\partial t}(x + (\alpha - \tau)v, v, \alpha - \tau)d\tau + \theta(x, v)\Bigg],$$

where

$$\theta(x, v) = (v, \nabla)\varphi + \frac{\partial \mu}{\partial t}(x + \alpha v, v, \alpha) - h(x, v, 0) + \Sigma(x, v, 0)\varphi(x, v) - \int\limits_V J(x, v', 0, v)\varphi(x, v')dv',$$

$$(Pu)(x, v, t) = -\Sigma(x, v, t)u(x, v, t) + \int\limits_V J(x, v', t, v)u(x, v', t)dv'.$$

By [1], there is $a > 0$ such that if $d < a$, then A^2 is a contracting operator on V. So $\exists!(u, f) \in V$ such that $A^2(u, f) = (u, f)$ and thus, $A(u, f) = (u, f)$.

We now show that (u,f) depends continuously of μ, φ and h. For this, consider

$$Y=\Bigg\{(h,\mu,\varphi)\in\mathcal{C}_t^1(\overline{D})\times\mathcal{C}_t^1(\Gamma_+)\times\mathcal{C}_{(v,\nabla)}^1(\overline{\Omega}):\ \frac{\partial\mu}{\partial t}(x,v,T)=h(x,v,T),$$
$$\mu(x,v,0)=\varphi(x,v),\ \mu(x,v,T)=0,\ \frac{\partial\mu}{\partial t}(x,v,0)+(v,\nabla)\varphi+\Sigma(x,v,0)\varphi(x,v)$$
$$-\int\limits_V J(x,v',0,v)\varphi(x,v')dv'-h(x,v,0)=0,\quad (\forall(x,v)\in\Upsilon_+)\Bigg\}$$

under the norm
$$\|(h,\mu,\varphi)\|_Y=\|\mu\|_{\mathcal{C}_t^1(\Gamma_+)}+\|h\|_{\mathcal{C}_t^1(\overline{D})}+\|\varphi\|_{\mathcal{C}_{(v,\nabla)}^1(\overline{\Omega})}.$$

Let us remark that $A(u,f)$ can be written as $A(u,f)=\varrho(u,f)+\beta(h,\mu,\varphi)$, where $(h,\mu,\varphi)\in Y$, $(u,f)\in V$ and ϱ, β are such that

$$\varrho(u,f)=(\varrho_1(u,f),\varrho_2(u,f)),$$
$$\beta(h,\mu,\varphi)=(\beta_1(h,\mu,\varphi),\beta_2(h,\mu,\varphi)),$$

with ϱ_1, ϱ_2, β_1, β_2 are defined by:

$$[\varrho_1(u,f)](x,v,t)=\begin{cases}-\int\limits_0^\alpha(Pu+fg)(x+v(\alpha-\tau),v,t+\alpha-\tau)d\tau, & if\ t+\alpha<T,\\ -\int\limits_{\alpha+t-T}^{\alpha}(Pu+fg)(x+v(\alpha-\tau),v,t+\alpha-\tau)d\tau, & if\ t+\alpha\geqslant T,\end{cases}$$

$$[\varrho_2(u,f)](x,v)=-\frac{1}{g(x,v,0)}\Bigg[\int\limits_0^\alpha f(x+(\alpha-\tau)v,v)\frac{\partial g}{\partial t}(x+(\alpha-\tau)v,v,\alpha-\tau)d\tau$$
$$+\int\limits_0^\alpha\frac{\partial Pu}{\partial t}(x+(\alpha-\tau)v,v,\alpha-\tau)d\tau\Bigg],$$

$$[\beta_1(h,\mu,\varrho)](x,v,t)=\begin{cases}\mu(x+\alpha v,v,t+\alpha)-\int\limits_0^\alpha h(x+v(\alpha-\tau),v,t+\alpha-\tau)d\tau, & if\ t+\alpha<T,\\ -\int\limits_{\alpha+t-T}^{\alpha}h(x+v(\alpha-\tau),v,t+\alpha-\tau)d\tau, & if\ t+\alpha\geqslant T,\end{cases}$$

$$[\beta_2(h,\mu,\varrho)](x,v)=-\frac{1}{g(x,v,0)}\Bigg[\int\limits_0^\alpha\frac{\partial h}{\partial t}(x+(\alpha-\tau)v,v,\alpha-\tau)d\tau-\theta(x,v)\Bigg].$$

We remark clearly that $\varrho(u,f)\in V$ and $\beta(h,\mu,\varphi)\in V$. Note that ϱ is a linear bounded operator from V into V and β is a linear bounded operator from Y into V. On the other hand, reasoning as in [1], we may conclude that if $d<a$, then

$$\|\varrho^2\|<1$$

and consequently,

$$\exists!(u_0,f_0)\in V:(u_0,f_0)=\varrho^2(u_0,f_0)+(\varrho\beta+\beta)(h,\mu,\varphi)$$

and

$$\|(u_0, f_0)\|_V \leqslant \frac{1}{1 - \|\varrho^2\|} \|\varrho\beta + \beta\| \left[\|\mu\|_{\mathcal{C}_t^1(\Gamma_+)} + \|\varphi\|_{\mathcal{C}^1_{(v,\nabla)}(\overline{\Omega})} + \|h\|_{\mathcal{C}_t^1(\overline{D})} \right].$$

Since the solution (u, f) of the problem (1)–(4) satisfies the condition

$$(u, f) = \varrho^2(u, f) + (\varrho\beta + \beta)(h, \mu, \varphi) \text{ and } (u, f) \in V,$$

then $(u, f) = (u_0, f_0)$. This completes the proof. ▷

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