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LINEAR OPERATORS ON ABRAMOVICH–WICKSTEAD TYPE SPACE

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In this note, we define and investigate Abramovich–Wickstead type spaces the elements of which are the sums of continuous functions and discrete functions. We give an analytic representation of regular and order continuous regular operators on these spaces with values in a Dedekind complete vector lattice.

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CD_0 -type spaces were firstly introduced by Yu. A. Abramovich and A. W. Wickstead in [1] and [2] and further investigated by S. Alpay and Z. Ercan in [3]. CD_0 -type spaces deserve to be called Abramovich–Wickstead spaces, or briefly AW -space as in [4], since they mainly stem from the works of Yu. A. Abramovich and A. W. Wickstead. In this note we construct a new type AW -space and call it CD_0^r for the sake of convenience.

Throughout this note, the symbols L^r and L_n^r denote the space of regular and order continuous regular operators respectively. For unexplained terminology about vector lattice theory, we refer to [5].

The first section is devoted to some introductory knowledge about vector-valued measures. This section will be useful in obtaining main results. For more detailed information about vector-valued measures, we refer to [6]. The second section of this note contains the definition of CD_0^r -spaces. The third section is devoted to description of regular operators characterizations about linear operators on CD_0^r -space by means of vector measures and order summation. The fourth section contains two main results of the paper. In this section we are mainly interested in regular and order continuous regular operators on CD_0^r -space with values in Dedekind complete vector lattices.

1. Vector Measures

Consider a nonempty set K and a σ -algebra \mathcal{A} of the subsets of K . Let E be a Dedekind complete vector lattice. We shall call the mapping $\mu : \mathcal{A} \rightarrow E$ a *vector measure* if $\mu(\emptyset) = 0$ and for every sequence (A_n) of pairwise disjoint sets $A_n \in \mathcal{A}$ the equality holds

$$\mu \left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n) := o\text{-}\lim_n \sum_{k=1}^n \mu(A_k).$$

We say that a measure μ is *positive* and write $\mu \geq 0$ if $\mu(A) \geq 0$ for all $A \in \mathcal{A}$. We denote the set of all order bounded E -valued measures on a σ -algebra \mathcal{A} by $\text{cab}(K, \mathcal{A}, E)$. If $\mu, \nu \in \text{cab}(K, \mathcal{A}, E)$ and $t \in \mathbb{R}$, then we put by definition

- (1) $(\mu + \nu)(A) := \mu(A) + \nu(A)$ ($A \in \mathcal{A}$),
- (2) $(t\mu)(A) := t\mu(A)$ ($A \in \mathcal{A}$),
- (3) $\mu \geq \nu \Leftrightarrow \mu - \nu \geq 0$.

One can prove that $\text{cab}(K, \mathcal{A}, E)$ is a Dedekind complete vector lattice. In particular, every measure $\mu : \mathcal{A} \rightarrow E$ has the positive part $\mu^+ := \mu \vee 0$ and the negative part $\mu^- := (-\mu)^+ = -\mu \wedge 0$. It is easy to verify that

$$\mu^+(A) = \sup\{\mu(A') : A' \in \mathcal{A}, A' \subset A\} \quad (A \in \mathcal{A}).$$

In the sequel, we shall consider special E -valued measures. Suppose that K is a compact topological space and \mathcal{A} is the Borel σ -algebra. A positive measure $\mu : \mathcal{A} \rightarrow E$ is said to be *regular* if for every $A \in \mathcal{A}$ we have

$$\mu(A) = \inf\{\mu(U) : A \subset U, U \in \text{Op}(K)\}$$

where $\text{Op}(K)$ is the collection of all open subsets of K . If the latter condition is true only for closed $A \in \mathcal{A}$, then μ is called *quasiregular*. Finally, an arbitrary measure $\mu : \mathcal{A} \rightarrow E$ is said to be *regular* (quasiregular) if the positive measures μ^+ and μ^- are regular (quasiregular). Let $\text{rca}(K, E)$ and $\text{qca}(K, E)$ be the sets of regular and quasiregular E -valued Borel measures respectively. It is seen from the definitions that $\text{rca}(K, E)$ and $\text{qca}(K, E)$ are vector sublattices in $\text{cab}(K, \mathcal{A}, E)$. Clearly, the supremum (infimum) of the increasing (decreasing) family of quasiregular measures bounded in $\text{cab}(A, \mathcal{A}, E)$ will also be quasiregular. The same holds for regular measures. Thus $\text{qca}(K, E)$ and $\text{rca}(K, E)$ are Dedekind complete vector lattices.

We now define the integral with respect to an arbitrary measure $\mu \in \text{cab}(K, \mathcal{A}, E)$. Let us denote by $\text{St}(K, \mathcal{A})$ the set of step functions $\varphi : K \rightarrow \mathbb{R}$ of the form $\varphi = \sum_{k=1}^n a_k \chi_{A_k}$, where $A_1, \dots, A_n \in \mathcal{A}$, $a_1, \dots, a_n \in \mathbb{R}$, and χ_A is the characteristic function of a set A . Construct the operator $I_\mu : \text{St}(K, \mathcal{A}) \rightarrow E$ by putting

$$I_\mu \left(\sum_{k=1}^n a_k \chi_{A_k} \right) := \sum_{k=1}^n a_k \mu(A_k).$$

As it is seen I_μ is a linear operator. Moreover the following normative inequality holds

$$|I_\mu(f)| \leq \|f\|_\infty |\mu|(K) \quad (f \in \text{St}(K, \mathcal{A})),$$

where $\|f\|_\infty = \sup_{k \in K} |f(k)|$. The subspace $\text{St}(K, \mathcal{A})$ is dense with respect to the norm in the space $l_\infty(K, \mathcal{A})$ of all bounded \mathcal{A} -measurable functions on K . Therefore I_μ admits a unique linear extension by continuity to $l_\infty(K, \mathcal{A})$, with the above-mentioned normative inequality being preserved. In particular, if K is a compact space and \mathcal{A} is the Borel σ -algebra, then $I_\mu(f)$ is defined for every continuous function $f \in C(K)$. Note also that $I_\mu \geq 0$ if and only if $\mu \geq 0$.

Finally, we give the following result of J. D. M. Wright [7] about analytical representation of linear operators which will play an important role to obtain main results of this note.

Theorem 1. *Let K be a compact topological space and let E be an arbitrary Dedekind complete vector lattice. The mapping $\mu \mapsto I_\mu$ implements a linear and lattice isomorphism of Dedekind complete vector lattices $\text{qca}(K, E)$ and $L^r(C(K), E)$.*

2. $CD_0^r(K, E)$ -spaces

In this section we introduce a new class of Abramovich–Wickstead type spaces. We start with the following definition which contains the building blocks of this space.

DEFINITION 2. For a compact space K and a relatively uniformly complete vector lattice E , we set

(1) $C(K, E(e))$ the space of all mappings from K into $E(e)$ which are continuous in the sense of the norm $\|\cdot\|_e$ where $E(e)$ denotes the ideal generated by $e \in E^+$ and

$$\|u\|_e := \inf\{\lambda > 0 : |u| \leq \lambda e\} \quad (u \in E(e)).$$

Then, we set

$$C_r(K, E) := \bigcup \{C(K, E(e)) : e \in E^+\}$$

and call the elements of this set r -continuous or uniformly continuous functions on K .

It is clear that $C_r(K, E)$ is contained in $l_\infty(K, E)$, the space of order bounded functions from K into E , since in $E(e)$ norm boundedness coincides with order boundedness. Moreover, $C_r(K, E)$ is a vector sublattice in $l_\infty(K, E)$.

(2) $c_0(K, E(e))$ the space of all mappings d from K into $E(e)$ such that for all $\varepsilon > 0$ the set $\{k \in K : \|d(k)\|_e \geq \varepsilon\}$ is finite. Then we set

$$c_0(K, E) := \bigcup \{c_0(K, E(e)) : e \in E^+\}.$$

It is clear that $c_0(K, E)$ is contained in $l_\infty(K, E)$. Moreover, $c_0(K, E)$ is a vector sublattice in $l_\infty(K, E)$.

Now we give the following theorem which will be useful in the sequel.

Theorem 3. Let K be a compact space. For any $f \in C_r(K, E)$ and $\varepsilon > 0$ there exist $e \in E^+$ and finite collections $\varphi_1, \dots, \varphi_n \in C(K)$ and $e_1, \dots, e_n \in E$ such that

$$\sup_{\alpha \in K} \left| f(\alpha) - \sum_{k=1}^n \varphi_k(\alpha) e_k \right| \leq \varepsilon e.$$

◁ If $f \in C_r(K, E)$, then $f \in C(K, E(e))$ for some $e \in E^+$. According to the Kakutani–Kreĭn Theorem, $E(e)$ is linearly isometric and lattice isomorphic to $C(Q)$ for some compact Hausdorff space Q . Therefore one can assume that $f \in C(K, C(Q))$. However, the spaces $C(K, C(Q))$ and $C(K \times Q)$ are isomorphic as Banach lattices. It remains to note that, according to the Stone–Weierstrass Theorem, the subspace of the functions $(\alpha, q) \mapsto \sum_{k=1}^n \varphi_k(\alpha) e_k(q)$, where $\varphi_1, \dots, \varphi_n \in C(K)$ and $e_1, \dots, e_n \in C(Q)$, is dense in $C(K \times Q)$. ▷

DEFINITION 4. Let K be a compact Hausdorff space without isolated points and E be a relatively uniformly complete vector lattice. We denote by $CD_0^r(K, E)$ the set of E -valued functions on K each of which is the sum of two E -valued functions f and d with $f \in C_r(K, E)$ and $d \in c_0(K, E)$.

For a finite subset S of K and $e \in E$, $\chi_S \otimes e$ is in $CD_0^r(K, E)$. It is easy to see that $CD_0^r(K, E)$ is an ordered vector space under the pointwise order.

Lemma 5. Let K be a compact Hausdorff space without isolated points and E be a relatively uniformly complete vector lattice. Then, $C_r(K, E) \cap c_0(K, E) = \{0\}$.

\triangleleft Suppose the contrary; let $0 \neq f \in C_r(K, E) \cap c_0(K, E)$. Assume that $f(x) \neq 0$. So there exists $e \in E^+$ such that $f \in C(K, E(e))$. Then there exists a neighborhood V of x such that for $y \in V$ we have $\|f(y)\|_e > \|f(x)\|_e/2$. But since x is not isolated, V is uncountable, which is a contradiction since $f \in c_0(K, E)$. \triangleright

It now follows that the decomposition of an element of $CD_0^r(K, E)$ -space into a sum of an r -continuous function and one with countable support is unique. So $CD_0^r(K, E)$ deserves to be an Abramovich–Wickstead type space.

Lemma 6. *Let K be a compact Hausdorff space without isolated points and E be a relatively uniformly complete vector lattice. Let $p \in CD_0^r(K, E)$. Then $p^+ = \sup(p, 0)$ exists in $CD_0^r(K, E)$.*

\triangleleft Let $p \in CD_0^r(K, E)$. Let $r(k) = f^+(k) + [-f^-(k) + h(k)] \vee (-f^+(k))$ for each $k \in K$ where f and h are continuous and discrete parts of p , respectively. By definitions f and h take their values in $E(e)$ for some $e \in E^+$. Let $s(k) = (-f^-(k) + h(k)) \vee (-f^+(k))$. Let $\varepsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that

$$\{k \in K : \varepsilon \leq \|s(k)\|_e\} \subset \left\{k \in K : \frac{1}{n_0} \leq \|h(k)\|_e\right\}.$$

Indeed, if this were not true, then for some sequence (k_n) in K , we would have $\varepsilon \leq \|s(k_n)\|_e$ while $\|h(k_n)\|_e < \frac{1}{n}$ for all $n \in \mathbb{N}$. By compactness of K , we can find a subnet (k_α) of k_n that converges to some $k_0 \in K$. As $\|h(k_\alpha)\|_e \rightarrow 0$ in $E(e)$, we have that

$$\varepsilon \leq \|s(k_\alpha)\|_e = \|(-f^-(k_\alpha) + h(k_\alpha)) \vee (-f^+(k_\alpha))\|_e \rightarrow \| -f^-(k_0) \vee (-f^+(k_0))\|_e = 0$$

which is a contradiction. Hence $r \in CD_0^r(K, E)$ whenever $p \in CD_0^r(K, E)$. On the other hand,

$$r(k) = f^+(k) + [-f^-(k) + h(k)] \vee (-f^+(k)) = [f^+(k) - f^-(k) + h(k)] \vee 0 = (p(k))^+$$

for each $k \in K$. So r is indeed p^+ . Continuous part of r is f^+ , where $f^+(k) = (f(k))^+$ by uniqueness of decomposition. \triangleright

We summarize what we have from the previous proposition as follows:

Proposition 7. *Let K be a compact Hausdorff space without isolated points and E be a relatively uniformly complete vector lattice. Then $CD_0^r(K, E)$ is a vector lattice under the pointwise ordering: $0 \leq p \in CD_0^r(K, E) \Leftrightarrow 0 \leq p(k)$ in E for all $k \in K$.*

Just like real-valued function space $CD_0(K)$ in [2], suprema and infima are easy to identify in $CD_0^r(K, E)$. We shall write $h_\gamma \uparrow h$ if the net h_γ is increasing and $\sup(h_\gamma) = h$.

Proposition 8. *Let K be a compact Hausdorff space without isolated points and E be a relatively uniformly complete vector lattice. If $h_\gamma \uparrow h$ in $CD_0^r(K, E)$, then $h_\gamma(k) \uparrow h(k)$ in E for all $k \in K$.*

\triangleleft Let k_0 be an arbitrary but fixed point of K . Then $h(k_0)$ is an upper bound of $\{h_\gamma(k_0) : \gamma \in \Gamma\}$ in $E(e)$ for some $e \in E^+$. Let ν be another upper bound for $\{h_\gamma(k_0) : \gamma \in \Gamma\}$. If $\nu \wedge h(k_0) = h(k_0)$, then the proof is obvious. On the other hand, if $\nu \wedge h(k_0) < h(k_0)$, then we can find some $0 < e_1 \in E(e)$ such that $\nu \wedge h(k_0) + e_1 \leq h(k_0)$. Then $h - \chi_{k_0} \otimes e_1$ is an upper bound in $CD_0^r(K, E)$ for the family $\{h_\gamma : \gamma \in \Gamma\}$, contradicting the definition of h . \triangleright

From the proposition above, we conclude that order convergence in $CD_0^r(K, E)$ is pointwise, order convergence in E .

3. Linear operators on $C_r(K, E)$ and $c_0(K, E)$

Throughout this section, unless stated otherwise, E will denote a relatively uniformly complete vector lattice and for a vector valued function f , $\chi_k \otimes f$ will denote the function which takes $f(k)$ at k and 0 otherwise. In this section we give two characterizations about the regular and order continuous linear operators from $CD_0^r(K, E)$ into a Dedekind complete vector lattice F .

We start with the following lemma which will be used in the sequel.

Lemma 9. *Let K be a compact space and F be a Dedekind complete vector lattice. Then for every regular operator $T : C_r(K, E) \rightarrow F$ there exists a regular operator $T' : C(K) \rightarrow L^r(E, F)$ such that*

$$T(\varphi \otimes e) = T'(\varphi)e \text{ for all } \varphi \in C(K) \text{ and } e \in E.$$

The correspondence $T \rightarrow T'$ is linear positive, and one-to-one.

◁ It is sufficient to consider positive linear operators. Let $T : C_r(K, E) \rightarrow F$ be a positive linear operator. For each $\varphi \in C(K)$ and $e \in E$, the function $\varphi \otimes e$ defined by $\varphi \otimes e(k) = \varphi(k)e$ belongs to $C_r(K, E)$. We put

$$T(\varphi \otimes e) = T'(\varphi)e \text{ for all } \varphi \in C(K) \text{ and } e \in E.$$

For fixed $\varphi \in C(K)$, the mapping $T'(\varphi) : e \mapsto T'(\varphi)e$ of E into F is evidently linear. Moreover, if $0 \leq e \in E$ and $0 \leq \varphi \in C(K)$, then $T'(\varphi)e = T(\varphi \otimes e) \geq 0$, therefore $T'(\varphi) \in L_+(E, F)$. Thus, the mapping $T' : \varphi \rightarrow T'(\varphi)$ of $C(K)$ into $L^r(E, F)$ is linear and positive.

It is easy to verify that the mapping $T \mapsto T'$ is linear and positive. In order to prove that this mapping is one-to-one, let $S : C_r(K, E) \rightarrow F$ be a positive linear operator such that

$$S(\varphi \otimes e) = T'(\varphi)e, \text{ for } \varphi \in C(K) \text{ and } e \in E.$$

Let $f \in C_r(K, E)$. Then by Theorem 3, there exists a sequence (f_n) of the form $\sum \varphi_i \otimes e_i$ (finite sum) with $\varphi_i \in C(K)$ and $e_i \in E$ converging relatively uniformly to f . Then $T(f_n) = S(f_n)$ for every n . On the other hand T and S are relatively uniformly continuous on $C_r(K, E)$, therefore

$$T(f) = \lim_{n \rightarrow \infty} T(f_n) = \lim_{n \rightarrow \infty} S(f_n) = S(f),$$

consequently $T = S$. ▷

Theorem 10. *Let K be a compact topological space and F be a Dedekind complete vector lattice. Then there exists a lattice isomorphism $T' \leftrightarrow \mu$ between the set of regular operators $T' : C(K) \rightarrow L^r(E, F)$ and the set of countably additive quasiregular Borel measures $\mu : K \rightarrow L^r(E, F)$ given by the equality*

$$T'(f) = \int f d\mu, \text{ for every } f \in C(K).$$

◁ Proof directly follows from Theorem 1, since $L^r(E, F)$ is a Dedekind complete vector lattice. ▷

Let F be another Dedekind complete vector lattice and $\mu \in \text{qca}(K, L^r(E, F))$. Then the integral $I_\mu : C(K) \rightarrow E$ can be extended to $C_r(K, E)$. We can view the algebraic tensor

product $C(K) \otimes E$ as the subspace in $C_r(K, E)$, consisting of the mappings $k \mapsto \sum_{i=1}^n \varphi_i(k)e_i$ ($k \in K$) where $e_i \in E$ and $\varphi_i \in C(K)$. Define I_μ on $C(K) \otimes E$ by the formula

$$I_\mu \left(\sum_{i=1}^n \varphi_i \otimes e_i \right) := \sum_{i=1}^n e_i \int_K \varphi_i d\mu.$$

If $f \in C_r(K, E)$, then by Theorem 3 there exist $e \in E^+$ and a sequence $(f_n) \subset C(K) \otimes E$ such that

$$\sup_{k \in K} |f(k) - f_n(k)| \leq \frac{1}{n}e.$$

Put by definition

$$\int_K f d\mu := I_\mu(f) := o\text{-lim } I_\mu(f_n).$$

It can be easily seen that this definition is correct even for an arbitrary order bounded finitely additive vector measure μ .

Theorem 11. *For any linear operator $T \in L^r(C_r(K, E), F)$ there exists a unique vector measure $\mu := \mu_T \in \text{qca}(K, L^r(E, F))$ such that*

$$Tf = \int_K f(k) d\mu(k) \quad (f \in C_r(K, E)).$$

The correspondence $T \mapsto \mu_T$ is a lattice isomorphism of $L^r(C_r(K, E), F)$ onto $\text{qca}(K, L^r(E, F))$.

◁ See [6, Theorem 2.1.14 (5)]. ▷

Theorem 12. *There exists a lattice isomorphism $T \leftrightarrow T'$ between the space of regular operators $T : C_r(K, E) \rightarrow F$ and the space of regular operators $T' : C(K) \rightarrow L^r(E, F)$ given by the equality*

$$T(\varphi \otimes e) = T'(\varphi)e, \quad \text{for } \varphi \in C(K) \text{ and } e \in E.$$

If T and T' are in correspondence, then there exists a unique common countably additive quasiregular Borel measure $\mu := \mu_T : K \rightarrow L^r(E, F)$ such that

$$T(f) = \int f d\mu, \quad \text{for } f \in C_r(K, E),$$

and

$$T'(\varphi) = \int \varphi d\mu, \quad \text{for } \varphi \in C(K).$$

In particular, the correspondence $T \mapsto \mu_T$ is a lattice isomorphism of $L^r(C_r(K, E), F)$ onto $\text{qca}(K, L^r(E, F))$.

◁ Let first $T : C_r(K, E) \rightarrow F$ be a regular operator. Let $T' : C(K) \rightarrow L^r(E, F)$ be the regular operator corresponding to T (Lemma 9) by the equality

$$T(\varphi \otimes x) = T'(\varphi)x, \quad \text{for } \varphi \in C(K) \text{ and } x \in E.$$

We know that the correspondence $T \mapsto T'$ is linear, positive, and one-to-one. We have

$$T(\varphi \otimes x) = T'(\varphi)x = \left(\int \varphi d\mu \right) x$$

for every $x \in E$, therefore

$$T(\varphi \otimes x) = \int \varphi \otimes x d\mu, \quad \text{for every } \varphi \in C(K) \text{ and } x \in E.$$

Conversely, let $T' : C(K) \rightarrow L^r(E, F)$ be a regular operator, and let $\mu : K \rightarrow L^r(E, F)$ be the countably additive quasiregular measure corresponding to T' by Theorem 10. If we put

$$T(f) = \int f d\mu, \quad f \in C_r(K, E),$$

then $T : C_r(K, E) \rightarrow F$ is a regular operator and we have

$$T(\varphi \otimes x) = T'(\varphi)x, \quad \text{for } \varphi \in C(K) \text{ and } x \in E.$$

It remains to apply Theorem 11. \triangleright

Now we give the following definition which will be useful in the sequel.

DEFINITION 13. Let K be a non-empty set and F be a Dedekind complete vector lattice. Then we define

(1) $c_0^r(\mathbb{N}, E) = \{(e_n) \subset E : \exists e \in E^+ \text{ such that } e_n \in E(e) \forall n \text{ and } \|e_n\|_e \rightarrow 0\}$,

(2) $l_1[K, L^r(E, F)]$ is the space of maps $\alpha : K \rightarrow L^r(E, F)$ such that $\sum_{n=1}^{\infty} |\alpha(k_n)|(|e_n|)$

exists in F for all sequences $(k_n) \in K$ and $(e_n) \in c_0^r(\mathbb{N}, E)$.

As usual, $\sum_{n=1}^{\infty} |\alpha(k_n)|(|e_n|)$ is the supremum of the sums $\sum_{n=1}^m |\alpha(k_n)|(|e_n|)$. Clearly, $l_1[K, L^r(E, F)]$ is a vector lattice under the pointwise operations and ordering.

Theorem 14. Let K and F be as above. Then $L^r(c_0(K, E), F)$ is lattice isomorphic to $l_1[K, L^r(E, F)]$.

\triangleleft Let $\phi : L^r(c_0(K, E), F) \rightarrow l_1[K, L(E, F)]$ be defined by $\phi(G)(k)(e) = G(\chi_k \otimes e)$ for each $G \in L^r(c_0(K, E), F)$, $k \in K$ and $e \in E$. Then $\phi(G)(k)$ is a regular operator from E into F as $\phi(G^+)(k)$ and $\phi(G^-)(k)$ are positive for each regular operator G . Thus $\phi(G)$ is a map from K into $L^r(E, F)$ and $\phi(G)(k)(e) \geq 0$ whenever $e \geq 0$ and $G \geq 0$, i. e. $\phi(G)(k)$ is positive for all $G \geq 0$.

Let us recall that $\phi(G)$ should also satisfy $\sum_{n=1}^{\infty} |\phi(G)(k_n)|(|e_n|) \in F$ for all sequences $(k_n) \in K$ and $(e_n) \in c_0^r(\mathbb{N}, E)$. Let $G \in L^r(c_0(K, E), F)$. Then we have

$$\sum_{n=1}^m |\phi(G)(k_n)|(|e_n|) = \sum_{n=1}^m |G|(\chi_{k_n} \otimes |e_n|) = |G| \left(\sum_{n=1}^m \chi_{k_n} \otimes |e_n| \right) \leq |G| \left(\sum_{n=1}^{\infty} \chi_{k_n} \otimes |e_n| \right) \in F,$$

therefore

$$\sum_{n=1}^{\infty} |\phi(G)(k_n)|(|e_n|) = \sup_m \sum_{n=1}^m |G|(\chi_{k_n} \otimes |e_n|) \in F.$$

Thus the map $\phi(G)$ we have defined belongs to $l_1[K, L(E, F)]$.

It is easy to verify that ϕ is a linear mapping. We now show that it is bipositive. Suppose that $\phi(G) \geq 0$ for some $G \in L^r(c_0(K, E), F)$, and take $0 \leq f \in c_0(K, E)$. As $\sum_{k \in S} \chi_k \otimes f \uparrow_S f$ in $c_0(K, E)$, we have $\sum_{k \in S} G(\chi_k \otimes f) \rightarrow G(f)$. By definition $G(\chi_k \otimes f) = \phi(G)(k)(f(k)) \geq 0$ and thus $G(f) \geq 0$ for each $0 \leq f \in c_0(K, E)$, i. e. $G \geq 0$.

Let now $\phi(G) = 0$ for some $G \in L^r(c_0(K, E), F)$. Then $G(\chi_k \otimes f) = 0$ for each $k \in K$ and and $0 \leq f \in c_0(K, E)$. As $\sum_{k \in S} \chi_k \otimes f \uparrow_S f$ in $c_0(K, E)$, we have $0 = \sum_{k \in S} G(\chi_k \otimes f) \rightarrow G(f)$ or $G(f) = 0$. The fact that $c_0(K, E)$ is vector lattice leads to $G = 0$.

To show that ϕ is surjective, let $0 \leq \alpha \in l_1[K, L^r(E, F)]$. Let $f \in c_0(K, E)$. Then there exists an at most countable subset (k_n) of K such that $f(k) = 0$ for all $k \neq k_n$ and there exists some $e \in E^+$ such that $f(k_n) \in E(e)$ for each n and $\|f(k_n)\|_e \rightarrow 0$. Hence we can define

$$G(f) = \sum_{n \in \mathbb{N}} \alpha(k_n)(f(k_n)),$$

which certainly belongs to F as $f(k_n) \in c_0^r(\mathbb{N}, E)$. We now verify that $\phi(G) = \alpha$. Let $0 \leq e \in E$, then

$$\phi(G)(k_0)(e) = G(\chi_{k_0} \otimes e) = \sum_{n \in \mathbb{N}} \alpha(k_0)(\chi_{k_0} \otimes e) = \alpha(k_0)(e).$$

Since $e \in E$ is arbitrary, we conclude that $\phi(G)(k_0) = \alpha(k_0)$ and k_0 is arbitrary, we have $\phi(G) = \alpha$. Since $l_1[K, L^r(E, F)]$ is a vector lattice, the proof of surjectivity of ϕ is now complete. \triangleright

4. Main results

Now we are in a position to give one of the first main result of this note as follows:

Theorem 15. *Let K be a compact Hausdorff space without isolated points and F be a Dedekind complete vector lattice. Then $L^r(CD_0^r(K, E), F)$ is lattice isomorphic to $\text{qca}(K, L^r(E, F)) \oplus l_1[K, L^r(E, F)]$ with the dual order on this direct sum defined by*

$$\langle \mu, \alpha \rangle \geq 0 \Leftrightarrow \mu \geq 0 \text{ and } \alpha \geq 0 \text{ and } \mu(\{k\}) \geq \alpha(k)$$

for all $k \in K$, which if we identify α with a discrete measure on K , is precisely requiring that $\mu \geq \alpha \geq 0$.

\triangleleft Let $T \in L^r(CD_0^r(K, E), F)$. Then certainly T splits into two regular operators T_1 and T_2 , where $T_1 : C_r(K, E) \rightarrow F$ and $T_2 : c_0(K, E) \rightarrow F$. By Theorem 12 there exists a measure $\mu \in \text{qca}(K, L^r(E, F))$ such that T_1 can be identified with μ . On the other hand, by Theorem 14 there exists a map $\alpha \in l_1[K, L^r(E, F)]$ such that T_2 can be identified with α . We thus have a map from $L^r(CD_0^r(K, E), F)$ into $\text{qca}(K, L^r(E, F)) \oplus l_1[K, L^r(E, F)]$.

Now suppose that $\mu \in \text{qca}(K, L^r(E, F))$ and $\alpha \in l_1[K, L^r(E, F)]$. We can certainly define an operator φ by

$$\varphi(f) = \int f_1 d\mu + \sum_{n \in \mathbb{N}} \alpha(k_n)(f_2(k_n)),$$

for $f = f_1 + f_2 \in C_r(K, E) \oplus c_0(K, E)$. The map we have defined from $\text{qca}(K, L^r(E, F)) \oplus l_1[K, L^r(E, F)]$ into $L^r(CD_0^r(K, E), F)$ is easily seen to be lattice isomorphism by Theorem 12 and Theorem 14. \triangleright

Now we give the following definition which will be used in our final result.

DEFINITION 16. Let K be a compact space and F be a Dedekind complete vector lattice. Then we set $l^1(K, L_n^r(E, F))$ the set of all maps β from K into $L_n^r(E, F)$ satisfying

- (1) $\sup_{\|f\|_e \leq 1} \sum_{k \in K} |\beta(k)|(|f(k)|)$ exist in F for all $e \in E^+$ and $f \in C(K, E(e)) \oplus c_0(K, E(e))$,
- (2) $\sum_{k \in K} |\beta(k)|(f_\alpha(k)) \downarrow_\alpha 0$ whenever $f_\alpha \downarrow 0$.

As usual, $\sum_{k \in K} |\beta(k)|(|f(k)|)$ is the supremum of the sums $\sum_{k \in S} |\beta(k)|(|f(k)|)$ where S is a finite subset of K . Evidently, $l^1(K, L_n^r(E, F))$ is a vector lattice under pointwise operations.

We close this section with a result of this note about order continuous operators on $CD_0^r(K, E)$ -spaces.

Theorem 17. *Let K be a compact Hausdorff space without isolated points and F be a Dedekind complete vector lattice. Then $L_n^r(CD_0^r(K, E), F)$ is lattice isomorphic to $l^1(K, L_n^r(E, F))$.*

◁ Define $\phi : L_n^r(CD_0^r(K, E), F) \rightarrow l^1(K, L_n^r(E, F))$ via $\phi(G)(k)(e) = G(\chi_k \otimes e)$ for each $G \in L_n^r(CD_0^r(K, E), F)$, $k \in K$ and $e \in E$. Then $\phi(G)(k)$ is order bounded, since $\phi(G^+)(k)$ and $\phi(G^-)(k)$ are order bounded F -valued operators for each G on $CD_0^r(K, E)$. If $e_\alpha \downarrow 0$ in E , then $\chi_k \otimes e_\alpha \downarrow 0$ in $CD_0^r(K, E)$ for each $k \in K$. This gives that $\phi(G)(k)(e_\alpha) = G(\chi_k \otimes e_\alpha)$ is order convergent to 0 so that $\phi(G)(k) \in L_n^r(E, F)$ for each $G \in L_n^r(CD_0^r(K, E), F)$. Thus $\phi(G)$ is a map from K into $L_n^r(E, F)$ and $\phi(G)(k)(e) \geq 0$ whenever $e \geq 0$ and $G \geq 0$, i. e., $\phi(G)(k)$ is positive for all $G \geq 0$.

Now we will show that $\phi(G)$ is an element of $l^1(K, L_n^r(E, F))$. Let S be a finite subset of K and $G \in L_n^r(CD_0^r(K, E), F)$. Then

$$\begin{aligned} \sum_{k \in S} |\phi(G)(k)|(|f(k)|) &= \sum_{k \in S} |\phi(G^+ - G^-)(k)|(|f(k)|) \\ &\leq \sum_{k \in S} \phi(G^+)(k)(|f(k)|) + \sum_{k \in S} \phi(G^-)(k)(|f(k)|) \\ &= \sum_{k \in S} G^+(\chi_k \otimes |f|) + \sum_{k \in S} G^-(\chi_k \otimes |f|) = G^+ \left(\sum_{k \in S} \chi_k \otimes |f| \right) + G^- \left(\sum_{k \in S} \chi_k \otimes |f| \right) \end{aligned}$$

for each $f \in CD_0^r(K, E)$. Hence we get

$$\sum_{k \in S} |\phi(G)(k)|(|f(k)|) \leq G^+(|f|) + G^-(|f|) = |G|(|f|)$$

as $\sum_{k \in S} \chi_k \otimes |f| \uparrow_S |f|$, G^+ and G^- are order continuous.

Let e be an arbitrary but fixed element of E^+ . Then

$$\sup_{\|f\|_e \leq 1} \sum_k |\phi(G)(k)|(|f(k)|) \leq \sup_{\|f\|_e \leq 1} |G|(|f|) \leq |G|(e) \in F,$$

as $|f| \leq \|f\|_e e$.

So far we have shown that $\phi(G)$ satisfies the first condition of Definition 16. Also we have to show that

$$\sum_k |\phi(G)(k)|(|f_\alpha(k)|) \downarrow_\alpha 0$$

for each $f_\alpha \in CD_0^r(K, E)$ such that $f_\alpha \downarrow 0$. It is sufficient to verify the claim for a positive $G \in L_n^r(CD_0^r(K, E), F)$. Let $0 \leq G \in L_n^r(CD_0^r(K, E), F)$ and $f_\alpha \downarrow 0$ in $CD_0^r(K, E)$. For a fixed α , we have $\sum_{k \in S} \chi_k \otimes f_\alpha \uparrow_S f_\alpha$. Since G is order continuous and positive, we have

$$G \left(\sum_{k \in S} \chi_k \otimes f_\alpha \right) = \sum_{k \in S} G(\chi_k \otimes f_\alpha) \uparrow G(f_\alpha).$$

Thus

$$\sum_{k \in K} |\phi(G)(k)|(f_\alpha(k)) = \sum_{k \in K} \phi(G)(k)(f_\alpha(k)) = \sum_{k \in K} G(\chi_k \otimes f_\alpha) = G(f_\alpha) \downarrow 0.$$

Hence the map $\phi(G)$ we have defined belongs to $l^1(K, L_n^r(E, F))$.

It is easy to see that ϕ is linear. We now show that it is bipositive. Certainly $\phi(G) \geq 0$ whenever $G \geq 0$. Now assume that $\phi(G) \geq 0$ for some $G \in L_n^r(CD_0^r(K, E), F)$ and take $0 \leq f \in CD_0^r(K, E)$. As $\sum_{k \in S} \chi_k \otimes f \uparrow_S f$ in $CD_0^r(K, E)$, we have $\sum_{k \in S} G(\chi_k \otimes f) \rightarrow G(f)$. By definition, $G(\chi_k \otimes f) = \phi(G)(k)(f) \geq 0$ and thus $G(f) \geq 0$ for each $0 \leq f \in CD_0^r(K, E)$, i. e., $G \geq 0$. We now show that ϕ is one-to-one. Let $\phi(G) = 0$ for some $G \in L_n^r(CD_0^r(K, E), F)$. Then $G(\chi_k \otimes f) = 0$ for each $k \in K$ and $0 \leq f \in CD_0^r(K, E)$. As G is order continuous and $\sum_{k \in S} \chi_k \otimes f \uparrow_S f$, this gives that $0 = \sum_{k \in S} G(\chi_k \otimes f) \rightarrow G(f)$ or $G(f) = 0$. As $CD_0^r(K, E)$ is a vector lattice, we get $G = 0$.

To show that ϕ is surjective, take an arbitrary $0 \leq \alpha \in l^1(K, L_n^r(E, F))$ and define $G : CD_0^r(K, E)_+ \rightarrow F$ by $G(f) = \sum_{k \in K} \alpha(k)(f(k))$. Then G is additive on $CD_0^r(K, E)$ and $G(f) = G(f^+) - G(f^-)$ extends G to $CD_0^r(K, E)$. We now verify that $\phi(G) = \alpha$. If $0 \leq e \in E$, then

$$\phi(G)(k_0)(e) = G(\chi_{k_0} \otimes e) = \sum_{k \in K} \alpha(k)(\chi_{k_0} \otimes e)(k) = \alpha(k_0)e.$$

Since $e \in E$ is arbitrary, we conclude that $\phi(G)(k_0) = \alpha(k_0)$ and k_0 is arbitrary, we have $\phi(G) = \alpha$. \triangleright

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References

1. Abramovich Y. A., Wickstead A. W. Regular operators from and into a small Riesz space // *Indag. Math. N. S.*—1991.—V. 2, № 3.—P. 257–274.
2. Abramovich Y. A., Wickstead A. W. Remarkable classes of unital AM-spaces // *J. of Math. Analysis and Appl.*—1993.—V. 180.—P. 398–411.
3. Ercan Z., Alpay S. $CD_0(K, E)$ and $CD_w(K, E)$ -spaces as Banach lattices // *Positivity*.—2000.—V. 4.—P. 213–225.
4. Ercan Z., Onal S. Kakutani-Krein compact space of $CD_w(X)$ -spaces interms of $X \otimes \{0, 1\}$ // *J. of Math. Anal. and Appl.*—2006.—V. 313, № 2.—P. 611–631.
5. Aliprantis C. D., Burkinshaw O. *Positive Operators*.—New York: Acad. press, 1985.—xvi, 367 p.
6. Kusraev A. G., Kutateladze S. S. *Subdifferentials: Theory and Applications*.—Dordrecht: Kluwer Academic Publishers, 1995.—408 p.
7. Wright J. D. M. An algebraic characterization of vector lattices with the Borel regularity property // *J. London Math. Soc.*—1973.—V. 7, № 2.—P. 277–285.

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