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THE EDGE C_K GRAPH OF A GRAPH

P. Siva Kota Reddy, K. M. Nagaraja, V. M. Siddalingaswamy

For any integer $k \geqslant 4$, the *edge C_k graph $E_k(G)$* of a graph $G = (V, E)$ has all edges of G as it vertices, two vertices in $E_k(G)$ are adjacent if their corresponding edges in G are either incident or belongs to a copy of C_k. In this paper, we obtained the characterizations for the edge C_k graph of a graph G to be connected, complete, bipartite etc. It is also proved that the edge C_4 graph has no forbidden subgraph characterization. Mereover, the dynamical behavior such as convergence, periodicity, mortality and touching number of $E_k(G)$ are studied.

1. Introduction

For graph theory terminology and notation in this paper we follow the book [3]. All graphs considered in this paper are finite, unoriented, without loops and multiple edges.

Graph theory [3] is an established area of research in combinatorial mathematics. It is also one of the most active areas of mathematics that has found large number of applications in diverse areas including not only computer science, but also chemistry, physics, biology, anthropology, psychology, geography, history, economics, and many branches of engineering. Graph theory has been especially useful in computer science, since after all, any data structure can be represented by a graph. Furthermore, there are applications in networking, in the design of computer architectures, and in general, in virtually every branch of computer science. However, to date most of the research in graph theory has only considered graphs that remain static, i. e., they do not change with time. A wealth of such literature has been developed for static graph theory. Our purpose is to classify dynamic graphs, i. e., graphs that change with time. Dynamic graphs appear in almost all fields of science. This is especially true of computer science, where almost always the data structures (modeled as graphs) change as the program is executed. Very little is known about the properties of dynamic graphs.

The study of graph dynamics has been receiving wide attention, since Ore's work on the line graph operator $L(G)$ (see [5, 6]). The *edge C_k graph $E_k(G)$* of a graph G is defined in [5] as follows: The edge C_k graph of a graph $G = (V, E)$ is a graph $E_k(G) = (V', E')$, with vertex set $V' = E(G)$ such that two vertices e and f are adjacent if, and only if, the corresponding edges in G either incident or opposite edges of some cycle C_k. So for any two edges in G are adjacent if, and only if, they belongs to a common P_3 or C_k in G. When $k = 3$, the definition coincides with triangular line graph of a graph [2], and when $k = 4$, the definition coincides with E_4-graph of a graph [4].

Throughout this paper we denote by P_n (respectively C_n), a path (respectively cycle) on n vertices. The graph obtained by deleting any edge of K_n is denoted by $K_n - e$. A graph G is H-free if G does not contain H as an induced subgraph. A graph H is a forbidden subgraph for a property P of graphs if no graph having property P contains an induced subgraph isomorphic to H. The cross product $G_1 \times G_2$ of two graphs G_1 and G_2 is a simple graph with $V(G_1) \times V(G2)$ as its vertex set and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ if, and only if, either $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2, or u_1 is adjacent to u_2 in G_1 and $v_1 = v_2$.

Clearly, the edge C_k graph coincides with the line graph for any acyclic graph. But they differ in many properties. As a case, for a connected graph G, $E_k(G) = G$ if, and only if, $G = C_n$, $n \neq k$. Also Beineke has proved in [1] that the line graph has nine forbidden subgraphs. In this paper, we see that $E_k(G)$ has no forbidden subgraphs.

In the following sections, we presented the characterizations for the edge C_k graph of a graph G is connected, complete, bipartite etc. We have also proved that the edge C_4 graph has no forbidden subgraph characterization. The dynamical behavior such as convergence, periodicity, mortality and touching number of $E_k(G)$ are also studied.

2. Edge C_k Graph of a Graph

The *edge C_k graph* $E_k(G)$ of a graph G is defined in [5] as follows: The edge C_k graph of a graph $G = (V, E)$ is a graph $E_k(G) = (V', E')$, with vertex set $V' = E(G)$ such that two vertices e and f are adjacent if, and only if, the corresponding edges in G either incident or opposite edges of some cycle C_k. So for any two edges in G are adjacent if, and only if, they belongs to a common P_3 or C_k in G. When $k = 3$, the definition coincides with triangular line graph of a graph [2], and when $k = 4$, the definition coincides with E_4-graph of a graph [4]. Clearly the edge C_k graph coincides with the line graph for any acyclic graph. But they differ in many properties. As a case, for a connected graph G, $E_k(G) = G$ if and only if $G = C_n$, $n \neq k$. The following result characterizes graphs whose E_k graph is isomorphic to their line graph.

Theorem 1. *For a graph G, $E_k(G) \cong L(G)$ if, and only if, G is C_k-free.*

Theorem 2. *For any graph G, $E_k(G)$ is connected if, and only if, exactly one component of G contains edges.*

Theorem 3. *For any graph G, the edge E_k graph is complete then $diam(G) \leqslant \lfloor \frac{k}{2} \rfloor$.*

$\lhd$ Since $E_k(G)$ is complete then by the definition of $E_k(G)$ any two edges must either incident or belongs to a cycle of length k. Suppose that $diam(G) > \lfloor \frac{k}{2} \rfloor$. That is there exists two vertices u and v in G with $d(u, v) > \lfloor \frac{k}{2} \rfloor$. Clearly u and v can not be in the some cycle of length $\lfloor \frac{k}{2} \rfloor$. Let u' and v' be any two vertices adjacent to u and v respectively. Then the edges uu' and vv' are not adjacent and does not belongs to a cycle of length k. This proves, a contradiction. $\rhd$

REMARK. The converse need not be true for example C_5 has diameter 2 but $E_4(G)$ is not complete.

In [4], the authors prove that: For a connected graph G, $E_4(G)$ is complete if, and only if, G is complete multipartite. But the same can not be generalized for $k \geqslant 5$. For example for the peterson graph $\mathscr{P}$, $E_5(\mathscr{P})$ is clearly complete graph K_{15}. But $\mathscr{P}$ is not complete bipartite. However, we have the following:

Theorem 4. *For a connected graph G, $E_k(G)$ is complete if, and only if, every edge of G belongs to a C_k.*

Corollary 5. *Let G be a complete r-multipartite graph for some $r \geqslant \frac{k}{2}$, $E_k(G)$ is complete.*

In [4], the authors proved that the edge C_4 graph $E_4(G)$ has no forbidden subgraphs. We now prove that the edge C_k graph $E_k(G)$ also have no forbidden subgraph characterization.

Theorem 6. *There is no forbidden subgraph characterization for $E_k(G)$ for any $k \geqslant 3$.*

$\lhd$ We can assume that $k \geqslant 5$, since the edge C_3 of a graph is nothing but triangular line graph and when $k = 4$, the result follows from the above result. We shall prove that given any graph G, we can find a graph H such that G is an induced subgraph of $E_k(H)$. For any graph G, let $H = G \times K_2$. Clearly H contains 2 copies of G say G and G'. Now let H' be the graph obtained from H by subdivide each edge of one copy of G in H into $k - 4$ edges.

We claim that G is induced subgraph of $E_k(H')$. For any $v \in V(G)$, $E_k(G)$ contains vertices of the form vv', where v' is the corresponding vertex in G'. Now for any two adjacent vertices u and v, the corresponding vertices uu' and vv' are also adjacent in $E_k(G)$, since the vertices $u, u', u_1, u_2, \cdots, u_{k-4}, v', v$ forms a cycle of length k in H'. Now if u and v are non adjacent adjacent vertices of G then uu' and vv' are also non adjacent vertices in $E_k(H')$. Thus the subgraph induced by the set $\{uu' : u \in G\}$ of vertices in $H'_k G$) contains G as a induced subgraph. This completes the proof. $\rhd$

Theorem 7. *For a connected graph G, $E_k(G)$ is bipartite if, and only if, G is either a path or an even cycle of length $r \neq k$.*

$\lhd$ Suppose that $E_k(G)$ is bipartite. Suppose that G has a vertex of degree at least 3, then G contains a cycle of length 3. Hence, the degree of every vertex is at least 2. Since G connected, G must be a path or a cycle. Now, if G is an odd cycle of length r, then r can not be odd or equal to k. since, if r is odd then $L(G)$ is also a cycle which is a subgraph of $E_k(G)$ and if $r = k$, then $E_k(G) = K_r$ and hence $E_k(G)$ cannot be bipartite in both cases. Finally if r is even and $r \neq k$ then $E_k(G) = G$, which is bipartite.

Conversely, suppose that G is either a path or an even cycle of length $r \neq k$, then $E_k(G)$ is either a path or an even cycle respectively. Hence $E_k(G)$ is bipartite. $\rhd$

Corollary 8. *For a connected graph G, $E_k(G)$ is a tree if, and only if, G is a path.*

3. Dynamical Properties

First we recall some graph dynamical terminologies from [6]. Let G be any graph. The nth-iterated graph is iteratively defined as follows: $E_k^0(G) = G$, $E_k^1(G) = E_k(G)$, $E_k^n(G) = E_k(E_k^{n-1}(G))$, $n \geqslant 2$. We say that G is convergent under E_k if $\{E_k^n(G) : n \in N\}$ is finite. If G is not convergent under E_k, then G is *divergent* under E_k. A graph G is *periodic* if there is some natural number n with $G = E_k^n(G)$. The smallest such number is called the period of G. The transition number $t(x)$ of a convergent graph G is defined as zero if G is periodic and as the smallest number n such that $E_k^n(G)$ is periodic. A graph G is *mortal* if for some $n \geqslant N$, $E_k^n(G) = \phi$ the empty graph.

Theorem 9. *The graphs P_n, $K_{1,3}$, C_n $(n \neq k)$ are the only E_k convergent graphs.*

$\lhd$ If G contains a vertex of degree > 3, then $E_k(G)$ contains K_4. In the subsequent iterations the clique size goes on increasing and hence G diverges. So, for convergent graphs $\delta(G) \leqslant 3$.

If G is a tree which is neither P_n nor $K_{1,3}$, then K_4 is contained at least in the third iterated graph and hence G cannot converge. $\rhd$

Corollary 10. *For $E_k(G)$, the only periodic graphs are the cycles C_n, $n \neq k$ and they have period one.*

◁ The paths P_n converge to and $K_{1,3}$ converges to the triangle. Consider the graphs which are not trees. If G is not a cycle, then G contains a cycle with a pendant edge as a subgraph (need not be induced). Then K_4 is a subgraph at least in the second iteration and hence in the subsequent iterations the clique size will go on increasing and hence cannot converge. All cycles except C_k are fixed under E_k and C_k is not convergent. Thus, the proof follows from the fact that a graph G is convergent if, and only if, G is either periodic or there is some positive integer n with $E_k^n(G)$ periodic. ▷

Corollary 11. *The transition number $t(K_{1,3}) = 1$ and for $n \neq k$, $t(C_n) = 0$.*

Corollary 12. *For $E_k(G)$, the paths are the only mortal graphs.*

◁ Among the convergent graphs, cycles other than C_k are fixed and $K_{1,3}$ converges to K_3. The paths are the only graphs converging to ϕ. ▷

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Siva Kota Reddy P.
Department of Mathematics
Siddaganga Institute of Technology, B. H. Road
Tumkur-572 103, India
E-mail: reddy_math@yahoo.com; pskreddy@sit.ac.in

Nagaraja K. M.
Department of Mathematics
J.S.S. Academy of Technical Education
Uttarahalli-Kengeri Main Road, Bangalore-560 060, India
E-mail: nagkmn@gmail.com

Siddalingaswamy V. M.
Department of Mathematics
J.S.S. Academy of Technical Education
Uttarahalli-Kengeri Main Road, Bangalore-560 060, India
E-mail: swamyvms@yahoo.com

РЕБЕРНЫЙ C_k-ГРАФ ГРАФА

Сива Кота Редди П., Нагарайя К. М., Сиддалингасвами В. М.

Для любого целого $k \geqslant 4$ реберный C_k-граф $E_k(G)$ графа G содержит все ребра графа G в качестве вершин, при этом две вершины смежны в $E_k(G)$, если соответствующие им ребра в графе G либо инцидентны, либо принадлежат копии C_k. В статье установлено, что реберный C_k-граф графа G является связным, полным, двудольным и т. д. Доказано также, что реберный C_4-граф не имеет характеризаций запрещенными подграфами. Кроме того, исследованы такие характеристики динамических графов как сходимость, периодичность, мортальность и число переходов графа $E_k(G)$.

Ключевые слова: реберный C_k-граф, треугольный линейный граф, сходимость, периодичность, мортальность, число переходов.