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TWO MEASURE-FREE VERSIONS OF THE BREZIS–LIEB LEMMA

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Dedicated to Professor A. E. Gutman
on the occasion of his 50th anniversary

We present two measure-free versions of the Brezis–Lieb lemma for uo-convergence in Riesz spaces.

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1. Introduction

The Brezis–Lieb lemma [2, Theorem 2] has numerous applications mainly in calculus of variations (see, for example [3, 6]). We begin with its statement. Let $j : \mathbb{C} \to \mathbb{C}$ be a continuous function with $j(0) = 0$. In addition, let j satisfy the following hypothesis: for every sufficiently small $\varepsilon > 0$, there exist two continuous, nonnegative functions φ_ε and ψ_ε such that

$$|j(a+b) - j(a)| \leqslant \varepsilon\varphi_\varepsilon(a) + \psi_\varepsilon(b) \tag{1}$$

for all $a, b \in \mathbb{C}$. The following result has been stated and proved by H. Brezis and E. Lieb in [2].

Theorem 1.1 (Brezis–Lieb lemma [2, Theorem 2]). *Let (Ω, Σ, μ) be a measure space. Let the mapping j satisfy the above hypothesis, and let $f_n = f + g_n$ be a sequence of measurable functions from Ω to $\mathbb{C}$ such that:*

(i) $g_n \xrightarrow{\text{a.e.}} 0$;
(ii) $j \circ f \in L^1$;
(iii) $\int \varphi_\varepsilon \circ g_n d\mu \leqslant C < \infty$ *for some C independent of ε and n;*
(iv) $\int \psi_\varepsilon \circ f d\mu < \infty$ *for all $\varepsilon > 0$.*

Then, as $n \to \infty$,

$$\int (j(f+g_n) - j(g_n) - j(f))d\mu \to 0. \tag{2}$$

Here we reproduce its proof from [2, Theorem 2] with several simple remarks.

$\lhd$ Fix $\varepsilon > 0$ and let $W_{\varepsilon,n} = [|j \circ f_n - j \circ g_n - j \circ f| - \varepsilon\varphi_\varepsilon \circ g_n]_+$. As $n \to \infty$, $W_{\varepsilon,n} \xrightarrow{\text{a.e.}} 0$. On the other hand,

$$|j \circ f_n - j \circ g_n - j \circ f| \leqslant |j \circ f_n - j \circ g_n| + |j \circ f| \leqslant \varepsilon\varphi_\varepsilon \circ g_n + \psi_\varepsilon \circ f + |j \circ f|.$$

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Therefore, $0 \leqslant W_{\varepsilon,n} \leqslant \psi_\varepsilon \circ f + |j \circ f| \in L^1$. By dominated convergence,

$$\lim_{n\to\infty} \int W_{\varepsilon,n} d\mu = 0. \tag{3}$$

However,

$$|j \circ f_n - j \circ g_n - j \circ f| \leqslant W_{\varepsilon,n} + \varepsilon\varphi_\varepsilon \circ g_n \tag{4}$$

and thus

$$I_n := \int |j \circ f_n - j \circ g_n - j \circ f| d\mu \leqslant \int [W_{\varepsilon,n} + \varepsilon\varphi_\varepsilon \circ g_n] d\mu.$$

Consequently, $\limsup I_n \leqslant \varepsilon C$. Now let $\varepsilon \to 0$. $\rhd$

Remark 1.1. (i) The conditions (3) and (4) mean that the sequence $|j \circ f_n - j \circ g_n|$ lies eventually in the set $[-|j \circ f|, |j \circ f|] + \frac{3\varepsilon C}{2} B_{L^1}$, where B_{L^1} is the unit ball of L^1. In other words, the sequence $j \circ f_n - j \circ g_n$ is almost order bounded.

(ii) The superposition operator $J_j : L^0 \to L^0$, $J_j(f) := j \circ f$ induced by the mapping j in the proof above can be replaced by a mapping $J : L^0 \to L^0$ satisfying some reasonably mild conditions for keeping the statement of the Brezis–Lieb lemma.

(iii) Theorem 1.1 is equivalent to its partial case when the $\mathbb{C}$-valued functions are replaced by $\mathbb{R}$-valued ones.

The following proposition is motivated directly by the proof of [2, Theorem 2].

Proposition 1.2 (Brezis–Lieb lemma for mappings on L^0). *Let (Ω, Σ, μ) be a measure space, $f_n = f + g_n$ be a sequence in L^0 such that $g_n \xrightarrow{\text{a.e.}} 0$, and $J : L^0 \to L^0$ be a mapping satisfying $J(0) = 0$ and such that the sequence $J(f_n) - J(g_n)$ is almost order bounded. Then*

$$\lim_{n\to\infty} \int \big(J(f + g_n) - (J(g_n) + J(f))\big)\, d\mu = 0. \tag{5}$$

$\lhd$ As in the proof of the Brezis–Lieb lemma above, denote $I_n := \int |J(f + g_n) - (J(f) + J(g_n))|\, d\mu$. By the conditions, the sequence

$$J(f + g_n) - (J(f) + J(g_n)) = (J(f_n) - J(g_n))) - J(f)$$

a.e.-converges to 0 and is almost order bounded. Therefore, by the generalized dominated convergence, $\lim\limits_{n\to\infty} I_n = 0$. $\rhd$

Since almost order boundedness is equivalent to uniform integrability in finite measure spaces, the following corollary is immediate.

Proposition 1.3 (Brezis–Lieb lemma for uniform integrable sequence $J(f_n) - J(g_n)$). *Let (Ω, Σ, μ) be a finite measure space, $f_n = f + g_n$ be a sequence in L^0 such that $g_n \xrightarrow{\text{a.e.}} 0$, and $J : L^0 \to L^0$ be a mapping satisfying $J(0) = 0$ and such that the sequence $J(f_n) - J(g_n)$ is uniformly integrable. Then*

$$\lim_{n\to\infty} \int (J(f + g_n) - (J(g_n) + J(f))) d\mu = 0\,. \tag{6}$$

2. Two variants of the Brezis–Lieb lemma in Riesz spaces

Recall that a sequence x_n in a Riesz space E is *order convergent* (or o-convergent, for short) to $x \in E$ if there is a sequence z_n in E satisfying $z_n \downarrow 0$ and $|x_n - x| \leqslant z_n$ for all $n \in \mathbb{N}$ (we write $x_n \xrightarrow{\text{o}} x$). In a Riesz space E, a sequence x_n is *unbounded order convergent* (or *uo*-convergent, for short) to $x \in E$ if $|x_n - x| \wedge y \xrightarrow{\text{o}} 0$ for all $y \in E_+$ (we write $x_n \xrightarrow{\text{uo}} x$).

Here we give two variants of the Brezis–Lieb lemma in Riesz space setting by replacing *a.e.*-convergence by *uo*-convergence, integral functionals by strictly positive functionals and the continuity of the scalar function j (in Theorem 1.1) by the so called σ-unbounded order continuity of the mapping $J : E \to F$ between Riesz spaces E and F. As standard references for basic notions on Riesz spaces we adopt the books [1, 7, 8] and on unbounded order convergence the papers [4, 5].

It is well known that if (Ω, Σ, μ) be a σ-finite measure space, then in L^p $(1 \leqslant p \leqslant \infty)$, *uo*-convergence of sequences is the same as the almost everywhere convergence (see, for example [5]). Therefore, in order to obtain versions of Brezis–Lieb lemma in Riesz spaces, we shall replace the *a.e.*-convergence by the *uo*-convergence.

A mapping $f : E \to F$ between Riesz spaces is said to be σ-unbounded order continuous (in short, σuo-continuous) if $x_n \xrightarrow{\text{uo}} x$ in E implies $f(x_n) \xrightarrow{\text{uo}} f(x)$ in F. Clearly this definition is parallel to the well-known notion of σ-order continuous mappings between Riesz spaces.

Let F be a Riesz space and l be a strictly positive linear functional on F. Define the following norm on F:

$$\| x \|_l := l(|x|). \tag{7}$$

Recall that a Banach lattice E is said to be *order continuous* if every order null net is norm null, and a subset A of E is said to be *almost order bounded* if for any $\varepsilon > 0$ there exists $u_\varepsilon \in E_+$ such that $A \subset [-u_\varepsilon, u_\varepsilon] + \varepsilon B_E$, where B_E is the closed unit ball in E. We say that a net x_α is almost order bounded if the set of its members is almost order bounded.

The next lemma will be used to prove a version of Brezis–Lieb lemma for arbitrary strictly positive linear functionals.

Lemma 2.1 (See [5, Proposition 3.7])**.** *Let X be an order continuous Banach lattice. If a net x_α is almost order bounded and uo-convergent to x, then x_α converges to x in norm.*

Suppose that F is a Riesz space and l is a strictly positive linear functional on F, then the $\| \cdot \|_l$-completion $(F_l, \| \, . \, \|_l)$ of $(F, \| \cdot \|_l)$ is an AL-space, and so it is order continuous Banach lattice. The following result is a measure-free version of Proposition 1.2.

Proposition 2.2 (A Brezis–Lieb lemma for strictly positive linear functionals)**.** *Let E be a Riesz space and F_l be the AL-space constructed above. Let $J : E \to F_l$ be σuo-continuous with $J(0) = 0$, and x_n be a sequence in E such that:*

(i) $x_n \xrightarrow{\text{uo}} x$ *in* E;

(ii) *the sequence* $(J(x_n) - J(x_n - x))_n$ *is almost order bounded in* F_l.

Then

$$\lim_{n\to\infty} \| J(x_n) - J(x_n - x) - J(x) \|_l = 0. \tag{8}$$

$\lhd$ Since $x_n \xrightarrow{\text{uo}} x$ and J is σuo-continuous, then $J(x_n) \xrightarrow{\text{uo}} J(x)$ and $J(x_n - x) \xrightarrow{\text{uo}} J(0) = 0$. Thus, $J(x_n) - J(x_n - x) \xrightarrow{\text{uo}} J(x)$. It follows from Lemma 2.1 that $\lim\limits_{n\to\infty} \| J(x_n) - J(x_n - x) - J(x) \|_l = 0$. $\rhd$

In the following Brezis–Lieb type lemma, the σuo-continuity of mappings between Riesz spaces is used.

Proposition 2.3 (A Brezis–Lieb lemma for σuo-continuous linear functionals)**.** *Let E, F be Riesz spaces, l a σuo-continuous linear functional on F, $J : E \to F$ a σuo-continuous mapping with $J(0) = 0$, and $x_n \xrightarrow{\text{uo}} x$ in E. Then*

$$\lim_{n\to\infty} l(J(x_n) - J(x_n - x) - J(x)) = 0\,. \tag{9}$$

$\lhd$ Since $x_n \xrightarrow{\text{uo}} x$ and J is σuo-continuous, then $J(x_n) \xrightarrow{\text{uo}} J(x)$ and $J(x_n - x) \xrightarrow{\text{uo}} J(0) = 0$. Thus, $(J(x_n) - J(x_n - x) - J(x)) \xrightarrow{\text{uo}} 0$. But l is σuo-continuous, so $l(J(x_n) - J(x_n - x) - J(x)) \xrightarrow{\text{uo}} 0$. Since in $\mathbb{R}$ the uo-convergence, the o-convergence, and the standard convergence are all equivalent, then $\lim\limits_{n\to\infty} l(J(x_n) - J(x_n - x) - J(x)) = 0$. $\rhd$

Note that in opposite to Proposition 2.3, in Proposition 2.2 we do not suppose the functional l to be σuo-continuous.

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ДВА ВАРИАНТА ЛЕММЫ БРЕЗИСА — ЛИБА БЕЗ ИСПОЛЬЗОВАНИЯ СХОДИМОСТИ ПОЧТИ ВСЮДУ

Емельянов Э. Ю., Мараби М. А. А.

Рассматриваются две версии леммы Брезиса — Либа для uo-сходимости в пространствах Рисса.

Ключевые слова: лемма Брезиса — Либа, равномерно интегрируемая последовательность, пространство Рисса, uo-сходимость, почти порядково ограниченное множество, σuo-сходимость.