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ON GENERALIZATION OF FOURIER AND HARTLEY TRANSFORMS FOR SOME QUOTIENT CLASS OF SEQUENCES

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In this paper we consider a class of distributions and generate two spaces of Boehmians for certain class of integral operators. We derive a convolution theorem and generate two spaces of Boehmians. The integral operator under concern is well-defined, linear and one-to-one in the class of Boehmians. An inverse problem is also discussed in some details.

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1. Introduction

Integral transforms have been introduced and found their applications in applied mathematics and diverse fields of science. The Hartley transform is an integral transformation that maps a real-valued temporal or spacial function into a real-valued frequency function via the kernel $\operatorname{cas}(\cdot) = \cos(\cdot) + \sin(\cdot)$. Advantages of Hartley transforms comes over that of Fourier transforms since they avoid the use of complex arithmetic which results in faster algorithms. Hartley transforms can further be analytically continued into the complex plane, and for real functions they are Hermitian symmetry or reflection in the real axis. In this article we consider an integral transform related to Hartley and Fourier transforms defined for functions of two variables as

$$H^{\rho,\eta}_{\alpha,\beta} f(\zeta,\xi) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} f(x,y)(\alpha\cos\zeta x + \beta\sin\zeta x)(\rho\cos\xi y + \eta\sin\xi y)\,dx\,dy, \tag{1}$$

where $(\zeta,\xi) \in \mathbb{R}^2$, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ are the transform variables and α, β, ρ, η are arbitrary constants.

A inversion formula of the cited integral be can easily recovered from (1) giving

$$f(x,y) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} H^{\rho,\eta}_{\alpha,\beta}(\zeta,\xi)(\alpha\cos\zeta x + \beta\sin\zeta x)(\rho\cos\xi y + \eta\sin\xi y)\,d\zeta\,d\xi. \tag{2}$$

In a special case, for $\alpha = \beta = 1$, $\rho = \eta = 1$, the integral transform (1) and the inversion formula (2) are respectively reduced to the double Hartley transform A^d pair (see [10])

$$A^d(\zeta,\xi) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} f(x,y)(\cos\zeta x + \sin\zeta x)(\cos\xi y + \sin\xi y)\,dx\,dy \tag{3}$$

and

$$f(x,y)=\frac{1}{2\pi}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} A^d(\zeta,\xi)(\cos\zeta x+\sin\zeta x)(\cos\xi y+\sin\xi y)\,d\zeta\,d\xi. \tag{4}$$

Further, with simple computations, the kernel function

$$(\cos\zeta x+\sin\zeta x)(\cos\xi y+\sin\xi y)=\operatorname{cas}\zeta x\operatorname{cas}\xi y \tag{5}$$

inside the integral signs can be written as

$$\operatorname{cas}\zeta x\operatorname{cas} vy=\cos(\zeta x-\xi y)+\sin(\zeta x+\xi y). \tag{6}$$

Hence, the integral Equations (3) and (4) can also be rearranged in terms of (6) as

$$A^d(\zeta,\xi)=\frac{1}{2\pi}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} f(x,y)(\cos(\zeta x-\xi y)+\sin(\zeta x+\xi y))\,dx\,dy \tag{7}$$

and

$$f(x,y)=\frac{1}{2\pi}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} A^d(\zeta,\xi)(\cos(\zeta x-\xi y)+\sin(\zeta x+\xi y))\,d\zeta\,d\xi, \tag{8}$$

respectively.

By setting $\alpha=1,\beta=i,\rho=1$ and $\eta=i$, we derive the double Fourier transform F^d pair,

$$F^d(\zeta,\xi)=\frac{1}{2\pi}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} f(x,y)(\cos\zeta x+i\sin\xi x)(\cos\zeta \mathrm{y}+i\sin\xi y)\,dx\,dy \tag{9}$$

and

$$f(x,y)=\frac{1}{2\pi}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} F^d(\zeta,\xi)(\cos\zeta x+i\sin\xi x)(\cos\zeta y+i\sin\xi y)\,d\zeta\,d\xi. \tag{10}$$

By factoring $A^d(\zeta,\xi)$ into even and odd components, $A^d(\zeta,\xi)=E_d(\zeta,\xi)+O_d(\zeta,\xi)$, where

$$E_d(\zeta,\xi)=\frac{1}{2\pi}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} f(x,y)\cos(\zeta x-\xi y)\,dx\,dy \tag{11}$$

and

$$O_d(\zeta,\xi)=\frac{1}{2\pi}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} f(x,y)\sin(\zeta x-\xi y)\,dx\,dy \tag{12}$$

we get

$$F^d(\zeta,\xi)=E_d(\zeta,\xi)-iO_d(\zeta,\xi)\ \text{and}\ \ A^d(\zeta,\xi)=\operatorname{Re}F^d(\zeta,\xi)+\operatorname{Im}F^d(\zeta,\xi). \tag{13}$$

Denote by $\mathscr{L}^2$ the Lebesgue space of integrable functions over $\mathbb{R}^2$; then the convolution product of $f(x,y)$ and $g(x,y)$ in $\mathscr{L}^2$ is defined by

$$(f*^2 g)(x,y)=\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} f(t,w)g(x-t,y-w)\,dt\,dw.$$

We state and prove the following theorem.

Theorem 1 (Convolution Theorem). *Let $f(x,y), g(x,y) \in \mathscr{L}^2$. Then we have*

$$H^{\rho,\eta}_{\alpha,\beta}(f *^2 g)(\zeta,\xi) = J(\zeta,\xi)G(\zeta,\xi),$$

where $J(\zeta,\xi)$ and $G(\zeta,\xi)$ are given by the integrals

$$J(\zeta,\xi) = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} f(t,w)\cos(t\zeta)\cos(w\xi)\,dt\,dw$$

and

$$G(\zeta,\xi) = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} 4\beta\eta\sin(\zeta z)\sin(r\xi)g(z,r)\,dz\,dr.$$

$\lhd$ Let $f(x,y), g(x,y) \in \mathscr{L}^2$. Then by using the convolution product formula we have

$$\begin{aligned} H^{\rho,\eta}_{\alpha,\beta}\left(f *^2 g\right)(\zeta,\xi) &= \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \left(f *^2 g\right)(x,y)(\alpha\cos(x\zeta)+\beta\sin(x\zeta)) \\ &\quad\times(\rho\cos(y\xi)+\eta\sin(y\xi))\,dx\,dy \\ &= \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\left(\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} f(t,w)g(x-t,y-w)\,dt\,dw\right) \\ &\quad\times(\alpha\cos(x\zeta)+\beta\sin(x\zeta))(\rho\cos(y\xi)+\eta\sin(y\xi))\,dx\,dy. \end{aligned}$$

Change of variables $x-t=z$ and $y-w=r$ imply $dx=dz$ and $dy=dr$ and hence

$$\begin{aligned} H^{\rho,\eta}_{\alpha,\beta}\left(f *^2 g\right)(\zeta,\xi) &= \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} f(t,w)\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} g(z,r)(\alpha\cos\zeta(z+t)+\beta\sin\zeta(z+t)) \\ &\quad\times(\rho\cos\zeta(r+w)+\eta\sin(r+w)\xi)\,dz\,dr\,dt\,dw. \end{aligned}$$

By aid of the facts $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ and $\sin(\alpha+\beta)=\sin\alpha\cos\beta-\sin\beta\cos\alpha$ and using simple computations we get

$$H^{\rho,\eta}_{\alpha,\beta}\left(f *^2 g\right)(\zeta,\xi) = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} f(t,w)\,\sigma(t,w)\,dw\,dt, \tag{14}$$

where

$$\begin{aligned} \sigma(t,w) &= \cos(t\zeta)\cos(wv)\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} g(z,r)(\alpha\cos(z\zeta)+\beta\sin(z\zeta))\times(\rho\cos(r\xi)+\eta\sin(r\xi))\,dz\,dr \\ &\quad-\cos(t\zeta)\sin(w\xi)\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} g(z,r)(\alpha\cos(z\zeta)+\beta\sin(z\zeta))\times(\rho\sin(r\xi)-\eta\cos(r\xi))\,dz\,dr \\ &\quad-\sin(t\zeta)\cos(w\xi)\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} g(z,r)(\alpha\sin(z\zeta)-\beta\cos(z\zeta))(\rho\cos(r\xi)+\eta\sin(r\xi))\,dz\,dr \\ &\quad+\sin(t\zeta)\sin(w\xi)\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} g(z,r)(\alpha\sin(z\zeta)-\beta\cos(z\zeta))\times(\rho\sin(r\xi)-\eta\cos(r\xi))\,dz\,dr. \end{aligned}$$

Hence, in view of (15), we get

$$H_{\alpha,\beta}^{\rho,\eta}\left(f *^2 g\right)(\zeta,\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(t,w) \begin{pmatrix} \cos(t\zeta)\cos(w\xi)\left(H_{\alpha,\beta}^{\rho,\eta} g(\zeta,\xi)\right) - \\ \cos(t\zeta)\sin(w\xi)\left(H_{\alpha,\beta}^{\rho,-\eta} g(\zeta,\xi)\right) - \\ \sin(t\zeta)\cos(w\xi)\left(H_{\alpha,-\beta}^{\rho,\eta} g(\zeta,\xi)\right) + \\ \sin(t\zeta)\sin(w\xi)\left(H_{\alpha,-\beta}^{\rho,-\eta} g(\zeta,\xi)\right) \end{pmatrix} dt\, dw$$

$$= \left(\left(H_{\alpha,\beta}^{\rho,\eta} - H_{\alpha,\beta}^{\rho,-\eta} - H_{\alpha,-\beta}^{\rho,\eta} + H_{\alpha,-\beta}^{\rho,-\eta}\right) g\right)(\zeta,\xi) \int_{\mathbb{R}}\int_{\mathbb{R}} f(t,w)\,\cos(t\zeta)\,\cos(w\xi)\, dt\, dw.$$

This can be put into the form

$$H_{\alpha,\beta}^{\rho,\eta}\left(f *^2 g\right)(\zeta,\xi) = \Psi(g)(\zeta,\xi) \times J(\zeta,\xi), \tag{15}$$

where $\Psi = H_{\alpha,\beta}^{\rho,\eta} - H_{\alpha,\beta}^{\rho,-\eta} - H_{\alpha,-\beta}^{\rho,\eta} + H_{\alpha,-\beta}^{\rho,-\eta}$. To complete the proof of the theorem, it is sufficiently enough we show that $\Psi(g)(\zeta,\xi) = J(\zeta,\xi)$.

By aid of (15) we derive

$$\begin{aligned}\Psi(g)(\zeta,\xi) &= \left(\left(H_{\alpha,\beta}^{\rho,\eta} - H_{\alpha,\beta}^{\rho,-\eta} - H_{\alpha,-\beta}^{\rho,\eta} + H_{\alpha,-\beta}^{\rho,-\eta}\right) g\right)(\zeta,\xi) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} (\alpha\cos(z\zeta) + \beta\sin(z\zeta))2\eta\,\sin(r\xi)\, g(z,r)\, dz\, dr \\ &- \int_{\mathbb{R}}\int_{\mathbb{R}} (\alpha\cos(z\zeta) - \beta\sin(z\zeta))2\eta\,\sin(r\xi)\, g(z,r)\, dz\, dr.\end{aligned}$$

Hence, it follows that

$$\Psi(g)(\zeta,\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}} 4\beta\eta\,\sin(z\zeta)\sin(r\xi)\, g(z,r)\, dz\, dr = J(\zeta,\xi).$$

Hence the theorem is completely proved. ▷

2. Distributional $H_{\alpha,\beta}^{\rho,\eta}$ transforms

Denote by $\mathscr{T}^2$ the space of smooth functions over φ defined on $\mathbb{R}^2$ such that

$$\wp_{k,K}(\varphi) = \sup_{\mathbf{x}\in K}\left|\mathscr{D}^k\varphi(\mathbf{x})\right| < \infty,$$

where the supremum traverses all compact subsets K of $\mathbb{R}^2$. Denote by $\mathscr{T}'^2$ the conjugate space $\mathscr{T}^2$ of distributions of compact supports over $\mathbb{R}^2$. Then, due to Pathak [13], $\mathscr{T}^2$ defines a norm and the collection $\wp_{k,K}$ is separating. Hence it defines a Hausdörff topology on $\mathscr{T}^2$. It is easy to notice that the kernel function

$$K(\zeta,\xi,x,y) = (\alpha\cos(\zeta x) + \beta\sin(\zeta x))(\rho\cos(\xi y) + \eta\sin(\xi y)) \tag{16}$$

of (1) is a member of $\mathscr{T}^2$ and hence, leads to the generalized definition

$$\widehat{H_{\alpha,\beta}^{\rho,\eta}} f(\zeta,\xi) = \langle f(x,y), K(\zeta,\xi,x,y)\rangle, \tag{17}$$

where f is an arbitrary distribution in $\mathscr{T}'^2$.

Further simple properties of $\widehat{H_{\alpha,\beta}^{\rho,\eta}}$ can be derived from (17) as follows:

Theorem 2. *Let $f(x,y) \in \mathscr{T}^{'2}$, then we have*
(i) $\widehat{H_{\alpha,\beta}^{\rho,\eta}}$ *is well-defined;*
(ii) $\widehat{H_{\alpha,\beta}^{\rho,\eta}}$ *is linear;*
(iii) $\widehat{H_{\alpha,\beta}^{\rho,\eta}}$ *is one to one;*
(iv) $\widehat{H_{\alpha,\beta}^{\rho,\eta}}$ *is analytic and*

$$\frac{\partial \widehat{H_{\alpha,\beta}^{\rho,\eta}}}{\partial \zeta} f(\zeta,\xi) = \left\langle f(x,y), \frac{\partial}{\partial \zeta} K(\zeta,\xi,x,y) \right\rangle$$

and

$$\frac{\partial \widehat{H_{\alpha,\beta}^{\rho,\eta}}}{\partial \xi} f(\zeta,\xi) = \left\langle f(x,y), \frac{\partial}{\partial \xi} K(\zeta,\xi,x,y) \right\rangle .$$

⊲ Proof of Part (i) follows from (16). To prove Part (ii) Let $\alpha \in \mathbb{R}$ and $\widehat{H_{\alpha,\beta}^{\rho,\eta}}f$, $\widehat{H_{\alpha,\beta}^{\rho,\eta}}g$ be the $\widehat{H_{\alpha,\beta}^{\rho,\eta}}$ transforms of f and $g \in \mathscr{T}^{'2}$, respectively. Then we have

$$\alpha^* \left(\widehat{H_{\alpha,\beta}^{\rho,\eta}}f + \widehat{H_{\alpha,\beta}^{\rho,\eta}}g \right) = \langle \alpha^*(g(x,y) + f(x,y)), K(\zeta,\xi,x,y) \rangle .$$

By the concept of addition of distributions we get

$$\alpha^* \left(\widehat{H_{\alpha,\beta}^{\rho,\eta}}f + \widehat{H_{\alpha,\beta}^{\rho,\eta}}g \right)(\zeta,\xi) = \langle \alpha^* f(x,y), K(\zeta,\xi,x,y) \rangle + \langle \alpha^* g(x,y), K(\zeta,\xi,x,y) \rangle .$$

Hence, scalar multiplication in the space $\mathscr{T}^{'2}$ implies

$$\alpha^* \left(\widehat{H_{\alpha,\beta}^{\rho,\eta}}f + \widehat{H_{\alpha,\beta}^{\rho,\eta}}g \right)(\zeta,\xi) = \alpha^* \widehat{H_{\alpha,\beta}^{\rho,\eta}}f + \alpha^* \widehat{H_{\alpha,\beta}^{\rho,\eta}}g.$$

This completes the proof of the linearity axiom of $\widehat{H_{\alpha,\beta}^{\rho,\eta}}$.

To prove that $\widehat{H_{\alpha,\beta}^{\rho,\eta}}$ is one-to-one, we assume $\widehat{H_{\alpha,\beta}^{\rho,\eta}}f = \widehat{H_{\alpha,\beta}^{\rho,\eta}}g$. Then we have $\langle f(x,y), K(\zeta,\xi,x,y) \rangle = \langle g(x,y), K(\zeta,\xi,x,y) \rangle$. Hence

$$\langle f(x,y) - g(x,y), K(\zeta,\xi,x,y) \rangle = 0$$

in the distributional sense. Therefore, it follows that $f(x,y) = g(x,y)$. This proves Part (iii). To prove Part (iv) we refer to [13]. Hence the proof is completed. ⊳

The operation $*^2$ can be extended to $\mathscr{T}'^2$ as

$$\left\langle f(x,y) *^2 g(x,y), \varphi(x,y) \right\rangle = \langle f(x,y), \langle g(t,w), \varphi(t+x, y+w) \rangle \rangle .$$

We state without proof the following theorem.

Theorem 3. *Let $f(x,y), g(x,y) \in \mathscr{T}'^2$. Then we have*

$$\widehat{H_{\alpha,\beta}^{\rho,\eta}} \left(f(x,y) *^2 g(x,y) \right)(\zeta,\xi) = J(\zeta,\xi) G(\zeta,\xi),$$

where

$$G(\zeta,\xi) = 4\beta\eta \langle g(t,w), \sin(t\zeta)\sin(w\xi) \rangle , \quad J(\zeta,\xi) = \langle f(t,w), \cos(t\zeta)\cos(w\xi) \rangle .$$

For similar proof see Theorem 1. Hence we delete the details.

3. The quotient space of Boehmians

The idea of construction of Boehmians was initiated by the concept of regular operators. Construction of Boehmians is similar to that of field of quotients and in some cases, it gives just the field of quotients. The construction of Boehmians consists of the following elements:

(i) A set $\boldsymbol{A}$;

(ii) A commutative semigroup $(\boldsymbol{B}, *)$;

(iii) An operation $\odot : \boldsymbol{A} \times \boldsymbol{B} \to \boldsymbol{A}$ such that for each $x \in \boldsymbol{A}$ and $\upsilon_1, \upsilon_2, \in \boldsymbol{B}$,

$$x \odot (\upsilon_1 * \upsilon_2) = (x \odot \upsilon_1) \odot \upsilon_2;$$

(iv) A set $\Delta \subset \boldsymbol{B}^{\mathbb{N}}$ satisfying:

(a) If $x, y \in \boldsymbol{A}$, $(\upsilon_n) \in \Delta$, $x \odot \upsilon_n = y \odot \upsilon_n$ for all n, then $x = y$;

(b) If $(\upsilon_n), (\sigma_n) \in \Delta$, then $(\upsilon_n * \sigma_n) \in \Delta$ (Δ is the set of all delta sequences).

Consider

$$\mathscr{A} = \{(x_n, \upsilon_n) : \ x_n \in \mathbf{A},\ (\upsilon_n) \in \Delta,\ x_n \odot \upsilon_m = x_m \odot \upsilon_n, \forall\, m, n \in \mathbb{N}\}.$$

If $(x_n, \upsilon_n), (y_n, \sigma_n) \in \mathscr{A}$, $x_n \odot \sigma_m = y_m \odot \upsilon_n$, $\forall\, m, n \in \mathbb{N}$, then we say $(x_n, \upsilon_n) \sim (y_n, \sigma_n)$. The relation $\sim$ is an equivalence relation in $\mathscr{A}$. The space of equivalence classes in $\mathscr{A}$ is denoted by $\boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B}, *), \odot, \Delta)$. Elements of $\boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B},*), \odot, \Delta)$ are called Boehmians.

Between $\boldsymbol{A}$ and $\boldsymbol{\kappa}\left(\boldsymbol{A}, (\boldsymbol{B},*), \odot, \Delta\right)$ there is a canonical embedding expressed as

$$x \to \frac{x \odot s_n}{s_n} \quad \text{as} \quad n \to \infty.$$

The operation $\odot$ can be extended to $\boldsymbol{\kappa}\left(\boldsymbol{A}, (\boldsymbol{B},*), \odot, \Delta\right) \times \boldsymbol{A}$ by

$$\frac{x_n}{\upsilon_n} \odot t = \frac{x_n \odot t}{\upsilon_n}.$$

In $\boldsymbol{\kappa}\left(\boldsymbol{A}, (\boldsymbol{B}, *), \odot, \Delta\right)$, two types of convergence:

1) A sequence $(h_n) \in \boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B}, *), \odot, \Delta)$ is said to be δ convergent to $h \in \boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B},*), \odot, \Delta)$, denoted by $h_n \xrightarrow{\delta} h$ as $n \to \infty$, if there exists a delta sequence (υ_n) such that $(h_n \odot \upsilon_n)$, $(h \odot \upsilon_n) \in \boldsymbol{A}$, $\forall\, k, n \in \mathbb{N}$, and $(h_n \odot \upsilon_k) \to (h \odot \upsilon_k)$ as $n \to \infty$, in $\boldsymbol{A}$, for every $k \in \mathbb{N}$.

2) A sequence $(h_n) \in \boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B},*), \odot, \Delta)$is said to be Δ convergent to $h \in \boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B},*), \odot, \Delta)$, denoted by $h_n \xrightarrow{\Delta} h$ as $n \to \infty$, if there exists a $(\upsilon_n) \in \Delta$ such that $(h_n - h) \odot \upsilon_n \in \boldsymbol{A}, \forall\, n \in \mathbb{N}$, and $(h_n - h) \odot \upsilon_n \to 0$ as $n \to \infty$ in $\boldsymbol{A}$.

For further details we refer to [1–9] and [11–14].

Let $\mathscr{D}^2$ be the Schwartz space of test functions of bounded supports over $\mathbb{R}^2$ and Δ^2 be the subset of $\mathscr{D}^2$ of sequences $(\theta_n(x, y))$ such that

(i) $\int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \theta_n(x, y)\, dx\, dy = 1$;

(ii) $\int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} |\theta_n(x, y)|\, dx\, dy \leqslant M, M$ is positive real number;

(iii) $\operatorname*{supp}\limits_{(x,y)\in\mathbb{R}^2} \theta_n(x, y) \to (0, 0)$ as $n \to \infty$.

Then Δ^2 is a set of delta sequences which correspond to the delta distribution $\delta(x,y)$. It is know from literature that $\delta(x,y)=0$, $x\neq 0$, $y\neq 0$ and $\int_{\mathbb{R}}\int_{\mathbb{R}}\delta(x,y)\,dx\,dy=1$ ($\Rightarrow \delta(x,y)=\delta(x)\delta(y)$). It also verified that

$$\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\delta(x-\alpha,y-\beta)\,f(x,y)\,dx\,dy=f(\alpha,\beta),$$

where α and β are constants.

Let $(\delta_n(x,y))\in\Delta^2$. Then it is easy to see that

$$\left(H^{\rho,\eta}_{\alpha,\beta}\delta_n(x,y)\right)(\zeta,\xi)\to\frac{\alpha\rho}{2\pi}\text{ as } n\to\infty.$$

Let $\mathscr{B}\left(\mathscr{T}^2,\mathscr{D}^2,\Delta^2,*^2\right)$ be the Boehmian space having $\mathscr{T}^2$ as a group, $\mathscr{D}^2$ as a subgroup of $\mathscr{T}^2$, $\mathscr{D}^2$ as the set of delta sequences and $*^2$ being the operation on $\mathscr{T}^2$ then we introduce the following definitions.

Let $f(t,w)\in\mathscr{T}^2$, $\theta(t,w)\in\mathscr{D}^2$ and $(\theta_n(t,w))\in\Delta^2$. We will usually choose $\mathfrak{h}(\zeta,\xi)$, $\mathfrak{g}(\zeta,\xi)$ and $\mathfrak{e}_n(\zeta,\xi)$ to denote

$$\mathfrak{h}(\zeta,\xi)=4\beta\eta\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}f(t,w)\sin(t\zeta)\sin(w\xi)\,dt\,dw, \tag{18}$$

$$\mathfrak{g}(\zeta,\xi)=\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\theta(t,w)\cos(t\zeta)\cos(w\xi)\,dt\,dw, \tag{19}$$

$$\mathfrak{e}_n(\zeta,\xi)=\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\theta_n(t,w)\cos(t\zeta)\cos(w\xi)\,dt\,dw \tag{20}$$

provided the integrals exist.

Let $\mathscr{H}_1^2(\zeta,\xi)$ or $\mathscr{H}_1^2$ be the space of all $H^{\rho,\eta}_{\alpha,\beta}$ transforms of smooth functions $\mathfrak{h}(\zeta,\xi)$ such that for some $f(t,w)\in\mathscr{T}^2$ (18) satisfies. By $\mathscr{H}_2^2(\zeta,\xi)$ or $\mathscr{H}_2^2$ denote the set of transforms of $\mathfrak{g}(\zeta,\xi)$ such that $\theta(t,w)\in\mathscr{D}^2$ and (19) satisfies and, similarly, $\Delta_3^2(\zeta,\xi)$ or Δ_3^2 denote the set of all sequences $\mathfrak{e}_n(\zeta,\xi)$ such that for some $(\theta_n(t,w))\in\Delta^2$ where (20) holds.

Remark 1. Let $(\theta_n(t,w))\in\Delta^2$. Then we have

$$\mathfrak{e}_n(\zeta,\xi)=\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\theta_n(t,w)\cos(t\zeta)\cos(w\xi)\,dt\,dw\to 1\text{ as } n\to\infty. \tag{21}$$

This remark is a straightforward result of (20). Now we are generating the Boehmian space $\mathscr{B}\left(\mathscr{H}_1^2,\mathscr{H}_2^2,\Delta_3^2,\times^2\right)$.

To this aim, we define an operation between $\mathscr{H}_1^2$ and $\mathscr{H}_2^2$ as

$$\mathfrak{h}(\zeta,\xi)\times^2\mathfrak{g}(\zeta,\xi)=\mathfrak{h}(\zeta,\xi)\mathfrak{g}(\zeta,\xi). \tag{22}$$

We proceed to establish the axioms of the first construction.

Theorem 4. *Let $\mathfrak{h}(\zeta,\xi)\in\mathscr{H}_1^2(\zeta,\xi)$ and $\mathfrak{g}(\zeta,\xi)\in\mathscr{H}_2^2(\zeta,\xi)$. Then we have $\mathfrak{h}(\zeta,\xi)\times^2\mathfrak{g}(\zeta,\xi)\in\mathscr{H}_1^2(\zeta,\xi)$.*

$\lhd$ Let $\mathfrak{h}(\zeta,\xi)\in\mathscr{H}_1^2(\zeta,\xi)$, $\mathfrak{g}(\zeta,\xi)\in\mathscr{H}_2^2(\zeta,\xi)$. Then $\mathfrak{h}(\zeta,\xi)\mathfrak{g}(\zeta,\xi)=H^{\rho,\eta}_{\alpha,\beta}\left(f*^2\theta\right)(\zeta,\xi)$ for every $f(t,w)\in\mathscr{T}^2$ and $\theta(t,w)\in\mathscr{D}^2$. But since $f*^2\theta\in\mathscr{T}^2$ it follows that $\mathfrak{h}(\zeta,\xi)\times^2\mathfrak{g}(\zeta,\xi)\in\mathscr{H}_1^2$. This completes the proof of the theorem. $\rhd$

Theorem 5. *Let $\mathfrak{h}_1(\zeta,\xi), \mathfrak{h}_2(\zeta,\xi) \in \mathscr{H}_1^2(\zeta,\xi)$. Then for all $\mathfrak{g}(\zeta,\xi) \in \mathscr{H}_2^2(\zeta,\xi)$ we have $(\mathfrak{h}_1(\zeta,\xi) + \mathfrak{h}_2(\zeta,\xi)) \times^2 \mathfrak{g}(\zeta,\xi) = \mathfrak{h}_1(\zeta,\xi) \times^2 \mathfrak{g}(\zeta,\xi) + \mathfrak{h}_2(\zeta,\xi) \times^2 \mathfrak{g}(\zeta,\xi)$.*

$\lhd$ Let $f_1(t,w), f_2(t,w) \in \mathscr{T}^2$ and $\theta(\zeta,\xi) \in \mathscr{D}^2$ be such that

$$\mathfrak{h}_1(\zeta,\xi) = 4\beta\eta \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} f_1(t,w) \sin(t\zeta) \sin(w\xi)\, dt\, dw,$$

$$\mathfrak{h}_2(\zeta,\xi) = 4\beta\eta \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} f_2(t,w) \sin(t\zeta) \sin(w\xi)\, dt\, dw$$

and

$$\mathfrak{g}(\zeta,\xi) = \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \theta(t,w) \cos(t\zeta) \cos(w\xi)\, dt\, dw;$$

then using the definition of $\times^2$

$$(\mathfrak{h}_1(\zeta,\xi) + \mathfrak{h}_2(\zeta,\xi)) \times^2 \mathfrak{g}(\zeta,\xi) = (\mathfrak{h}_1(\zeta,\xi) + \mathfrak{h}_2(\zeta,\xi))\, \mathfrak{g}(\zeta,\xi)$$
$$= \mathfrak{h}_1(\zeta,\xi)\mathfrak{g}(\zeta,\xi) + \mathfrak{h}_2(\zeta,\xi)\mathfrak{g}(\zeta,\xi) = \mathfrak{h}_1(\zeta,\xi) \times^2 \mathfrak{g}(\zeta,\xi) + \mathfrak{h}_2(\zeta,\xi) \times^2 \mathfrak{g}(\zeta,\xi).$$

This completes the proof. $\rhd$

Theorem 6. *Let$\mathfrak{h}_1(\zeta,\xi), \mathfrak{h}_2(\zeta,\xi) \in \mathscr{H}_1^2(\zeta,\xi)$. Then for all $\mathfrak{g}(\zeta,\xi) \in \mathscr{H}_2^2(\zeta,\xi)$ we have $(\alpha^* \mathfrak{h}_1(\zeta,\xi)) \times^2 \mathfrak{g}(\zeta,\xi) = \alpha^* \big(\mathfrak{h}_1(\zeta,\xi) \times^2 \mathfrak{g}(\zeta,\xi)\big)$.*

$\lhd$ Proof of this theorem is analogous to the previous proof. Details are omitted. $\rhd$

Theorem 7. *Let $\mathfrak{h}_n(\zeta,\xi) \to \mathfrak{h}(\zeta,\xi)$ in $\mathscr{H}_1^2(\zeta,\xi)$ and $\mathfrak{g}(\zeta,\xi) \in \mathscr{H}_2^2(\zeta,\xi)$; then $\mathfrak{h}_n(\zeta,\xi) \times^2 \mathfrak{g}(\zeta,\xi) \to \mathfrak{h}(\zeta,\xi) \times^2 \mathfrak{g}(\zeta,\xi)$.*

$\lhd$ Let $\mathfrak{h}_n(\zeta,\xi), \mathfrak{h}(\zeta,\xi) \in \mathscr{H}_1^2(\zeta,\xi)$ and $\mathfrak{g}(\zeta,\xi) \in \mathscr{H}_2^2(\zeta,\xi)$ satisfy for some $f_n, f \in \mathscr{T}^2$ and $\theta \in \mathscr{D}^2$. Then ofcourse $f_n \to f$ as $n \to \infty$. Therefore,

$$(\mathfrak{h}_n - \mathfrak{h})(\zeta,\xi) \times^2 \mathfrak{g}(\zeta,\xi) = (\mathfrak{h}_n - \mathfrak{h})(\zeta,\xi)\mathfrak{g}(\zeta,\xi)$$
$$= \mathfrak{g}(\zeta,\xi) \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} (f_n - f)(t,w) \sin(t\zeta) \sin(w\xi)\, dt\, dw \to\ 0 \text{ as } \ n \to \infty.$$

Hence, $(\mathfrak{h}_n - \mathfrak{h})(\zeta,\xi) \times^2 \mathfrak{g}(\zeta,\xi) \to 0$ as $n \to \infty$.

From which we write,

$$(\mathfrak{h}_n - \mathfrak{h})(\zeta,\xi)\mathfrak{g}(\zeta,\xi) = \mathfrak{h}_n(\zeta,\xi)\mathfrak{g}(\zeta,\xi) - \mathfrak{h}(\zeta,\xi)\mathfrak{g}(\zeta,\xi) \to 0 \text{ as } n \to \infty.$$

Thus

$$\mathfrak{h}_n(\zeta,\xi) \times^2 \mathfrak{g}(\zeta,\xi) \to \mathfrak{h}(\zeta,\xi) \times^2 \mathfrak{g}(\zeta,\xi) \text{ as } n \to \infty.$$

This completes the proof of the theorem. $\rhd$

Theorem 8. *Let $\mathfrak{h}_n(\zeta,\xi) \to \mathfrak{h}(\zeta,\xi)$ and $(\mathfrak{e}_n(\zeta,\xi)) \in \Delta_3^2$. Then $\mathfrak{h}_n(\zeta,\xi) \times^2 \mathfrak{e}_n(\zeta,\xi) \to \mathfrak{h}(\zeta,\xi)$.*

$\lhd$ Let $\mathfrak{h}_n(\zeta,\xi), \mathfrak{h}(\zeta,\xi) \in \mathscr{H}_1^2(\zeta,\xi)$ and $\mathfrak{e}_n(\zeta,\xi) \in \Delta_3^2$ satisfy for some $f_n, f \in \mathscr{T}^2$ and $(\theta_n) \in \Delta^2$. Then employing Remark 1 gives

$$\mathfrak{h}_n(\zeta,\xi) \times^2 \mathfrak{e}_n(\zeta,\xi) = \mathfrak{h}_n(\zeta,\xi)\mathfrak{e}_n(\zeta,\xi) \to \mathfrak{h}_n(\zeta,\xi) \to \mathfrak{h}(\zeta,\xi) \text{ as } n \to \infty.$$

This completes the proof of the Theorem. $\rhd$

Theorem 9. *Let $(\mathfrak{e}_n(\zeta,\xi)), (\mathfrak{r}_n(\zeta,\xi)) \in \Delta_3^2$. Then $\mathfrak{e}_n(\zeta,\xi) \times^2 \mathfrak{r}_n(\zeta,\xi) \in \Delta_3^2$.*

◁ By (22) we have

$$\mathfrak{e}_n(\zeta,\xi) \times^2 \mathfrak{r}_n(\zeta,\xi) = \mathfrak{e}_n(\zeta,\xi)\mathfrak{r}_n(\zeta,\xi) = j^{\rho,\eta}_{\alpha,\beta}\left(\theta_n *^2 \epsilon_n\right).$$

Hence by the fact that $\theta_n *^2 \epsilon_n \in \Delta^2$ it follows that $H^{\rho,\eta}_{\alpha,\beta}\left(\theta_n *^2 \epsilon_n\right) \in \Delta^2_3$. Hence the Theorem 9 is proved. ▷

The Boehmian space $\mathscr{B}\left(\mathscr{H}^2_1, \mathscr{H}^2_2, \Delta^2_3, \times^2\right)$ is therefore constructed.

A typical element in $\mathscr{B}\left(\mathscr{H}^2_1, \mathscr{H}^2_2, \Delta^2_3, \times^2\right)$ is of the form $\left[\frac{\mathfrak{h}_n}{\mathfrak{e}_n}\right]$. Addition, multiplication by a scalar, convolution and differentation in the space $\mathscr{B}\left(\mathscr{H}^2_1, \mathscr{H}^2_2, \Delta^2_3, \times^2\right)$ are defined as

$$\left[\frac{\mathfrak{h}_n}{\mathfrak{e}_n}\right] + \left[\frac{\mathfrak{d}_n}{\mathfrak{r}_n}\right] = \left[\frac{\mathfrak{h}_n \times^2 \mathfrak{r}_n + \mathfrak{d}_n \times^2 \mathfrak{e}_n}{\mathfrak{e}_n \times^2 \mathfrak{r}_n}\right]$$

$$\kappa\left[\frac{\mathfrak{h}_n}{\mathfrak{e}_n}\right] = \left[\frac{\kappa\mathfrak{h}_n}{\mathfrak{e}_n}\right], \quad \kappa \text{ being complex number.}$$

$$\left[\frac{\mathfrak{h}_n}{\mathfrak{e}_n}\right] \times^2 \left[\frac{\mathfrak{d}_n}{\mathfrak{r}_n}\right] = \left[\frac{\mathfrak{h}_n \times^2 \mathfrak{d}_n}{\mathfrak{e}_n \times^2 \mathfrak{r}_n}\right] \text{ and } \mathscr{D}^\alpha\left[\frac{\mathfrak{h}_n}{\mathfrak{e}_n}\right] = \left[\frac{\mathscr{D}^\alpha\mathfrak{h}_n}{\mathfrak{e}_n}\right].$$

Δ and δ-convergence are defined as usual for Boehmian spaces.

4. $H^{\rho,\eta}_{\alpha,\beta}$ of generalized Boehmians

From previous analysis given in this article we define the $H^{\rho,\eta}_{\alpha,\beta}$ transform of $\left[\frac{f_n}{\theta_n}\right]$ as

$$\widehat{H^{\rho,\eta}_{\alpha,\beta}}\left[\frac{f_n}{\theta_n}\right] = \left[\frac{\mathfrak{h}_n}{\mathfrak{e}_n}\right], \tag{23}$$

where $\mathfrak{h}_n$, $\mathfrak{e}_n$ has the representation of (18) and (20).

It is clear that $\left[\frac{\mathfrak{h}_n}{\mathfrak{e}_n}\right] \in \mathscr{B}\left(\mathscr{H}^2_1, \mathscr{H}^2_2, \Delta^2_3, \times^2\right)$. Let $\left[\frac{f_n}{\theta_n}\right] = \left[\frac{g_n}{\epsilon_n}\right]$, then $f_n *^2 \epsilon_m = g_m *^2 \theta_n$. Applying $H^{\rho,\eta}_{\alpha,\beta}$ transform and using the convolution theorem yield

$$\mathfrak{h}_n \times^2 \mathfrak{r}_m = \mathfrak{d}_m \times^2 \mathfrak{e}_n,$$

where $\mathfrak{h}_n$, $\mathfrak{r}_m$, $\mathfrak{d}_m$, $\mathfrak{e}_n$ have similar representations as in (18) and (20). Therefore $\frac{\mathfrak{h}_n}{\mathfrak{e}_n} \sim \frac{\mathfrak{d}_n}{\mathfrak{r}_n}$. Hence $\left[\frac{\mathfrak{h}_n}{\mathfrak{e}_n}\right] = \left[\frac{\mathfrak{d}_n}{\mathfrak{r}_n}\right]$. Therefore, we have $\widehat{H^{\rho,\eta}_{\alpha,\beta}}\left[\frac{f_n}{\theta_n}\right] = \widehat{H^{\rho,\eta}_{\alpha,\beta}}\left[\frac{g_n}{\epsilon_n}\right]$. Therefore (23) is well-defined.

Following two theorem are straightforward proofs. We prefer we omit details.

Theorem 10. $\widehat{H^{\rho,\eta}_{\alpha,\beta}} : \mathscr{B}\left(\mathscr{T}^2, \mathscr{D}^2, \Delta^2, *^2\right) \to \mathscr{B}\left(\mathscr{H}^2_1, \mathscr{H}^2_2, \Delta^2_3, \times^2\right)$ *is linear.*

Theorem 11. $\widehat{H^{\rho,\eta}_{\alpha,\beta}} : \mathscr{B}\left(\mathscr{T}^2, \mathscr{D}^2, \Delta^2, *^2\right) \to \mathscr{B}\left(\mathscr{H}^2_1, \mathscr{H}^2_2, \Delta^2_3, \times^2\right)$ *is one-one.*

Theorem 12. $\widehat{H^{\rho,\eta}_{\alpha,\beta}} : \mathscr{B}\left(\mathscr{T}^2, \mathscr{D}^2, \Delta^2, *^2\right) \to \mathscr{B}\left(\mathscr{H}^2_1, \mathscr{H}^2_2, \Delta^2_3, \times^2\right)$ *is continuous with respect to δ convergence.*

◁ Let $\beta_n \to \beta$ in $\mathscr{B}\left(\mathscr{T}^2, \mathscr{D}^2, \Delta^2, *^2\right)$ as $n \to \infty$. We show that $\widehat{H^{\rho,\eta}_{\alpha,\beta}}\beta_n \to \widehat{H^{\rho,\eta}_{\alpha,\beta}}\beta$ in $\mathscr{B}\left(\mathscr{H}^2_1, \mathscr{H}^2_2, \Delta^2_3, \times^2\right)$ as $n \to \infty$. Let $\beta_n, \beta \in \mathscr{B}\left(\mathscr{T}^2, \mathscr{D}^2, \Delta^2, *^2\right)$, then we can find $f_{n,k}, f_k \in \mathscr{T}^2$ such that $\beta_n = \left[\frac{f_{n,k}}{\theta_k}\right]$ and $\beta = \left[\frac{f_k}{\theta_k}\right]$ and $f_{n,k} \to f_k$ as $n \to \infty$, $\forall\, k \in \mathscr{N}$.

Therefore $\overbrace{H_{\alpha,\beta}^{\rho,\eta}}\left[\frac{f_{n,k}}{\theta_k}\right]=\left[\frac{\mathfrak{h}_{n,k}}{\mathfrak{e}_k}\right]$ where $\mathfrak{h}_{n,k}$ and $\mathfrak{e}_k$ are the the corresponding integral equations of $f_{n,k}$ and θ_k, see (18) and (20). Hence, we have

$$\overbrace{H_{\alpha,\beta}^{\rho,\eta}}\left[\frac{f_{n,k}}{\theta_k}\right]=\left[\frac{\mathfrak{h}_{n,k}}{\mathfrak{e}_k}\right]\rightarrow\left[\frac{\mathfrak{h}_k}{\mathfrak{e}_k}\right]=\beta.$$

The proof is completed. $\rhd$

Theorem 13. $\overbrace{H_{\alpha,\beta}^{\rho,\eta}}:\mathscr{B}\left(\mathscr{T}^2,\mathscr{D}^2,\Delta^2,*^2\right)\rightarrow\mathscr{B}\left(\mathscr{H}_1^2,\mathscr{H}_2^2,\Delta_3^2,\times^2\right)$ *is continuous with respect to* Δ *convergence.*

$\lhd$ Let $\beta_n\overset{\Delta}{\rightarrow}\beta$ in $\mathscr{B}\left(\mathscr{T}^2,\mathscr{D}^2,\Delta^2,*^2\right)$, as $n\rightarrow\infty$. Then there is $f_n\in\mathscr{T}^2$ and $(\theta_k(\zeta,\xi))\in\Delta^2$ such that

$$(\beta_n-\beta)\times^2\theta_n=\left[\frac{f_n\times^2\theta_k}{\theta_k}\right]$$

and $f_n\rightarrow 0$ as $n\rightarrow\infty$. Hence

$$\overbrace{H_{\alpha,\beta}^{\rho,\eta}}((\beta_n-\beta)\times^2\theta_k)=\overbrace{H_{\alpha,\beta}^{\rho,\eta}}\left[\frac{f_n\times^2\theta_k}{\theta_k}\right]=\left[\frac{\mathfrak{h}_n\times^2\mathfrak{e}_k}{\mathfrak{e}_k}\right]\simeq\mathfrak{h}_n\rightarrow 0\ \text{ as }\ n\rightarrow\infty.$$

Hence the theorem is completely proved. $\rhd$

5. The inverse problem

Let $\left[\frac{\mathfrak{h}_n}{\mathfrak{e}_k}\right]\in\mathscr{B}\left(\mathscr{H}_1^2,\mathscr{H}_2^2,\Delta_3^2,\times^2\right)$. Then the inverse transform $\overbrace{H_{\alpha,\beta}^{\rho,\eta}}^{-1}$ of $\overbrace{H_{\alpha,\beta}^{\rho,\eta}}$ can be defined by

$$\overbrace{H_{\alpha,\beta}^{\rho,\eta}}^{-1}\left[\frac{\mathfrak{h}_n}{\mathfrak{e}_k}\right]=\left[\frac{f_n}{\theta_n}\right]$$

in the space $\mathscr{B}\left(\mathscr{T}^2,\mathscr{D}^2,\Delta^2,*^2\right)$.

Theorem 14. $\overbrace{H_{\alpha,\beta}^{\rho,\eta}}^{-1}:\mathscr{B}\left(\mathscr{H}_1^2,\mathscr{H}_2^2,\Delta_3^2,\times^2\right)\rightarrow\mathscr{B}\left(\mathscr{T}^2,\mathscr{D}^2,\Delta^2,*^2\right)$ *is a well-defined and linear.*

$\lhd$ Let $\left[\frac{\mathfrak{h}_n}{\mathfrak{e}_n}\right]=\left[\frac{\mathfrak{d}_n}{\mathfrak{r}_n}\right]\in\mathscr{B}\left(\mathscr{H}_1^2,\mathscr{H}_2^2,\Delta_3^2,\times^2\right)$. Then it follows that $\mathfrak{h}_n(\zeta,\xi)\times^2\mathfrak{r}_m(\zeta,\xi)=\mathfrak{d}_m(\zeta,\xi)\times^2\mathfrak{e}_n(\zeta,\xi)$, where

$$\mathfrak{h}_n(\zeta,\xi)=4\beta\eta\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}f_n(t,w)\sin(t\zeta)\sin(w\xi)\,dt\,dw,$$

$$\mathfrak{e}_n(\zeta,\xi)=\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\theta_n(t,w)\cos(t\zeta)\cos(w\xi)\,dt\,dw,$$

$$\mathfrak{d}_m(\zeta,\xi)=4\beta\eta\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}g_m(t,w)\sin(t\zeta)\sin(w\xi)\,dt\,dw,$$

and

$$\mathfrak{r}_n(\zeta,\xi)=\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\epsilon_n(t,w)\cos(t\zeta)\cos(w\xi)\,dt\,dw,\quad \epsilon_n,\theta_n\in\Delta^2,\quad f_n,g_n\in\mathscr{T}^2.$$

The meaning of $\times^2$ then leads to

$$\mathfrak{h}_n(\zeta,\xi)\mathfrak{r}_m(\zeta,\xi) = \mathfrak{d}_m(\zeta,\xi)\mathfrak{e}_n(\zeta,\xi).$$

Therefore, (22) gives

$$H^{\rho,\eta}_{\alpha,\beta}\left(f_n *^2 \theta_m\right)(\zeta,\xi) = H^{\rho,\eta}_{\alpha,\beta}\left(g_m *^2 \epsilon_n\right)(\zeta,\xi). \tag{24}$$

Since $H^{\rho,\eta}_{\alpha,\beta}$ is one-to-one, (24) yields $f_n *^2 \theta_m = g_m *^2 \epsilon_n$. Thus $\frac{f_n}{\theta_n} \sim \frac{g_n}{\epsilon_n}$, which then confirms $\left[\frac{f_n}{\theta_n}\right] = \left[\frac{g_n}{\epsilon_n}\right]$. This establishes that our transform is well-defined.

To establish linearity, we assume there are $\alpha_1^*, \alpha_2^* \in \mathbb{C}$, field of complex numbers, $\left[\frac{f_n}{\theta_n}\right], \left[\frac{g_n}{\epsilon_n}\right] \in \mathscr{B}\left(\mathscr{T}^2, \mathscr{D}^2, \Delta^2, *^2\right)$, then

$$\overbrace{H^{\rho,\eta}_{\alpha,\beta}}^{-1}\left(\left[\frac{\alpha_1^* f_n}{\theta_n}\right] + \left[\frac{\alpha_2^* g_n}{\epsilon_n}\right]\right) = \overbrace{H^{\rho,\eta}_{\alpha,\beta}}^{-1}\left(\left[\frac{\alpha_1^* f_n *^2 \epsilon_n + \alpha_2^* g_n *^2 \theta_n}{\theta_n *^2 \epsilon_n}\right]\right)$$

$$= \left[\frac{\alpha_1^* \mathfrak{h}_n \times^2 \mathfrak{r}_n + \alpha_2^* \mathfrak{d}_n \times^2 \mathfrak{e}_n}{\mathfrak{e}_n \times^2 \mathfrak{r}_n}\right] = \left[\frac{\alpha_1^* \mathfrak{h}_n}{\mathfrak{e}_n}\right] + \left[\frac{\alpha_2^* \mathfrak{d}_n}{\mathfrak{r}_n}\right] = \alpha_1^* \left[\frac{\mathfrak{h}_n}{\mathfrak{e}_n}\right] + \alpha_2^* \left[\frac{\mathfrak{d}_n}{\mathfrak{r}_n}\right].$$

This completes the proof of the theorem. ▷

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ОБ ОБОБЩЕНИИ ПРЕОБРАЗОВАНИЙ ФУРЬЕ И ХАРТЛИ ДЛЯ ОДНОГО ФАКТОР-КЛАССА ПОСЛЕДОВАТЕЛЬНОСТЕЙ

Эль-Омари Ш. Х.

В работе рассматривается некоторый класс распределений и строятся два пространства Боэхмианов для одного класса интегральных операторов. Устанавливается конволюционная теорема относительно пространств Боэхмианов. Возникающий при этом интегральный оператор корректно определен, линеен и однозначно задается соответствующим Боэхмианом. В работе также подробно рассматривается некоторая обратная задача.

Ключевые слова: интегральное преобразование, преобразование Хартли, преобразование Фурье, фактор пространство.